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New Perspectives on Interval Orders and Interval Graphs

William T. Trotter

Summary Interval orders and interval graphs are particularly natural examples of two widely studied classes of discrete structures: partially ordered sets and undirected graphs. So it is not surprising that researchers in such diverse fields as mathematics, computer science, engineering and the social sciences have investigated structural, algorithmic, enumerative, combinatorial, extremal and even experimental problems associated with them. In this article, we survey recent work on interval orders and interval graphs, including research on on-line coloring, dimension estimates, fractional parameters, balancing pairs, hamiltonian paths, ramsey theory, extremal problems and tolerance orders. We provide an outline of the arguments for many of these results, especially those which seem to have a wide range of potential applications. Also, we provide short proofs of some of the more classical results on interval orders and interval graphs. Our goal is to provide fresh insights into the current status of research in this area while suggesting new perspectives and directions for the future.

1 Introduction

A complex process (manufacturing computer chips, for example) is often broken into a series of tasks, each with a specified starting and ending time. Task A *precedes* Task B if A ends before B begins. When A precedes B, the output of A can safely be used as input to B, and resources dedicated to the completion of A, such as machines or personnel, can now be applied to B. When A and B have overlapping time periods, they may be viewed as *conflicting* tasks, in the sense that they compete for limited resources.

This short paragraph is intended to motivate the formal definition of two of the most widely studied classes of discrete structures in all of combinatorial mathematics: interval orders and interval graphs. The main point to the discussion is that interval orders and interval graphs are important from an applications standpoint. This much is inescapable. They occur so naturally and with such frequency that they *must* be studied. Fortunately, the study of interval orders and interval graphs has yielded work of intrinsic interest and beauty, work that can be appreciated for its elegance independent of the fact that many find it useful and important.

The remainder of this section includes a brief summary of the notation and terminology necessary for the balance of the paper. For a more comprehensive treatment of background material, the reader is referred to Peter Fishburn's monograph *Interval Orders and Interval Graphs* [36]. Other recommended sources for background information are the author's survey articles [115], [116], [120], [121] and monograph [118] and the books by Golumbic [48] and

Roberts [98].

Throughout this paper, we consider a *partially ordered set* (or *poset*) $\mathbf{P} = (X, P)$ as a structure consisting of a set X and a reflexive, antisymmetric and transitive binary relation P on X. We call X the *ground set* of the poset $\mathbf{P}$, and we call P a *partial order* on X. The notations $x \leq y$ in P, $y \geq x$ in P and $(x, y) \in P$ are used interchangeably, and the reference to the partial order P is often dropped when its definition is fixed throughout the discussion. We write $x < y$ in P and $y > x$ in P when $x \leq y$ in P and $x \neq y$. When $x, y \in X$, $(x, y) \notin P$ and $(y, x) \notin P$, we say x and y are incomparable and write $x \| y$ in P. When $\mathbf{P} = (X, P)$ is a poset, we call the partial order $P^d = \{(y, x) : (x, y) \in P\}$ the *dual* of P and we let $\mathbf{P}^d = (X, P^d)$.

When P is a binary relation on X and $Y \subseteq X$, we denote the *restriction* of P to Y by $P(Y)$. When P is a partial order on X, $Q = P(Y)$ is a partial order on Y and $\mathbf{Q} = (Y, Q)$ is called a *subposet* of $\mathbf{P} = (X, P)$. Also, we call $\mathbf{Q}$ the subposet *determined* by Y. When $X_1, X_2, \ldots, X_r \subseteq X$, we will find it convenient to denote the subposets they determine by $\mathbf{X}_1, \mathbf{X}_2 \ldots, \mathbf{X}_r$, respectively. In this article, we tend not to distinguish between isomorphic posets, so we abuse language slightly and say that a poset $\mathbf{Q}$ is *contained* in another poset $\mathbf{P}$ when $\mathbf{Q}$ is isomorphic to a subposet of $\mathbf{P}$.

Although we are concerned primarily with *finite* posets, i.e., those posets with finite ground sets, we find it convenient to use the familiar notation $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ to denote respectively the reals, rationals, integers and positive integers equipped with the usual orders. Note that these four infinite posets are *total* orders; in each case, any two distinct points are comparable. Total orders are also called *linear* orders, or *chains*. When $X = X_1 \cup X_2 \cup \cdots \cup X_t$ is a partition and L_i is a linear order on X_i for each $i = 1, 2, \ldots, r$, we let $L = L_1 < L_2 < \cdots < L_r$ denote the linear order on X defined by $x < y$ in L if and only if $x \in X_i$, $y \in X_j$ and either $i < j$ or both $i = j$ and $x < y$ in L_i.

For a positive integer n, we let $\mathbf{n}$ denote the n-element chain $0 < 1 < \cdots < n - 1$. Somewhat inconsistently, we let $[n]$ denote the n-element set $\{1, 2, \ldots, n\}$. Also, when X is a set, we let $\binom{X}{n}$ denote the set of all n-element subsets of X.

Let $\mathbf{P} = (X, P)$ be a poset, and let $\mathcal{F} = \{\mathbf{Q}_x = (Y_x, Q_x) : x \in X\}$ be a family of posets indexed by the elements of X. Define the *lexicographic sum* of $\mathcal{F}$ over $\mathbf{P}$, denoted $\sum_{x \in \mathbf{P}} \mathcal{F}$, as the poset $\mathbf{Q} = (Y, Q)$ where $Y = \{(x, y) : x \in X, y \in Y_x\}$ and $(x_1, y_1) < (x_2, y_2)$ in Q if and only if $x_1 < x_2$ in P, or if both $x_1 = x_2$ and $y_1 < y_2$ in Q_{x_1}. With this definition, a disjoint sum is just a lexicographic sum over a two-element antichain.

In the remainder of this article, we will assume some familiarity with the basic concepts for partially ordered sets. The author's survey article on partially ordered sets [120] provides a thorough overview of the combinatorial aspects. Other sources for background material on posets are Brightwell's survey article [17] and the author's other survey articles [115], [112], [117] and [121].

2 Interval orders and interval gra

A poset $\mathbf{P} = (X, P)$ is called an *inter*
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3 Classical representation theore

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Three distinct vertices x, y and z in a gra
triple when for each two vertices in $\{x, y, z\}$
no vertex on the path adjacent to the third.

2 Interval orders and interval graphs

A poset $\mathbf{P} = (X, P)$ is called an *interval order* if there is a function I assigning to each element $x \in X$ a closed interval $I(x) = [a_x, b_x]$ of a linearly ordered set $\mathbf{L}$ (usually, we take $\mathbf{L}$ as the real line $\mathbb{R}$) so that for all $x, y \in X$, $x < y$ in P if and only if $b_x < a_y$ in L. We call I an *interval representation* of $\mathbf{P}$, or just a *representation* for short. For brevity, whenever we say that I is a representation of an interval order $\mathbf{P} = (X, P)$, we will use the alternate notation $[a_x, b_x]$ for the closed interval $I(x)$. Also, we let $|I(x)|$ denote the *length* of the interval, i.e., $|I(x)| = b_x - a_x$.

