Solutions

Student Name and ID Number

MATH 3012 Quiz 3, November 22, 2010, WTT

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1. Write the general solution of the advancement operator equation: $(A-3)^2(A+1)(A+4)^3 f = 0$.

$f(n) = c_1 3^n + c_2 n 3^n + c_3 (-1)^n + c_4 (-4)^n + c_5 n (-4)^n + c_6 n^2 (-4)^n$

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2. Find a particular solution to the advancement operator equation: $(A-5)f(n) = 33(2)^n$.

We try $f(n) = c \cdot 2^n$

$(A-5) c \cdot 2^n = c \cdot 2^{n+1} - 5c \cdot 2^n$

$= 2c \cdot 2^n - 5c \cdot 2^n$

$= -3c \cdot 2^n$

This requires $-3c = 33$ or $c = -11$

Answer: $\boxed{f(n) = -11 \cdot 2^n}$

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3. Find the unique solution to the advancement operator equation: $(A-5)f(n) = 33(2)^n$ with $f(1) = 13$.

General solution to homogeneous equation $(A-5)f = 0$ is $f(n) = c_1 5^n$, so solution has form $f(n) = c_1 5^n - 11 \cdot 2^n$

Substituting $n = 1$, we see $f(1) = 13 = 5c_1 - 22$.

$35 = 5c_1 \Rightarrow c_1 = 7$.

Answer: $\boxed{f(n) = 7 \cdot 5^n - 11 \cdot 2^n}$

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4. For positive integers n and m, let $S(n,m)$ count the number of surjections from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, m\}$. Write the Inclusion-Exclusion formula for $S(n,m)$:

$$S(n,m) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n$$

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5. Use the formula from the preceding problem to find the value of $S(6,3)$.

$$S(6,3) = \binom{3}{0} 3^6 - \binom{3}{1} 2^6 + \binom{3}{2} 1^6 - \binom{3}{3} 0^6$$

$$\boxed{\begin{aligned} &= 729 - 3 \cdot 64 + 3 \\ &= 540 \end{aligned}}$$

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6. For an integer $n \geq 2$, let $\phi(n)$ denote the Euler Phi Function. Use Inclusion-Exclusion to find $\phi(441)$. *Hint:* $441 = 9 \times 49$.

Prime factors of 441 are 3 and 7. Therefore

$\phi(441) = \boxed{441 \left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{7}\right)} = 441 \cdot \frac{2}{3} \cdot \frac{6}{7} = 21 \cdot 12 = \boxed{252}$

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7. Write the generating function for the number of partitions of an integer into distinct parts. *Hint:* Your answer should be expressed as an infinite product.

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$$f(n) = (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6)(1+x^7)\cdots$$
$$= \prod_{n=1}^{\infty} 1+x^n$$

8. Write all partitions of the integer 7 into distinct parts.

3

7 = 7
= 6 + 1
= 5 + 2
= 4 + 3
= 4 + 2 + 1

Total 5

9. Write all partitions of the integer 7 into odd parts.

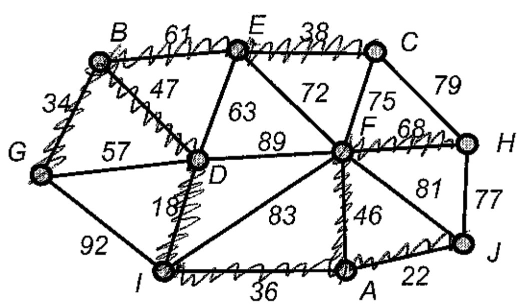
3

7 = 7
= 5 + 1 + 1
= 3 + 3 + 1
= 3 + 1 + 1 + 1 + 1
= 1 + 1 + 1 + 1 + 1 + 1 + 1

Total 5

10. Consider the weighted graph shown below. In the space below the figure, list *in order* the edges which make up a minimum weight spanning tree using, respectively Kruskal's Algorithm (avoid cycles) and Prim's Algorithm (build tree). For Prim, use vertex *A* as the root.

