Complete Solutions for MATH 3012 Quiz 3, November 22, 2011, WTT

Note. The answers given here are more complete than is expected on an actual exam. It is intended that the more comprehensive solutions presented here will be valuable to students in studying for the final exam. In a few places, the wording of a problem is changed slightly to reflect the modifed layout. A table providing point values for the problems is given at the very end.

1. Find the general solution to the advancement operator equation:
$A^2(A-4)^3(A+6)^4(A-5)f(n) = 0$

The term A^2 is ignored since this is just a shifting factor. The general solution is then

$$f(n) = c_1 4^n + c_2 n 4^n + c_3 n^2 4^n + c_4(-6)^n + c_5 n(-6)^n + c_6 n^2(-6)^n + c_7 n^3(-6)^n + c_8 5^n.$$

2. Find the solution to the advancement operator equation:
$(A^2 - 8A + 12)f(n) = 0, \quad f(0) = 1 \text{ and } f(1) = 34.$

First we note that $A^2-8A+12 = (A-6)(A-2)$ so the general solution is $f(n) = c_1 6^n + c_2 2^n$. The initial conditions lead to the equations $1 = c_1 + c_2$ and $34 = 6c_1 + 2c_2$. The unique solutions to this system of equations is $c_1 = 8$ and $c_2 = -7$. So the solution is $f(n) = 8 \cdot 6^n - 7 \cdot 2^n$.

3. Find a particular solution to the advancement operator equation:
$(A+3)g(n) = 7 \cdot 3^n$.

We search for a particular solution of the form: $c \cdot 3^n$. The equation requires

$$(A+3)c \cdot 3^n = c \cdot 3^{n+1} + 3c \cdot 3^n = 6c \cdot 3^n$$

to equal $7 \cdot 3^n$, so $c = 7/6$. The particular solution is then $(7/6)3^n$.

4. Find the form of a particular solution to (Express your answer with constants to be determined. Do not attempt to carry out the solution):
$(A-4)^3(A+6)^4(A-5)f(n) = 7 \cdot 5^n + 3n$.

Search for a particular solution of the form: $Cn \cdot 5^n + Dn + E$. Note that $(A-5)$ is one of the factors, so 5 is a root, and this requires the term $Cn \cdot 5^n$, rather than $C \cdot 5^n$.

5. How many permutations of $\{1, 2, 3, \ldots, 23\}$ satisfy the four requirements: $\sigma(3) = 3$, $\sigma(4) = 4$, $\sigma(11) = 11$ and $\sigma(19) = 19$?

Clearly, the answer is 19!, as the permutation may arrange the remaining $19 = 23 - 4$ integers in an arbitrary manner.

6. How many functions from $\{1, 2, 3, \ldots, 18\}$ to $\{1, 2, 3, \ldots, 11\}$ satisfy the three requirements: 2 is not in the range, 6 is not in the range, and 8 is not in the range.

The answer is 8^{18}, as each element of $\{1, 2, 3, \ldots, 18\}$ can be mapped to any of the eight elements of $\{1, 2, 3, \ldots, 11\} - \{2, 6, 8\}$.

7. The integer $65,000$ can be factored as $2^3 \cdot 5^4 \cdot 13$. Use the inclusion-exclusion formula to find $\phi(65,000)$.

$$\phi(65,000) = \phi(2^3 \cdot 5^4 \cdot 13) = 2^3 \cdot 5^4 \cdot 13(1 - \frac{1}{2})(1 - \frac{1}{5})(1 - \frac{1}{13} = 2^2 \cdot 5^3 \cdot 4 \cdot 12.$$

For the curious, this results in $\phi(65,000) = 24,000$.

8. Interpret the coefficients of the function $(1+x)(1+x^3)(1+x^7)/(1-x^2)$ in terms of partitions of an integer. Then write all the partitions of the integer 11 that correspond to this interpretation.

The factor $(1+x)$ means the partition includes at most one 1; similarly $(1+x^3)$ means the partition includes at most one 3; while $(1+x^7)$ means the partition includes at most one 7. On the other hand, the factor $1/(1-x^2)$ means the partition can contain arbitrarily many 2's. Here are the resulting partitions:

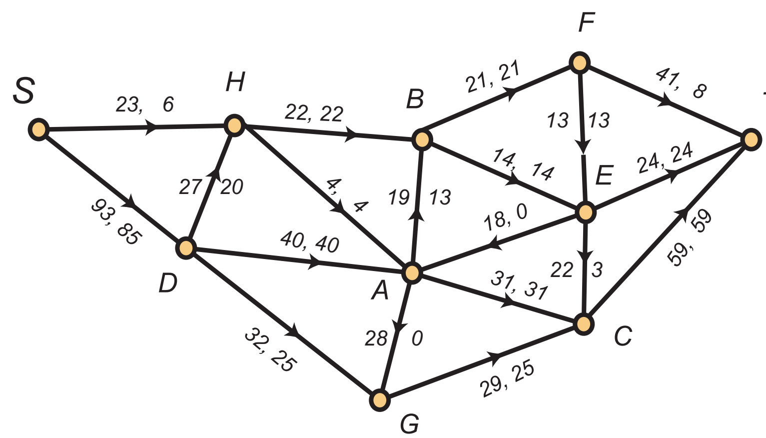
$$\begin{aligned} 11 &= 1+2+2+2+2+2 \\ &= 3+2+2+2+2 \\ &= 7+2+2 \\ &= 1+3+7 \end{aligned}$$

Accordingly, the coefficient of x^{11} is 4.

