Solutions

Student Name and ID Number

MATH 3012, Quiz 3, November 16, 2017, WTT

12

4x3

1. Consider the poset $\mathbf{2}^{17}$ consisting of all subsets of $[17] = \{1, 2, \ldots, 17\}$, partially ordered by inclusion.

a. The height of this poset is: 18

b. The width of this poset is: $\binom{17}{8}$ also $\binom{17}{9}$

c. The number of maximal chains in this poset is: $17!$

d. The number of maximal chains in this poset passing through the subset $\{2, 3, 8, 10, 14\}$ is:

$5!\ 12!$

18

6x3

2. This question concerns inclusion-exclusion.

a. How many permutations of the integers in $[17]$ satisfy the requirements that $\sigma(3) = 3$, $\sigma(7) = 7$ and $\sigma(14) = 14$?

$14!$

b. Write the inclusion-exclusion formula for d_{17}, the number of derangements of the integers in $[17]$.

$$d_{17} = \sum_{k=0}^{17} (-1)^k (17-k)! \binom{17}{k}$$

c. How many functions from $[10]$ to $[7]$ satisfy the requirement that neither 2 nor 6 is in the range of the function?

5^{10}

d. Write the inclusion-exclusion formula for $S(10, 7)$, the number of surjections from $[10]$ to $[7]$.

$$S(10,7) = \sum_{k=0}^{7} (-1)^k (7-k)^{10} \binom{7}{k}$$

e. For the integer $n = 3^4 \cdot 5^3 \cdot 7^6$, how many integers in $[n]$ are divisible by 3 and 7? You may give your answer in terms of n, using multiplication and division.

$\frac{n}{3 \cdot 7}$

f. Write out the inclusion-exclusion formula for $\phi(n)$ for this particular value of n. Again, you do not have to carry out any arithmetic in your answer.

$$\phi(n) = n \cdot (1 - \tfrac{1}{3})(1 - \tfrac{1}{5})(1 - \tfrac{1}{7})$$

15

3. This question concerns generating functions.

5×3

a. Find a partition of 37 into 5 distinct parts:

$25+6+3+2+1$ Many other correct answers

b. Find a partition of 37 into 5 odd parts:

$21+7+5+3+1$ MOCA

c. Write in product form the generating function for the number of partitions of an integer n into distinct parts:

$(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)(1+x^6)\cdots$

d. Write in product form the generating function for the number of partitions of an integer n into odd parts:

$$\frac{1}{1-x}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdot\frac{1}{1-x^7}\cdot\frac{1}{1-x^9}\cdots$$

e. Write in closed form the function whose Taylor series is $\sum_{n=0}^{\infty}\binom{2n}{n}x^n$.

$(1-4x)^{-\frac{1}{2}}$

15

4. This question concerns advancement operator equations.

3×5

a. Find the general solution to the advancement operator equation:
$(2-3i)\cdot(A-4+7i)^2(A-5)^4 f(n) = 0$

$f(n) = c_1(4-7i)^n + c_2 n(4-7i)^n + c_3 5^n + c_4 n 5^n + c_5 n^2 5^n + c_6 n^3 5^n$

b. Find a particular solution ~~the~~ for the advancement operator equation:

$$(A-7)f(n) = 3\cdot 7^n.$$

Try $f(n) = d\cdot n\cdot 7^n$

$(A-7)\, d\cdot n\cdot 7^n = d(n+1)7^{n+1} - 7dn7^n$

$= dn\cdot 7^{n+1} + d7^{n+1} - 7dn7^n$

$= 7dn\cdot 7^n + d7^{n+1} - 7dn7^n$

$= d7^{n+1}$

$= 7d7^n$ so $7d = 3$

$d = 3/7$

Answer $f(n) = \frac{3}{7}n\cdot 7^n$

c. Find the solution to the advancement operator equation:
$(A^2-8A+15)f(n) = 0, \quad f(0) = 2$ and $f(1) = 22.$