Note that end points of intervals used in a representation need not be distinct. In fact, distinct points x and y from X may satisfy $I(x) = I(y)$. We even allow degenerate intervals. On the other hand, a representation is said to be *distinguishing* if all intervals are non-degenerate and all end points are distinct. It is easy to see that every interval order has a distinguishing representation. In fact, since we are concerned only with finite posets, we could have just as well required that all intervals used in the representation be open.

Analogously, a graph $\mathbf{G} = (V, E)$ is an *interval graph* when there is a function I which assigns to each vertex $x \in V$ a closed interval $I(x) = [a_x, b_x]$ from a linearly ordered set $\mathbf{L}$ so that $\{x, y\} \in E$ if and only if $I(x) \cap I(y) \neq \emptyset$. As before, we call I an *interval representation* of $\mathbf{G}$ and note that, if desired, we may assume I is distinguishing.

Throughout this article, we will move back and forth between posets and graphs in discussions about a family of intervals. The interval graph determined by a family of intervals is just the incomparability graph of the interval order. Chains correspond to independent sets and antichains correspond to cliques.

3 Classical representation theorems

A good fraction of the early research on interval graphs and interval orders was focused on characterization issues. Recall that a graph is *triangulated* if it does not contain a cycle on four or more vertices as an induced subgraph. Also, a vertex x in a graph $\mathbf{G}$ is *simplicial* if its neighborhood is a complete subgraph of $\mathbf{G}$, so a graph is triangulated if and only if every induced subgraph has a simplicial vertex. Triangulated graphs are a well studied class of perfect graphs (see [58] and Chapter 4 of Golumbic's monograph [48], for example). Obviously, interval graphs are triangulated, but it is natural to ask whether all triangulated graphs are interval graphs. This is not true. In fact, not all trees are interval graphs, e.g. the subdivision of $\mathbf{K}(1, 3)$ is not an interval graph.

Three distinct vertices x, y and z in a graph $\mathbf{G}$ are said to form an *asteroidal triple* when for each two vertices in $\{x, y, z\}$, there is a path joining them, with no vertex on the path adjacent to the third. For example, the three leaves in a subdivision of $\mathbf{K}(1,3)$ form an asteroidal triple. In [83], Lekkerkerker and Boland proved that a triangulated graph is an interval graph if and only if it does not contain any asteroidal triples. They used this characterization theorem to provide a minimum list of forbidden subgraphs for interval graphs. This list includes the cycles on four or more vertices, three other infinite families and two isolated examples. One of these is the subdivision of $\mathbf{K}(1,3)$.

Other characterizations of interval graphs in terms of forbidden substructures have been provided by Gilmore and Hoffman [47] and by Ghouila-Houri [46]. Characterizations of interval graphs by forbidden subgraphs or forbidden substructures provide important structural information about the properties of interval graphs but do not necessarily yield a useful algorithm. Using a special kind of data structure called a *PQ-tree*, Booth and Lueker [15] produced an $O(n^2)$ algorithm for testing whether a graph $\mathbf{G}$ on n vertices is an interval graph and producing the representation when it is.

Characterization problems for interval graphs are closely related to characterization problems for comparability graphs. The classic paper of Gallai [45] provides a forbidden subgraph (again, in terms of induced subgraphs) characterization of comparability graphs with a minimum list including eight infinite families and 10 isolated examples. A comparability graph may have many different transitive orientations, but Gallai shows that if T_1 and T_2 are transitive orientations of the same comparability graph, then T_1 may be transformed into T_2 by a finite sequence of reversals applied to *autonomous sets*. Gallai's paper remains one of the deepest and most important contributions to this subject.

Next, we discuss three important representation theorems which are essential to understanding the material which follows. First, a finite poset $\mathbf{P} = (X, P)$ is called a *weak order* if there exists a function $f: X \to \mathbb{R}$ so that for all $x, y \in X$ with $x \neq y$,

1. $x < y$ in P if and only if $f(x) < f(y)$ in $\mathbb{R}$, and
2. $x \| y$ if and only if $f(x) = f(y)$.

The following elementary result is left as an exercise (see [36], e.g.).

Proposition 3.1 *Let $\mathbf{P} = (X, P)$ be a poset. Then the following are equivalent.*

1. *$\mathbf{P}$ is a weak order.*
2. *$\mathbf{P}$ does not contain $\mathbf{2}+\mathbf{1}$ as a subposet.*
3. *$\mathbf{P}$ is the lexicographic sum of a family of antichains over a chain.* ∎

Given a representation I of an interval order $\mathbf{P} = (X, P)$, there are two natural weak orders defined on X by the end points. The *ordering by left end points* L defined by $x < y$ in L if and only if $a_x < a_y$ in $\mathbb{R}$ and the *ordering by*

right end points R defined by $x < y$ in R
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Next, we have the following characteri
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For a poset $\mathbf{P} = (X, P)$ and a subset S
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say $S = \{x\}$, we write $D(x)$ and $D[x]$ rather
for a subset $S \subseteq X$, we define $U(S) = \{y \in$
$y > x$ in $P\}$. As before, set $U[S] = U(S) \cup$

Theorem 3.2 *Let $\mathbf{P} = (X, P)$ be a poset.*

1. *$\mathbf{P}$ is an interval order.*
2. *$\mathbf{P}$ does not contain $\mathbf{2}+\mathbf{2}$ as a subpos*
3. *Whenever $x < y$ and $z < w$ in P, the*
4. *For every $x, y \in X$, either $D(x) \subseteq D($*
5. *For every $x, y \in X$, either $U(x) \subseteq U($*

Proof The equivalence of the last four st
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An interval order $\mathbf{P} = (X, P)$ is called a
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right end points R defined by $x < y$ in R if and only if $b_x < b_y$ in $\mathbb{R}$. When the representation is distinguishing, these weak orders are linear orders.

Next, we have the following characterization theorem for interval orders due to Fishburn [35]. Our argument is motivated by proofs due to Bogart [5] and Greenough [51] and requires the following notation.

For a poset $\mathbf{P} = (X, P)$ and a subset $S \subset X$, let $D(S) = \{y \in X : \text{there exists some } x \in S \text{ with } y < x \text{ in } P\}$. Also, let $D[S] = D(S) \cup S$. When $|S| = 1$, say $S = \{x\}$, we write $D(x)$ and $D[x]$ rather than $D(\{x\})$ and $D[\{x\}]$. Dually, for a subset $S \subseteq X$, we define $U(S) = \{y \in X : \text{there exists some } x \in X \text{ with } y > x \text{ in } P\}$. As before, set $U[S] = U(S) \cup S$.