9. Consider the data file (shown on the left below) for the weights on the edges of a graph with vertex set $\{a, b, c, d, e, f, g, h\}$. In the space to the right, list in order the edges that would be selected in carrying out Kruskal's algorithm (avoid cycles) and Prim's algorithm to find a minimum weight spanning tree. For Prim, use vertex a as the root.

graphdata.txt			**Kruskal**			**Prim**		
e	h	10	e	h	10	a	d	17
c	e	12	c	e	12	c	d	14
b	c	13	b	c	13	c	e	12
c	d	14	c	d	14	e	h	10
b	h	15	f	g	16	b	c	13
f	g	16	a	d	17	b	g	20
a	d	17	b	g	20	f	g	16
a	b	18						
b	g	20						
e	d	21						
a	h	32						

10. Consider the following network



a. What is the current value of the flow?

A total of $91 = 85 + 6$ units leave the source, and $91 = 8 + 24 + 59$ units enter the sink. The value is 91 (either calculation would suffice).

b. What is the capacity of the cut $\{S, A, D, G, H\} \cup \{B, C, E, F, T\}$?

We add the capacities of the edges *from* the first set *to* the second. The capacity of this cut is then $22 + 19 + 31 + 29 = 101$. Recall that the value of any flow cannot exceed the capacity of any cut.

c. Write below the labels that are applied by carrying out the Ford-Fulkerson labeling algorithm.

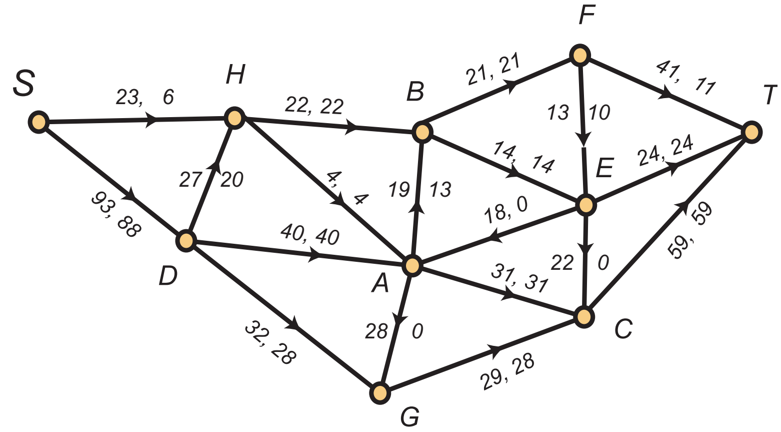
$$S(*, +, \infty)$$
$$D(S, +, 8)$$
$$H(S, +, 17)$$
$$G(D, +, 7)$$
$$C(G, +, 4)$$
$$A(C, -, 4)$$
$$E(C, -, 3)$$
$$B(A, +, 4)$$
$$F(E, -, 3)$$
$$T(F, +, 3)$$

d. Write the sequence of vertices that forms an augmenting path, as determined by the labeling done in the previous step.

Using back-tracking, we obtain the path (S, D, G, C, E, F, T).

e. Use the information gleaned from the previous two parts to update the flow.

We show the updated flow on the following revised figure. Note that there are four forward edges (first three and the last) and two backwards edges. edges. Increase the flow by 3 on the forward edges and decrease it by 3 on the backwards edges.



f. What is the new value value of the flow?

The value of the new flow is $94 = 91 + 3$.

g. Write below the labels that are applied by carrying out the Ford-Fulkerson labeling algorithm on the updated network. It should terminate without the sink being labeled.

$$S(*, +, \infty)$$
$$D(S, +, 5)$$
$$H(S, +, 17)$$
$$G(D, +, 4)$$
$$C(G, +, 1)$$
$$A(C, -, 1)$$
$$B(A, +, 1)$$

h. Find a cut whose capacity is the value of the current flow.

The algorithm has halted with the labeled vertices $\mathcal{L} = \{S, A, B, C, D, G, H\}$ and the unlabeled vertices $\mathcal{U} = \{T, E, F\}$. The capacity of this cut is $21 + 14 + 59 = 94$, which is the current value.

Point Totals

1. Eight points.
2. Ten points.
3. Eight points.
4. Eight points.
5. Eight points.
6. Eight points.
7. Eight points.
8. Eight points.
9. Ten points.
10. Twenty four points. $24 = 8 \times 3$.

The total is 100.