KEY

Student Name and ID Number

MATH 3012 Quiz 3, April 19, 2007, WTT

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1. Write the general solution of the advancement operator equation:
$(A+2)^3 4(A-3)^2 f = 0$.

$f(n) = c_1(-2)^n + c_2 n(-2)^n + c_3 n^2(-2)^n + c_4 3^n + c_5 n 3^n$

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2. Find a particular solution to the advancement operator equation:
$(A^2+2A-15)f(n) = 27(4)^n$.

Try $f(n) = c\cdot 4^n$.

$$\left.\begin{aligned} c\cdot 4^{n+2} + 2c4^{n+1} - 15\cdot c4^n &= 27\cdot 4^n \\ 16c4^n + 8c4^n - 15c4^n &= 27\cdot 4^n \\ 9c &= 27 \end{aligned}\right\} c = 3$$

$f(n) = 3(4)^n$

3. Find the unique solution to the advancement operator equation:
$(A^2+2A-15)f(n) = 27(4)^n$ with $f(0) = 18$ and $f(1) = 1$.

$A^2 + 2A - 15 = (A+5)(A-3)$

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general solution

$f(n) = c_1(-5)^n + c_2(3)^n + 3\cdot 4^n$

$c_1 + c_2 + 3 = 18$

$-5c_1 + 3c_2 + 12 = 1$

$$\left.\begin{aligned} c_1 + c_2 &= 15 \\ -5c_1 + 3c_2 &= -11 \end{aligned}\right\} \begin{aligned} -8c_1 &= -56 \\ c_1 &= 7 \quad c_2 = 8 \end{aligned}$$

$\boxed{f_n = 7(-5)^n + 8(3^n) + 3(4)^n}$

4. Write the Inclusion-Exclusion formula for d_n, the number of derangements of $\{1, 2, \ldots, n\}$:

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$$d_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)!$$

5. Use the formula in the preceding question to find the value of d_4.

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$$d_4 = \binom{4}{0}4! - \binom{4}{1}3! + \binom{4}{2}2! - \binom{4}{3}1! + \binom{4}{4}0!$$

$$= 24 - 24 + 6\cdot 2 - 4\cdot 1 + 1\cdot 1$$

$$= 12 - 4 + 1$$

$$= 9$$

6. Verify your answer to the previous question by listing all the derangements on $\{1, 2, 3, 4\}$.

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2 1 4 3	3 1 4 2	4 1 2 3
2 3 4 1	3 4 1 2	4 3 1 2
2 4 1 3	3 4 2 1	4 3 2 1

7. A data file `digraph_data.txt` has been read for a digraph whose vertex set is $[6]$. The weights on the directed edges are shown in the matrix below. In the space to the right, apply Dijkstra's algorithm to find the distance from vertex 1 to all other vertices in the graph. Also, for each x, find a shortest path from 1 to x.

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W	1	2	3	4	5	6
1	0	54	58	17	22	97
2	60	0	28	9	19	8
3	46	24	0	19	9	12
4	16	36	40	0	8	73
5	23	29	30	3	0	47
6	19	8	82	16	28	0

2	3	4	5	6
(1,2) 54	(1,3) 58	$\boxed{(1,4), 17}$	(1,5) 22	(1,6) 97
(1,4,2) 53	(1,4,3) 57		$\boxed{(1,5)\ 22}$	(1,4,6) 90
$\boxed{(1,5,2)\ 51}$	(1,5,3) 52			(1,5,6) 69
	$\boxed{(1,5,3)\ 52}$			(1,5,2,6) 59
				$\boxed{(1,5,2,6)\ 59}$

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8. The data file for a graph with vertex set $\{1, 2, \ldots, 7\}$ is shown below. In the space to the right, list in order the edges that would be found in carrying out Kruskal's algorithm (avoid cycles) and Prim's algorithm (build tree). Vertex 1 is the root.

graph1.txt	**Kruskal**	**Prim**
7		
2 7 24	2 7 24	6 1 35
6 1 35	6 1 35	2 1 38
2 1 38	2 1 38	2 7 24
7 6 39	1 4 45	1 4 45
1 4 45	3 1 53	3 1 53
6 4 47	5 2 58	5 2 58
3 1 53		
7 3 54		
4 7 56		
5 2 58		
4 5 60		

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9. Let $R(n, m)$ denote the least positive integer t so that every graph on t vertices contains a complete subgraph of size n or and independent set of size m. Bob has a new computer and prides himself on being a good programmer. One day, he boasts to Alice that with the assistance of his new computer, he has succeeded in verifying that $R(100, 150) \leq 2^{300}$. Alice is not impressed. Can you explain why?

We know $R(n,m) \leq \binom{n+m-2}{n-1}$ so

$$R(100,150) \leq \binom{248}{99} < 2^{248} << 2^{300}$$

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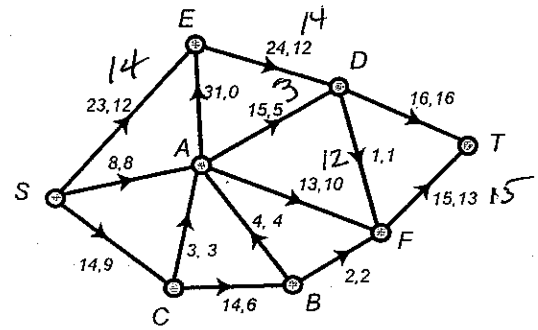
10.



a. What is the current value of the flow? $12 + 8 + 9 = 29$

b. What is the capacity of the cut $V = \{S, A, C, E\} \cup \{B, D, F, T\}$. $24 + 15 + 13 + 14 = 66$

c. Carry out the labeling algorithm, using the pseudo-alphabetic order on the vertices and list below the labels which will be given to the vertices.

S $(*, +, \infty)$
C $(S, +, 5)$
E $(S, +, 11)$
B $(C, +, 5)$
D $(E, +, 11)$
A $(D, -, 5)$
F $(A, +, 3)$
T $(F, +, 2)$

d. Use your work in part c to find an augmenting path and make the appropriate changes directly on the diagram.

Augmenting path (S E, D, A, F, T)

e. What is the value of the new flow?

$29 + 2 = 31$

Carry out the labeling algorithm a second time on the updated flow. It should halt without the sink being labeled. Find a cut whose capacity is equal to the value of the flow.

S $(*, +, \infty)$
C $(S, +, 5)$
E $(S, +, 9)$
B $(C, +, 5)$
D $(E, +, 9)$
A $(D, -, 3)$
F $(A, +, 1)$

Labelled

T

Unlabel

capacity =
$16 + 15$
$= 31$