Theorem 3.2 *Let $\mathbf{P} = (X, P)$ be a poset. Then the following are equivalent.*

1. *$\mathbf{P}$ is an interval order.*
2. *$\mathbf{P}$ does not contain $\mathbf{2} + \mathbf{2}$ as a subposet.*
3. *Whenever $x < y$ and $z < w$ in P, then either $x < w$ or $z < y$ in P.*
4. *For every $x, y \in X$, either $D(x) \subseteq D(y)$ or $D(y) \subseteq D(x)$.*
5. *For every $x, y \in X$, either $U(x) \subseteq U(y)$ or $U(y) \subseteq U(x)$.*

Proof The equivalence of the last four statements is immediate. We now show that statement 1 is equivalent to 4. Suppose first that $\mathbf{P} = (X, P)$ is an interval order and that I is an interval representation of $\mathbf{P}$. Let $x, y \in X$; without loss of generality, we may assume $a_x \leq a_y$ in $\mathbb{R}$. Then $D(x) \subseteq D(y)$.

Now suppose that statement 4 holds for a poset $\mathbf{P} = (X, P)$. We show that $\mathbf{P}$ is an interval order. Let $Y = \{D(x) : x \in X\}$, and let $m = |Y|$. Then define a linear order L on Y by $D < D'$ in L if $D \subsetneq D'$. Then label the sets in Y so that $D_1 < D_2 < \cdots < D_m$ in L. For each $x \in X$, let $F(x) = [i, j]$, where $D(x) = D_i$ and $j = m$ if x is maximal, and $D_{j+1} = \bigcap\{D(y) : x < y \text{ in } P\}$, otherwise. ■

One advantage to the proof given here for Fishburn's representation theorem for interval orders is that the total number of end points used in the representation is minimal. Also, note that $|\{D(x) : x \in X\}| = |\{U(x) : x \in X\}|$, when $\mathbf{P} = (X, P)$ is an interval order, as pointed out by Greenough [51].

An interval order $\mathbf{P} = (X, P)$ is called a *semi-order* if there is a constant c for which it has an interval representation F such that the length of the interval $F(x)$ is exactly c, for every $x \in X$. From a modern perspective, it would perhaps be more natural if these objects were called *constant length* interval orders; however, after rescaling, it may be assumed that the constant length of intervals used in the representation of a semi-order is 1.

For semi-orders, we have the following representation theorem, the principal part of which is due to Scott and Suppes [104]. Simple proofs have been given by several authors, including Bogart [4], Fishburn [36] and Rabinovitch [92, 91].

Theorem 3.3 *Let* $\mathbf{P} = (X, P)$ *be an interval order. Then the following statements are equivalent.*

1. $\mathbf{P}$ *is a semi-order.*
2. $\mathbf{P}$ *does not contain* $\mathbf{3}+\mathbf{1}$ *as a subposet.*
3. *Whenever* $x < y < z$ *and* $w \| y$ *in* P, *then either* $x < w$ *or* $w < z$ *in* P.
4. *The binary relation* W *is a weak order on* X, *where*

$$\begin{aligned} W = &\{(x,y) \in X \times X : x = y\} \\ &\cup \{(x,y) \in X \times X : D(x) \subseteq D(y), U(y) \subsetneq U(x)\} \\ &\cup \{(x,y) \in X \times X : D(x) \subsetneq D(y), U(y) \subseteq U(x)\}. \end{aligned}$$

Proof The equivalence of statements 2, 3 and 4 is immediate. We show that statements 1 and 4 are equivalent. First let $\mathbf{P} = (X, P)$ be a semi-order and let I be an interval representation in which all intervals have length c. Let $I(x) = [a_x, a_x + c]$, for every $x \in X$. Then $(x, y) \in W$ if and only if $a_x \leq a_y$ in $\mathbb{R}$, so that W is a weak order on X.

Now suppose that statement 4 holds for a poset $\mathbf{P} = (X, P)$. We show that $\mathbf{P}$ is a semi-order. We actually prove something stronger. Let L be any linear order on X extending the weak order W. Proceeding by induction on $|X|$, we show that there exists a distinguishing interval representation I of $\mathbf{P}$ which assigns to each $x \in X$ a unit length interval $I(x) = [a_x, a_x + 1]$ such that for all $x, y \in X$, $a_x < a_y$ in $\mathbb{R}$ if and only if $x < y$ in L.

Noting that the claim holds trivially when $|X| = 1$, consider the inductive step. Suppose that L orders X as $x_1 < x_2 < \cdots < x_n$. Let $Y = X - \{x_n\}$, let $Q = P(Y)$ and $L' = L(Y)$. In the poset $\mathbf{Q} = (Y, Q)$, let W'' be the binary relation defined in statement 4 for the subposet $\mathbf{Q}$. Then let $W' = W(Y)$. Then $W'' \subseteq W' \subseteq L'$.

It follows that $\mathbf{Q}$ is a semi-order and that there exists a distinguishing representation I' of $\mathbf{Q}$ so that for all $y, z \in Y$, $a_y < a_z$ if and only if $y < z$ in L'. Also, $y < z$ in Q if and only if $a_y + 1 < a_z$. We now show that this representation can be extended by an appropriate choice of an interval $I(x_0) = [a_{x_n}, a_{x_n}+1]$ for x_0. If $y < x_n$ for every $y \in Y$, let $a = \max\{a_y : y \in Y\}$ and set $a_{x_n} = 2 + a$.

So we may assume that $S = \{y \in Y : y \| x_n\} \neq \emptyset$. It follows that there is a positive integer i so that $S = \{x_i, x_{i+1}, \ldots, x_{n-1}\}$, and $[a, b] = \bigcap\{I(y) : y \in S\}$ is a nondegenerate interval. If $D(x_n) = \emptyset$, set $a' = a$; otherwise, set $a' = \max\{a_y + 1 : y < x_n\}$. In either case, note that $a' < b$ in $\mathbb{R}$. It follows that we may take a_n as any real number between a' and b distinct from any end point previously chosen. ∎

There is an important corollary to the Scott-Suppes theorem for semi-orders, a result which was first noted by Roberts [97]. Call an interval order

$\mathbf{P} = (X, P)$ *proper* if it admits an interval r
and $x \neq y$, then $I(x) \not\subseteq I(y)$ and $I(y) \not\subseteq$
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4 Dilworth's theorem for interva

The *width* of a finite poset is the max
and the *height* is the maximum cardinality
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a poset of height h can be partitioned into
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Theorem 4.1 *Let* $\mathbf{P} = (X, P)$ *be a poset*
partition

$$X = A_1 \cup A_2 \cup \cdot$$

with A_j *an antichain, for each* $j \in [h]$.

