Solutions

Student Name and ID Number

MATH 3012 Quiz 1, September 17, 2015, WTT

1. Consider the 15-element set consisting of the ten digits $\{0, 1, 2, \ldots, 9\}$ and the five capital letters $\{A, B, C, D, E\}$.

⑨
3×3

a. How many strings of length 10 can be formed if repetition of symbols is permitted?

15^{10}

b. How many strings of length 10 can be formed if repetition of symbols is *not* permitted?

$P(15, 10)$ OR $15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$

c. How many strings of length 10 can be formed using exactly two A's, five B's and three C's?

$\binom{10}{2,5,3}$ OR $\frac{10!}{2!\,5!\,3!}$

③

2. How many lattice paths from $(0, 0)$ to $(7, 7)$ do travel through any point above the diagonal?

$\frac{\binom{14}{7}}{8}$ (Catalan Number)

3. How many integer valued solutions to the following equations and inequalities:

⑱
6×3

a. $x_1 + x_2 + x_3 + x_4 = 52$, all $x_i > 0$.

$\binom{51}{3}$ (51 gaps, choose 3)

b. $x_1 + x_2 + x_3 + x_4 = 52$, all $x_i \geq 0$.

$\binom{55}{3}$ (4 artificial elements, 55 gaps)

c. $x_1 + x_2 + x_3 + x_4 < 52$, all $x_i > 0$.

$\binom{51}{4}$ (add positive slack variable x_5)

d. $x_1 + x_2 + x_3 + x_4 \leq 52$, all $x_i \geq 0$.

$\binom{56}{4}$ (add non-negative slack variable x_5)

e. $x_1 + x_2 + x_3 + x_4 = 52$, $x_1, x_3, x_4 > 0$, $x_2 \geq 8$.

$\binom{44}{3}$ (set aside 7)

f. $x_1 + x_2 + x_3 + x_4 = 52$, $x_1, x_3, x_4 > 0$, $0 < x_2 \leq 7$.

$\binom{51}{3} - \binom{44}{3}$ (part a - part e)

4. Find the coefficient of $a^5 b^{12} c^{21}$ in $(6a - 3b^2 - 4c^3)^{18}$

$\binom{18}{5,6,7} 6^5 (-3)^6 (-4)^7$

⑥

36

⑥ **5.** Use the Euclidean algorithm to find $d = \gcd(3960, 840)$.

$$\begin{array}{r} 4 \\ 840\overline{)3960} \\ 3360 \\ \hline 600 \end{array} \qquad \begin{array}{r} 1 \\ 600\overline{)840} \\ 600 \\ \hline 240 \end{array} \qquad \begin{array}{r} 2 \\ 240\overline{)600} \\ 480 \\ \hline \boxed{120} \end{array} \qquad \begin{array}{r} 2 \\ 120\overline{)240} \\ 0 \end{array}$$

$120 = g.c.d(3960, 840)$

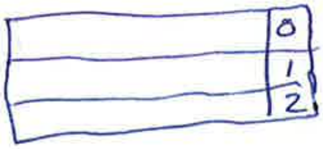
⑥ **6.** Use your work in the preceding problem to find integers a and b so that $d = 3960a + 840b$.

$$120 = 600 - 2\cdot240$$
$$240 = 840 - 1\cdot600$$
$$600 = 3960 - 4\cdot840$$

$$\begin{aligned} 120 &= 600 - 2[\cancel{2440}\ 840 - 1\cdot600] \\ &= 3\cdot600 - 2\cdot840 \\ &= -2\cdot840 + 3[3960 - 4\cdot840] \\ &= 3\cdot3960 - 14\cdot840 \end{aligned}$$

$\boxed{a = 3 \quad b = -14}$

⑧ **7.** For a positive integer n, let t_n count the number of ternary strings of length n that do not contain 200 as a substring. Note that $t_1 = 3$, $t_2 = 9$ and $t_3 = 26$. Develop a recurrence relation for t_n and use it to compute t_4, t_5 and t_6.



← some over count here. take off

$$t_n = 3\cdot t_{n-1} - t_{n-3}$$
$$t_4 = 3\cdot26 - 3 = \cancel{81}\ 75$$
$$t_5 = 3\cdot\cancel{81}^{75} - 9 = \cancel{243}^{225} - 9 = \cancel{234}^{216}$$
$$t_6 = 3\cdot\cancel{234}^{216} - 26 = \underset{648}{\cancel{702}} - 26 = \underset{622}{\cancel{676}}$$

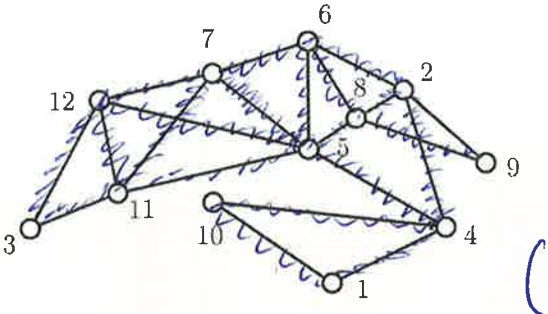
56

12

8. Use the greedy algorithm developed in class (always proceed to the lowest legal vertex) to find an Euler circuit in the graph G shown below (use node 1 as root):

INCOMPLETE - 8.