16 = 2×8



Kruskal's Algorithm 8

DI 18
AJ 22
BG 34
AI 36
EC 38
AF 46
BD 47
BE 61
FH 68

Prim's Algorithm 8

AJ 22
AI 36
DI 18
AF 46
BD 47
BG 34
BE 61
EC 38
FH 68

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11. Consider a directed graph G with vertex set $\{1,2,3,4,5,6\}$. In the matrix below, the entry $w(i,j)$ denotes the length of the directed edge from i to j in G. In the space to the right of the matrix, carry out Dijkstra's algorithm to find all shortest paths from root node 1 to the other five vertices.

W	1	2	3	4	5	6
1	0	32	92	25	12	99
2	60	0	20	1	30	26
3	46	60	0	19	42	6
4	16	13	17	0	8	24
5	23	11	35	13	0	38
6	19	1	82	16	10	0

1,2 32   1,3 92   1,4 25 $\boxed{1,5 \;\; 12}$ 1,6 99

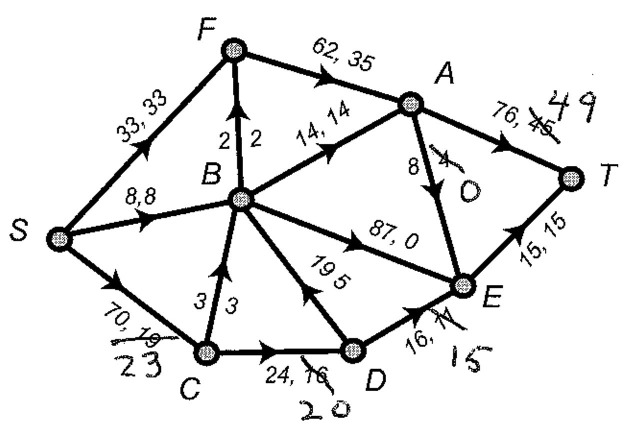
5→P $\boxed{1,5,2 \;\; 23 = 12+11}$ 1,5,3 47 = 12+35   1,4 25   1,5,6 50 = 12+38

2→P   1,5,2,3 43 = 23+20 $\boxed{1,5,2,4 \;\; 24 = 23+1}$ 1,5,2,6 49 = 23+26

4→P $\boxed{1,5,2,4,3 \;\; 41 = 24+17}$ 1,5,2,4,6 48 = 24+24

3→P $\boxed{1,5,2,4,3,6 \;\; 47 = 41+6}$

12. Consider the network flow problem illustrated below.



a. What is the current value of the flow?

3

value = $\boxed{33+8+19} = \boxed{45+15} = \boxed{60}$

b. What is the capacity of the cut $S = \{S, B, C\}$, $T = \{A, D, E, F, T\}$.

3

capacity = $\boxed{33+2+14+87+24} = \boxed{160}$

c. Carry out the labeling algorithm, using the pseudo-alphabetic order $S, T, A, B, C, \ldots$ on the vertices and list below the labels which will be given to the vertices.

10

S (*, +, ∞)
C (S, +, 51)
D (C, +, 8)
B (D, +, 8)
E (D, +, 5)
A (E, −, 4)
T (A, +, 4)

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d. Use your work in part c to find an augmenting path and make the appropriate changes to increase the flow directly on the diagram. What is the value of the new flow?

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$\boxed{64} = \boxed{60 + 4}$

e. Carry out the labeling algorithm a second time on the updated flow. It should halt without the sink being labeled. Find a cut whose capacity is equal to the value of the updated flow.

8

S $(*, +, \infty)$
C $(S, +, 47)$
D $(S, +, 4)$
B $(D, +, 4)$
E $(D, +, 1)$

$S = \{S, C, D, B, E\}$ $T = \{F, A, T\}$

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Grading Summary

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