Proof For each $x \in X$, let $d(x)$ denote the
by $D[x]$. Then for each $j \in [h]$, let $A_j = \{x$

Here is a sketch of how a bipartite matc
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Then let $\mathcal{M}$ be a maximum matching in
by putting distinct $x, y \in X$ in the same c
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For interval orders, we can be more exp
portant connection with graph coloring. Al
we summarize it briefly as the concepts wil
argument comes from Hajnal and Suranyi
graphs.

Theorem 4.2 *Interval graphs are perfect,*
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from applying First Fit to the vertices in th
in a distinguishing representation.

Proof Let I be a distinguishing represen
(X, E), and let L be the linear order on X
i.e., $x < y$ in L if and only if $b_x < b_y$ in
$\{y \in X : \{x, y\} \in E\}$. Note that $N_L(x)$ is a

$\mathbf{P} = (X, P)$ *proper* if it admits an interval representation I so that if $x, y \in X$ and $x \neq y$, then $I(x) \not\subseteq I(y)$ and $I(y) \not\subseteq I(x)$. A semi-order is obviously a proper interval order, but as Roberts pointed out, a proper interval order is also a semi-order. We will revisit this theme in Section 19.

4 Dilworth's theorem for interval orders

The *width* of a finite poset is the maximum cardinality of an antichain, and the *height* is the maximum cardinality of a chain. Dilworth's theorem [26] asserts that a poset of width w can be partitioned into w chains. Dually [58], a poset of height h can be partitioned into h antichains. Short proofs of these results have been provided by many authors, e.g. see [120] or [118]. Here is one for the second result.

Theorem 4.1 *Let $\mathbf{P} = (X, P)$ be a poset of height h. Then there exists a partition*

$$X = A_1 \cup A_2 \cup \cdots \cup A_h, \tag{1}$$

with A_j an antichain, for each $j \in [h]$.

Proof For each $x \in X$, let $d(x)$ denote the height of the subposet determined by $D[x]$. Then for each $j \in [h]$, let $A_j = \{x \in X : d(x) = j\}$. ∎

Here is a sketch of how a bipartite matching algorithm can be used to find the width w of a poset $\mathbf{P}$ and a partition into w chains. See Brightwell [17] for details. Given a poset $\mathbf{P} = (X, P)$, form a bipartite graph $G = (V, E)$ with $V = \{a_x : x \in X\} \cup \{b_x : x \in X\}$ and $E = \{\{a_x, b_y\} : x < y \text{ in } P\}$. Then let $\mathcal{M}$ be a maximum matching in $\mathbf{G}$. Define a chain partition of X by putting distinct $x, y \in X$ in the same chain when there exists a sequence $x = u_0, u_1, \ldots, u_s = y$ so that $\{a_{u_{i-1}}, b_{u_i}\} \in \mathcal{M}$, for every $i \in [s]$.

For interval orders, we can be more explicit by taking advantage of an important connection with graph coloring. Although this material is well known, we summarize it briefly as the concepts will be used later in this article. The argument comes from Hajnal and Suranyi's paper [58] on classes of perfect graphs.

Theorem 4.2 *Interval graphs are perfect, i.e., the chromatic number is equal to the maximum clique size. Furthermore, an optimal coloring always results from applying First Fit to the vertices in the order their right end points occur in a distinguishing representation.*

Proof Let I be a distinguishing representation of an interval graph $\mathbf{G} = (X, E)$, and let L be the linear order on X determined by the right end points, i.e., $x < y$ in L if and only if $b_x < b_y$ in $\mathbb{R}$. For each $x \in V$, let $N_L(x) = \{y \in X : \{x, y\} \in E\}$. Note that $N_L(x)$ is a complete subgraph of $\mathbf{G}$, for each

$x \in X$, as all the intervals corresponding to intervals in $N_L(x)$ contain the left end point a_x of $I(x)$.

When First Fit assigns color α to a vertex x, then x belongs to a complete subgraph of size α consisting of x and $\alpha - 1$ vertices from $N_L(x)$. It follows that if the maximum clique size of $\mathbf{G}$ is k, then First Fit will color $\mathbf{G}$ in exactly k colors. ■

Equivalently, First Fit partitions the associated interval order $\mathbf{P} = (X, P)$ for which I is a distinguishing representation into w chains, where w is its width.

In material to follow, we will find it convenient to be even more explicit than the argument used for Theorem 4.1 in applications of the dual to Dilworth's theorem. Set $A_0 = \emptyset$ and $X_0 = X$. Now suppose that we have defined A_{j-1} and X_{j-1} for some $j \geq 1$. Set $X_j = X_{j-1} - A_{j-1}$. If $X_j \neq \emptyset$, let y_j be the unique element of X_j for which the right end point $r_j = b_{y_j}$ is minimum. Then let $A_j = \{x \in X_j : b_{y_j} \in I(x)\}$. When the algorithm halts, we have a partition $X = A_1 \cup A_2 \cup \cdots \cup A_h$ into h antichains, and we have a chain $C = \{y_1, y_2, \ldots, y_h\}$ of cardinality h. Furthermore, every interval in the representation intersects the right end point of at least one interval in C. We call C the *lexicographically least* maximum chain of $\mathbf{P}$, and we call the associated partition into antichains the *canonical minimum partition.*

5 Linear extensions and dimension

When P and Q are binary relations on a set X, we say Q is an *extension* of P when $P \subseteq Q$; a linear order L on X is called a *linear extension* of a partial order P on X when $P \subseteq L$. A set $\mathcal{R}$ of linear extensions of P is called a *realizer* of $\mathbf{P}$ when $P = \bigcap \mathcal{R}$, i.e., for all x, y in X, $x < y$ in P if and only if $x < y$ in L, for every $L \in \mathcal{R}$. The minimum cardinality of a realizer of $\mathbf{P}$ is called the *dimension* of $\mathbf{P}$ and is denoted $\dim(\mathbf{P})$. Note that if $\mathbf{P}$ contains $\mathbf{Q}$, then $\dim(\mathbf{Q}) \leq \dim(\mathbf{P})$.