$(A-3)(A-5)$

$f(n) = c_1 3^n + c_2 5^n$

$f(0) = 2 = c_1 + c_2$

$f(1) = 22 = 3c_1 + 5c_2$

$6 = 3c_1 + 3c_2$

$16 = 2c_2$

$c_2 = 8$

$c_1 = -6$

Answer $f(n) = -6\cdot 3^n + 8\cdot 5^n$

16
2×8

5. Consider the data file (shown on the left below) for the weights on the edges of a graph with vertex set $\{a, b, c, d, e, f, g\}$. In the space to the right, list in order the edges that would be selected in carrying out Kruskal's algorithm (avoid cycles) and Prim's algorithm to find a minimum weight spanning tree. For Prim, use vertex a as the root.

graphdata.txt

b	c	22
d	e	23
f	g	24
b	f	25
c	g	26
b	e	27
c	d	28
a	e	29
c	f	30
e	c	31
a	g	32

Kruskal	Prim
bc	ae
de	de
fg	be
bf	bc
be	bf
ae	fg

15
5×3

6. Dijkstra's algorithm is being run on a weighted digraph with vertex set $\{1, 2, \ldots, 862\}$ to find shortest paths from vertex 1 to all other vertices. After 5 iterations, the vertices marked *permanent* are $\{1, 19, 38, 106, 392\}$ and scans have been completed from each of these five vertices. Here are the shortest paths the algorithm has found thus far:

$P(1) = (1)$ total length 0.
$P(19) = (1, 106, 19)$ total length 108.
$P(38) = (1, 106, 19, 38)$ total length 125.
$P(106) = (1, 106)$ total length 92.
$P(392) = (1, 392)$ total length 450.

Here are three specific candidate paths for the temporary vertices:

$P(52) = (1, 52)$ total length 978.
$P(309) = (1, 392, 309)$ total length 950.
$P(517) = (1, 106, 19, 38, 517)$ total length 980.

a. The weight $w(106, 19)$ of the edge $(106, 19)$ is 16.

b. The weight $w(19, 38)$ of the edge $(19, 38)$ is 17.

c. Suppose all temporary vertices other than 52, 309 and 517 have paths which currently have length at least 1000. The temporary vertex which is now marked permanent is 309.

d. If $w(309, 52) = 21$, is there a change in the path $P(52)$? If yes, write the new path $P(52)$ and give its length. YES. New Path P(52) = (1, 392, 309, 52) length 971

e. If $w(309, 517) = 30$, is there a change in the path $P(517)$? If yes, write the new path $P(517)$ and give its length. NO. ~~No~~ Alternative path has same length.

8

8×1

7. True–False. Mark in the left margin.

T 1. When $n \geq 2$, the cover graph of $\mathbf{2}^n$ has a hamiltonian cycle.

T 2. For all $n \geq 1$, the subset lattice $\mathbf{2}^n$ has a Dilworth partition using symmetric chains.

F 3. in $\mathbf{2}^9$, the following sets form a symmetric chain: $\{2,5\}$, $\{2,4,5\}$, $\{2,4,5,8\}$ and $\{1,2,4,5,8\}\}$.

F 4. The number of partitions of an integer n into even parts is equal to the number of partitions of n into parts, all of which have the same size.

F 5. Generating functions of the form $f(x) = \sum_{n=0}^{\infty} a_n x^n$ are only applied in combinatorics when they are a Taylor series.

F 6. When $p(A)$ is a polynomial in the advancement operator A and the degree of this polynomial is $d = 5^2 \cdot 7^4$, the solution space to the equation $p(A)f(n)$ is a vector space whose dimension is $d(1-1/5)(1-1/7)$.

F 7. To implement Kruskal's algorithm, it is not necessary to sort the edges by weight. One can simply take the edges in any order and take the first one avoiding a cycle when added to those edges already chosen.

F 8. Dijkstra's algorithm finds shortest paths having the maximum number of edges.

Fun!! 9. Weakly convergent generating operators which span Kruskal products have dimension at most the Sperner width of hamiltonian advancements.

Grading Summary

1.	12
2.	18
3.	15
4.	15
5.	16
6.	15
7.	8
TOTAL	99

Tests scored down from 100, i.e., blank test still gets +1.