Cover every edge once



(1)

(1, 4, 2, 6, 5, 4, 10, 1)

2, 8, 5, 7, 6, 8, 9, 2)

(1, 4, 2, 8, 5, 7, 6, 8, 9, 2, 6, 5, 4, 10, 1)

(5, 11, 3, 12, 5)

(1, 4, 2, 8, 5, 11, 13, 12, 5, 7, 6, 8, 9, 2, 6, 5, 4, 10, 1)

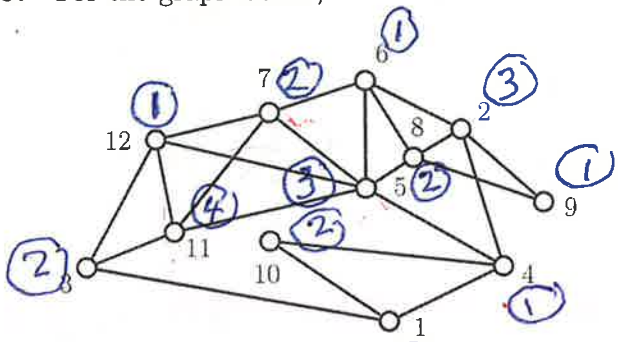
(11, 7, 12, 11)

Final (1, 4, 2, 8, 5, 11, 7, 12, 11, 13, 12, 5, 7, 6, 8, 9, 2, 6, 5, 4, 10, 1)

9. For the graph below,

12

3+4



(a) Find a clique of size 4. (3) - Fully connected

{5, 7, 11, 12}

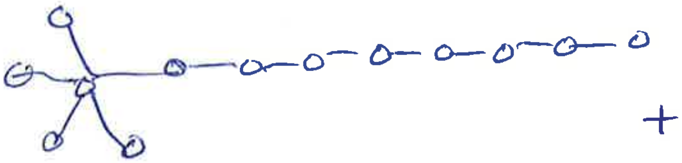
(b) Find an induced cycle of size 5. - Vertices not connected by another edge

{1, 4, 5, 11, 3}

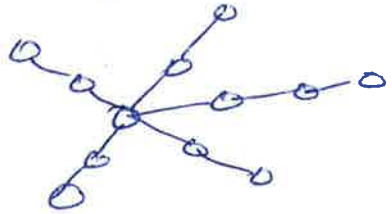
(c) Show that $\chi(G) \leq 4$ by producing a proper coloring using the elements of $\{1, 2, 3, 4\}$ as colors. You may write directly on the figure.

10. Draw a diagram of a tree on 12 vertices with exactly five leaves and exactly one vertex of degree 5.

5



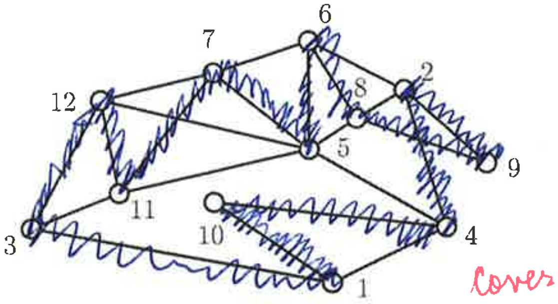
+ many others



85

8

11. Show that the following graph has a hamiltonian cycle. You may either darken the appropriate edges or provide a suitable permutation of the vertex set.



Several others

Covers every vertex once

7

12. True–False. Mark in the left margin.

F 1. $P(8, 3) = 330$. 336

F 2. $C(8, 3) = 65$. 56

T 3. If 67 pigeons are placed in 5 holes, then there is some hole with at least 13 pigeons. 5·12=60

T 4. If $f(n) = 624n^2 + 90n + 48n \log n$, and $g(n) = 3n^2 + 7n$, then $f(n) = O(g(n))$.

F 5. If $f(n) = 624n^2 + 90n + 48n \log n$, and $g(n) = 3n^2 + 7n$, then $g(n) = o(f(n))$.

T 6. $\log n = o(\sqrt{n})$, $\sqrt{n} = o(n)$, $n = o(n \log n)$, $n \log n = o(n^2)$, $n^2 = o(n^3)$ and $n^3 = o(2^n)$.

F 7. Any graph with 16 vertices and 153 edges has a hamiltonian cycle.

(similar to Dirac but not quite).

Typo should be 26 vertices. Error spotted after papers graded. All scores raised +1. wr

Scoring Chart

#1	9	#7	8
2	3	8	12
3	18	9	12
4	6	10	5
5	6	11	8
6	6	12	7
	48		52

100