It is natural to ask what causes a poset to have large dimension. Here is a partial answer. For integers $n \geq 2$ and $k \geq 0$, define the *crown* $\mathbf{S}_n^k$ as the poset of height two with $n+k$ minimal elements $a_1, a_2, \ldots, a_{n+k}$, $n+k$ maximal elements $b_1, b_2, \ldots, b_{n+k}$ and ordering $a_i < b_j$ if and only if $j \in \{i+k+1, i+k+2, \ldots, i-1\}$. In this definition, the subscripts are interpreted cyclically, so that $b_{n+k+1} = b_1$, etc. When $n \geq 3$, the dimension of the crown $\mathbf{S}_n^k$ is given by the following formula [110]:

$$\dim(\mathbf{S}_n^k) = \left\lceil \frac{2(n+k)}{k+2} \right\rceil \tag{2}$$

For each $k \geq 0$, the poset $\mathbf{S}_2^k$ is the disjoint sum of $k+2$ two-element chains, so these posets have dimension 2. When $n \geq 3$, the crown $\mathbf{S}_n^k$ always has dimension at least 3. Posets in the family $\mathcal{S} = \{\mathbf{S}_n^0 : n \geq 2\}$ are referred to as

standard examples. Note that the dimension
for each $n \geq 3$, $\mathbf{S}_n^0$ is n-irreducible, i.e., th
subposet having dimension $n-1$. Also not
example $\mathbf{S}_n^0$ is isomorphic to the family of 1-el
of $\{1, 2, \ldots, n\}$ ordered by inclusion. The s
of a special case. It has dimension two and
of two 2-element chains, but it is not irreduc

We summarize some basic facts about di
tion, referring the reader to [118] for proofs

Proposition 5.1 *Let* $\mathbf{P} = (X, P)$ *and* $\mathbf{Q} =$

1. $\dim(\mathbf{P} + \mathbf{Q}) = \max\{2, \dim(\mathbf{P}), \dim(\mathbf{Q}$
2. $\dim(\mathbf{P} \times \mathbf{Q}) \leq \dim(\mathbf{P}) + \dim(\mathbf{Q})$, *with greatest and least elements.*
3. *The removal of a point from* $\mathbf{P}$ *decrea*
4. *If A is a maximum antichain in* $\mathbf{P}$, *th* $\max\{2, |X - A|\}$.
5. $\dim(\mathbf{P}) = \dim(\mathbf{P}^d)$. ■

Note that the family of standard exampl
of Proposition 5.1 are best possible. We w
inequality 1 in the preceding theorem in a
general formula for dimension and lexicogra

Proposition 5.2 *Let* $\mathbf{P} = (X, P)$ *be a pos*
$x \in X\}$ *be a family of posets. Then*

$$\dim\left(\sum_{x \in \mathbf{P}} \mathcal{F}\right) = \max\{\dim(\mathbf{P}), \max$$

For additional background information o
to the author's monograph [118], the survey
and Trotter and the survey articles [115] a
and [119] also discuss combinatorial problem
dimension for posets and a wide range of co
in [123], with greater detail provided in the

6 Linear extensions of interval o

When $\mathbf{P} = (X, P)$ is a poset, $A, B \subset X$
extension of P, we say B is *over* A in L w
$b \in B$ and $a \| b$ in P. In applying this definiti
do not require that $b > a$ in L, for *all* $a \in A$
pairs. The following elementary result was fi

standard examples. Note that the dimension of $\mathbf{S}_n^0$ is exactly n. Furthermore, for each $n \geq 3$, $\mathbf{S}_n^0$ is n-irreducible, i.e., the removal of any point leaves a subposet having dimension $n-1$. Also note that when $n \geq 3$, the standard example $\mathbf{S}_n^0$ is isomorphic to the family of 1-element and $(n-1)$–element subsets of $\{1, 2, \ldots, n\}$ ordered by inclusion. The standard example $\mathbf{S}_2^0$ is somewhat of a special case. It has dimension two and is isomorphic to the disjoint sum of two 2-element chains, but it is not irreducible.

We summarize some basic facts about dimension in the following proposition, referring the reader to [118] for proofs and references.

Proposition 5.1 *Let $\mathbf{P} = (X, P)$ and $\mathbf{Q} = (Y, Q)$ be posets. Then:*

1. $\dim(\mathbf{P} + \mathbf{Q}) = \max\{2, \dim(\mathbf{P}), \dim(\mathbf{Q})\}$.
2. $\dim(\mathbf{P} \times \mathbf{Q}) \leq \dim(\mathbf{P}) + \dim(\mathbf{Q})$, *with equality holding if* $\mathbf{P}$ *and* $\mathbf{Q}$ *have greatest and least elements.*
3. *The removal of a point from* $\mathbf{P}$ *decreases* $\dim(\mathbf{P})$ *by at most one.*
4. *If A is a maximum antichain in* $\mathbf{P}$, *then* $\dim(\mathbf{P}) \leq |A|$ *and* $\dim(\mathbf{P}) \leq \max\{2, |X - A|\}$.
5. $\dim(\mathbf{P}) = \dim(\mathbf{P}^d)$. ∎

Note that the family of standard examples shows that inequalities 3 and 4 of Proposition 5.1 are best possible. We will also find it convenient to put inequality 1 in the preceding theorem in a more general setting. Here is the general formula for dimension and lexicographic sums (see [118]).

Proposition 5.2 *Let $\mathbf{P} = (X, P)$ be a poset, and let $\mathcal{F} = \{\mathbf{Q}_x = (Y_x, P_x) : x \in X\}$ be a family of posets. Then*

$$\dim\left(\sum_{x \in \mathbf{P}} \mathcal{F}\right) = \max\{\dim(\mathbf{P}), \max\{\dim(\mathbf{Q}_x) : x \in X\}\}. \quad ∎ \tag{3}$$

For additional background information on dimension, the reader is referred to the author's monograph [118], the survey article [63] on dimension by Kelly and Trotter and the survey articles [115] and [121]. The articles [112], [117] and [119] also discuss combinatorial problems for posets. Connections between dimension for posets and a wide range of combinatorial problems are discussed in [123], with greater detail provided in the monograph [118].

6 Linear extensions of interval orders

When $\mathbf{P} = (X, P)$ is a poset, $A, B \subset X$ and $A \cap B = \emptyset$, and L is a linear extension of P, we say B is *over* A in L when $b > a$ in L, whenever $a \in A$, $b \in B$ and $a \| b$ in P. In applying this definition, it is important to note that we do not require that $b > a$ in L, for *all* $a \in A$ and $b \in B$, only the incomparable pairs. The following elementary result was first discovered by Rabinovitch [94].

Theorem 6.1 *Let $\mathbf{P} = (X, P)$ be an interval poset, and let $A, B \subset X$ with $A \cap B = \emptyset$. Then there exists a linear extension L of P with B over A in L.*

Proof Let I be a distinguishing interval representation of $\mathbf{P}$. For each $x \in X$, let $p_x = a_x$ if $x \in A$ and $p_x = b_x$, otherwise. Then define a linear extension L by setting $x < y$ in L if and only if $p_x < p_y$ in $\mathbb{R}$. ∎

More generally, the following proposition, first noted by Felsner in [28], is an easy exercise.

Proposition 6.2 *Let $\mathbf{P} = (X, P)$ be an interval order, and let I be any distinguishing interval representation of $\mathbf{P}$. If L is a linear extension of P, then it is possible to choose for each $x \in X$ a point $p_x \in I(x)$ so that $x < y$ in L if and only if $p_x < p_y$ in $\mathbb{R}$.* ∎

7 Dimension of interval orders

It is natural to ask whether an interval order can have large dimension. If the answer is yes, it cannot be due to the presence of large standard examples, as no interval order contains any of them. Note that for each $n \geq 2$, the subposet of $\mathbf{S}_n^0$ determined by a_1, a_2, b_1 and b_2 is isomorphic to $\mathbf{2}+\mathbf{2}$.

Nevertheless, interval orders may have large dimension, and to explain how this may occur, we introduce a standard example of an interval order. For an integer $n \geq 2$, let $\mathbf{I}_n = (\binom{[n]}{2}, P_n)$ denote the interval order defined by the representation $I(\{i,j\}) = [i,j]$. To avoid confusion with the family of standard examples discussed previously, we call the interval orders in the family $\{\mathbf{I}_n : n \geq 2\}$ *canonical* interval orders.

The following result is due to Bogart, Rabinovitch and Trotter [10].

Theorem 7.1 *For every integer t, there exists an integer n_0 so that if $n \geq n_0$, then the dimension of the canonical interval order $\mathbf{I}_n$ is larger than t.*

Proof Evidently, $\dim(\mathbf{I}_n)$ is a non-decreasing function of n. We assume that $\dim(\mathbf{I}_n) \leq t$, for all $n \geq 2$ and obtain a contradiction when n is sufficiently large in terms of t. Let i, j, and k be distinct integers with $1 \leq i < j < k \leq n$. Then $\{i,j\} \| \{j,k\}$ in P_n, so if $\mathcal{R} = \{L_1, L_2, \dots, L_t\}$ is a realizer of P_n, then we may choose $\alpha \in \{1, 2, \dots, t\}$ so that $\{i,j\} > \{j,k\}$ in L_α. This is a coloring of the 3-element subsets of $\{1, 2, \dots, n\}$ with t colors. If n is sufficiently large, there exists a 4-element subset $S = \{i < j < k < l\}$ and an integer $\alpha \in \{1, 2, \dots, t\}$ so that all 3-element subsets of S are mapped to α. This implies that $\{i,j\} > \{j,k\} > \{k,l\} > \{i,j\}$ in L_α, which is a contradiction. ∎

Now that we know that interval orders can have large dimension, we pause to discuss some of the special properties interval orders exhibit.

Let $\mathbf{P} = (X, P)$ be a poset and let $X = X_1 \cup X_2$ be a partition of X into disjoint non-empty subsets. It is natural to ask whether one can say anything

about the dimension of $\dim(\mathbf{P})$ given informa
For posets in general, the answer is no. For
the partition of the point set of the standard
and maximal elements. The dimension of $\mathbf{S}_n^0$
dimension 2.

For interval orders, things are different. T
Proposition 6.2.

Lemma 7.2 *Let $\mathbf{P} = (X, P)$ be an interv
be a partition of X into disjoint non-empty
extensions of the subposets $\mathbf{P}_1$ and $\mathbf{P}_2$ induce
there exists a linear extension L of P so that*

Theorem 7.3 *Let $\mathbf{P} = (X, P)$ be an interv
a partition of X into disjoint non-empty sub*

$$\dim(\mathbf{P}) \leq 2 + \max\{\dim(\mathbf{X}$$

Proof Let $t = \max\{\dim(\mathbf{X}_1), \dim(\mathbf{X}_2)\}$.
there exists a family $\mathcal{F} = \{L_1, L_2, \dots, L_t\}$
$\mathcal{F}_i = \{L_1(X_i), L_2(X_i), \dots, L_t(X_i)\}$ is a reali
M_1 and M_2 be linear extensions of P so tha
over X_1 in M_2. It follows that $\{M_1, M_2\} \cup \mathcal{F}$

When one of the two sets in the partitio
we can do a little better. We leave the proof

Theorem 7.4 *Let $\mathbf{P} = (X, P)$ be an interva
If X_1 is the set of all maximal elements, and*

$$\dim(\mathbf{P}) \leq 1 + \dim($$

Now it is natural to ask whether we can s
contained in an interval order of large dime
answer. In Section 10, we will give a more c
content to show that an interval order of lar
chain.

Theorem 7.5 *An interval order of height h*

Proof Let $\mathbf{P} = (X, P)$ be an interval order of
representation of $\mathbf{P}$ and let $X = A_1 \cup A_2 \cup \cdots$
into antichains. Note that if $x < y$ in P, $x \in$

For each $i \in [h]$, let L_i be a linear extensio
Then let $L_{h+1} = L_1^d(A_1) < L_1^d(A_2) < \cdots < L$

e an interval poset, and let $A, B \subset X$ with
near extension L of P with B over A in L.

nterval representation of $\mathbf{P}$. For each $x \in X$,
otherwise. Then define a linear extension L
f $p_x < p_y$ in $\mathbb{R}$. ■

roposition, first noted by Felsner in [28], is

P) be an interval order, and let I be any
tion of $\mathbf{P}$. If L is a linear extension of P,
ach $x \in X$ a point $p_x \in I(x)$ so that $x < y$
■

rders

interval order can have large dimension. If
to the presence of large standard examples,
of them. Note that for each $n \geq 2$, the
a_2, b_1 and b_2 is isomorphic to $\mathbf{2}+\mathbf{2}$.
may have large dimension, and to explain
a standard example of an interval order.
$\binom{[n]}{2}), P_n)$ denote the interval order defined
$[i,j]$. To avoid confusion with the family
reviously, we call the interval orders in the
val orders.
Bogart, Rabinovitch and Trotter [10].

there exists an integer n_0 so that if $n \geq n_0$,
al interval order $\mathbf{I}_n$ is larger than t.

-decreasing function of n. We assume that
tain a contradiction when n is sufficiently
be distinct integers with $1 \leq i < j < k \leq n$.
$\{L_1, L_2, \ldots, L_t\}$ is a realizer of P_n, then we
at $\{i,j\} > \{j,k\}$ in L_α. This is a coloring
$., n\}$ with t colors. If n is sufficiently large,
$= \{i < j < k < l\}$ and an integer $\alpha \in$
subsets of S are mapped to α. This implies
$\}$ in L_α, which is a contradiction. ■

l orders can have large dimension, we pause
erties interval orders exhibit.
l let $X = X_1 \cup X_2$ be a partition of X into
atural to ask whether one can say anything

about the dimension of $\dim(\mathbf{P})$ given information about $\dim(\mathbf{X}_1)$ and $\dim(\mathbf{X}_2)$. For posets in general, the answer is no. For example, for each $n \geq 2$, consider the partition of the point set of the standard example $\mathbf{S}_n^0$ into minimal elements and maximal elements. The dimension of $\mathbf{S}_n^0$ is n but the two antichains have dimension 2.

For interval orders, things are different. The next result follows easily from Proposition 6.2.

Lemma 7.2 *Let $\mathbf{P} = (X, P)$ be an interval order, and let $X = X_1 \cup X_2$ be a partition of X into disjoint non-empty subsets. If L_1 and L_2 are linear extensions of the subposets $\mathbf{P}_1$ and $\mathbf{P}_2$ induced by X_1 and X_2 respectively, then there exists a linear extension L of P so that $L_1 = L(X_1)$ and $L_2 = L(X_2)$.* ■

Theorem 7.3 *Let $\mathbf{P} = (X, P)$ be an interval order, and let $X = X_1 \cup X_2$ be a partition of X into disjoint non-empty subsets. Then*

$$\dim(\mathbf{P}) \leq 2 + \max\{\dim(\mathbf{X}_1), \dim(\mathbf{X}_2)\}. \tag{4}$$

Proof Let $t = \max\{\dim(\mathbf{X}_1), \dim(\mathbf{X}_2)\}$. From Lemma 7.2, we know that there exists a family $\mathcal{F} = \{L_1, L_2, \ldots, L_t\}$ of linear extensions of P so that $\mathcal{F}_i = \{L_1(X_i), L_2(X_i), \ldots, L_t(X_i)\}$ is a realizer of $\mathbf{X}_i$, for $i = 1, 2$. Then let M_1 and M_2 be linear extensions of P so that X_1 is over X_2 in M_1 and X_2 is over X_1 in M_2. It follows that $\{M_1, M_2\} \cup \mathcal{F}$ is a realizer of P. ■

When one of the two sets in the partition is the set of maximal elements, we can do a little better. We leave the proof as an exercise.

Theorem 7.4 *Let $\mathbf{P} = (X, P)$ be an interval order which is not an antichain. If X_1 is the set of all maximal elements, and $X_2 = X - X_1$, then*

$$\dim(\mathbf{P}) \leq 1 + \dim(\mathbf{X}_2). \quad ■ \tag{5}$$

Now it is natural to ask whether we can say anything about what must be contained in an interval order of large dimension. Here we present a partial answer. In Section 10, we will give a more complete answer. For now, we are content to show that an interval order of large dimension must contain a long chain.

Theorem 7.5 *An interval order of height h has dimension at most $h+1$.*

Proof Let $\mathbf{P} = (X, P)$ be an interval order of height h, let I be a distinguishing representation of $\mathbf{P}$ and let $X = A_1 \cup A_2 \cup \cdots \cup A_h$ be the canonical partition into antichains. Note that if $x < y$ in P, $x \in A_i$ and $y \in A_j$, then $i < j$.

For each $i \in [h]$, let L_i be a linear extension of P with A_i over $X - A_i$ in L_i. Then let $L_{h+1} = L_1^d(A_1) < L_1^d(A_2) < \cdots < L_1^d(A_h)$. ■

Although interval orders can have large dimension, this is not true for semi-orders. The following result is due to Rabinovitch [93].

Theorem 7.6 *If* $\mathbf{P} = (X, P)$ *is a semi-order, then* $\dim(\mathbf{P}) \leq 3$.

Proof Let $\mathbf{P} = (X, P)$ be a semi-order, let I be a distinguishing representation of $\mathbf{P}$ and let let $X = A_1 \cup A_2 \cup \cdots \cup A_h$ be the canonical partition into antichains.

Let $\mathcal{O} = \bigcup\{A_j : 1 \leq j \leq h,\ j \text{ odd}\}$ and $\mathcal{E} = \bigcup\{A_j : 1 \leq j \leq h,\ j \text{ even}\}$. Let L_1 and L_2 be linear extensions of P with $\mathcal{E}$ over $\mathcal{O}$ in L_1 and $\mathcal{O}$ over $\mathcal{E}$ in L_2. Then let $L_3 = L_1^d(A_1) < L_1^d(A_2) < \cdots < L_1^d(A_h)$. ■

A semi-order has bounded dimension, not just because it has a representation in which all the intervals have the same length, but rather because there is no element incomparable with all the points in a long chain. We leave the following lemma as an exercise.

Lemma 7.7 *For every* $k \geq 1$, *there exists an integer* s_k *so that if* $\mathbf{P} = (X, P)$ *is an interval order in which the maximum size of a chain* C *for which there exists a point* x *incomparable to all points of* C *is at most* k, *then* $\dim(\mathbf{P}) \leq s_k$. ■

8 Critical pairs and alternating cycles

In arguments to follow, we will find it convenient to take advantage of a technical detail in the proof of Theorem 7.6. Let L be an arbitrary linear order on X. Define linear extensions L_d and L_u of P as follows. Set $x < y$ in L_d if and only if one of the following conditions holds:

1. $D(x) \subsetneq D(y)$.
2. $D(x) = D(y)$ and $U(y) \subsetneq U(x)$.
3. $D(x) = D(y)$, $U(y) = U(x)$, and $x < y$ in L.

Dually, set $x < y$ in L_u if and only if one of the following conditions holds:

1. $U(y) \subsetneq U(x)$.
2. $U(y) = U(x)$ and $D(x) \subsetneq D(y)$.
3. $U(y) = U(x)$, $D(x) = U(x)$, and $x > y$ in L.

Now let $\mathcal{F}$ be a family of linear extensions of P. Then $\{L_d, L_u\} \cup \mathcal{F}$ is a realizer of P if and only if for every $x, y \in X$ with $x \| y$ in P, $D(x) \subsetneq D(y)$, and $U(y) \subsetneq U(x)$, there exists $L \in \mathcal{F}$ with $x > y$ in L.

This last observation is a special case of a somewhat more general situation. For an arbitrary poset $\mathbf{P} = (X, P)$, let $\operatorname{inc}(\mathbf{P}) = \{(x, y) \in X \times X : x \| y \text{ in } P\}$. Then a family $\mathcal{R}$ of linear extensions of P is a realizer of P if and only if for every $(x, y) \in \operatorname{inc}(\mathbf{P})$, there exists $L \in \mathcal{R}$ so that $x > y$ in L. Call a pair

$(x, y) \in \operatorname{inc}(\mathbf{P})$ a *critical pair* if $u < x$ in
in P implies $v > x$ in P, for all $u, v \in X$.
all critical pairs. It follows that $\mathcal{R}$ is a re
$(x, y) \in \operatorname{crit}(\mathbf{P})$, there exists some $L \in \mathcal{R}$ s

We say L *reverses* the incomparable p
$S \subset \operatorname{inc}(\mathbf{P})$. We say that L *reverses* S wh
For an integer $k \geq 2$, a subset $S = \{(x_i,$
an *alternating cycle* when $x_i \leq y_{i+1}$ in
last definition, the subscripts are interpre
alternating cycle $S = \{(x_i, y_i) : 1 \leq i \leq k\}$
if $j = i + 1$, for all $i, j = 1, 2, \ldots, k$. When
following three statements hold:

1. The elements in $\{x_1, x_2, \ldots, x_k\}$ form
2. The elements in $\{y_1, y_2, \ldots, y_k\}$ form
3. If $i, j \in [k]$ and $x_i \geq y_j$, then $j = i +$

The following elementary result is due to
for a short proof and a number of applicati

Theorem 8.1 *Let* $\mathbf{P} = (X, P)$ *be a pose*
following statements are equivalent.

1. *There exists a linear extension* L *of*
2. S *does not contain an alternating cyc*
3. S *does not contain a strict alternatin*

9 Interval orders and shift graph

Although it has been known for many
height must contain long chains, it has on
relatively tight bounds have been found. T
dimension in interval orders is best explain
coloring problem. Fix integers n and k wi
pair (A, B) of k-element sets a (k, n)-*shift p*
subset $C = \{i_1 < i_2 < \cdots < i_{k+1}\} \subseteq \{1, 2,$
and $B = \{i_2, i_3, \ldots, i_{k+1}\}$. We then define t
graph whose vertex set consists of all k-elem
k-element set A adjacent to a k-element set
shift pair. The shift graph $\mathbf{S}(1, n)$ is just a
customary to call a $(2, n)$-shift graph just a
graph is called a *double shift graph*.

$(x, y) \in \text{inc}(\mathbf{P})$ a *critical pair* if $u < x$ in P implies $u < y$ in P and $v > y$ in P implies $v > x$ in P, for all $u, v \in X$. Then let $\text{crit}(\mathbf{P})$ denote the set of all critical pairs. It follows that $\mathcal{R}$ is a realizer of P if and only if for every $(x, y) \in \text{crit}(\mathbf{P})$, there exists some $L \in \mathcal{R}$ so that $x > y$ in L.

We say L *reverses* the incomparable pair (x, y) when $x > y$ in L. Let $S \subset \text{inc}(\mathbf{P})$. We say that L *reverses* S when $x > y$ in L, for every $(x, y) \in S$. For an integer $k \geq 2$, a subset $S = \{(x_i, y_i) : 1 \leq i \leq k\} \subset \text{inc}(\mathbf{P})$ is called an *alternating cycle* when $x_i \leq y_{i+1}$ in P, for all $i = 1, 2, \ldots, k$. In this last definition, the subscripts are interpreted cyclically, i.e., $y_{k+1} = y_1$. An alternating cycle $S = \{(x_i, y_i) : 1 \leq i \leq k\}$ is *strict* if $x_i \leq y_j$ in P if and only if $j = i + 1$, for all $i, j = 1, 2, \ldots, k$. When an alternating cycle is strict, the following three statements hold:

1. The elements in $\{x_1, x_2, \ldots, x_k\}$ form a k–element antichain.
2. The elements in $\{y_1, y_2, \ldots, y_k\}$ form a k–element antichain.
3. If $i, j \in [k]$ and $x_i \geq y_j$, then $j = i + 1$ and $x_i = y_j$.

The following elementary result is due to Trotter and Moore [124]. See [118] for a short proof and a number of applications.

Theorem 8.1 *Let* $\mathbf{P} = (X, P)$ *be a poset and let* $S \subseteq \text{inc}(\mathbf{P})$. *Then the following statements are equivalent.*

1. *There exists a linear extension L of P which reverses S.*
2. *S does not contain an alternating cycle.*
3. *S does not contain a strict alternating cycle.* ∎

9 Interval orders and shift graphs

Although it has been known for many years that interval orders of large height must contain long chains, it has only been in the last few years that relatively tight bounds have been found. The relationship between height and dimension in interval orders is best explained via a connection with a graph coloring problem. Fix integers n and k with $1 \leq k < n$. We call an ordered pair (A, B) of k-element sets a (k, n)*-shift pair* if there exists a $(k+1)$-element subset $C = \{i_1 < i_2 < \cdots < i_{k+1}\} \subseteq \{1, 2, \ldots, n\}$ so that $A = \{i_1, i_2, \ldots, i_k\}$ and $B = \{i_2, i_3, \ldots, i_{k+1}\}$. We then define the (k, n)*-shift graph* $\mathbf{S}(k, n)$ as the graph whose vertex set consists of all k-element subsets of $\{1, 2, \ldots, n\}$ with a k-element set A adjacent to a k-element set B exactly when (A, B) is a (k, n)-shift pair. The shift graph $\mathbf{S}(1, n)$ is just a complete graph on n vertices. It is customary to call a $(2, n)$-shift graph just a *shift* graph; similarly, a $(3, n)$-shift graph is called a *double shift graph.*

One of the folklore results of graph theory is the following formula for the chromatic number of the shift graph (throughout this paper, we use the notation $\lg n$ to denote the base 2 logarithm of n).

Theorem 9.1 *The chromatic number of the shift graph* $\mathbf{S}(2,n)$ *is* $\lceil \lg n \rceil$. ∎

The proof of Theorem 7.1 establishes the following lower bound.

Proposition 9.2 *The dimension of the canonical interval order* $\mathbf{I}_n$ *is at least as large as the chromatic number of the double shift graph* $\mathbf{S}(3,n)$. ∎

In turn, the next result relates the determination of the dimension of the family of canonical interval orders to the classical enumeration problem known as Dedekind's problem: estimate the number of antichains in the poset $\mathbf{2^t}$, the cartesian product of t two-element chains. This poset is just the subset lattice, the family of all subsets of $[t]$ partially ordered by inclusion. The next four results are due to Füredi, Hajnal, Rödl and Trotter [44].

Theorem 9.3 *The chromatic number of the double shift graph* $\mathbf{S}(3,n)$ *is the least positive integer* t *for which there are at least* n *antichains in the subset lattice* $\mathbf{2^t}$.

Proof Suppose that there are n antichains in the subset lattice $\mathbf{2^t}$. We show that the chromatic number of $\mathbf{S}(3,n) \le t$. Let Q be the partial order defined on the antichains of $\mathbf{2^t}$ by setting $\mathcal{A} \le \mathcal{B}$ in Q if and only if for every $S \in \mathcal{A}$, there exists $B \in \mathcal{B}$ so that $A \subseteq B$. Then let L be any linear extension of Q, and suppose that $\mathcal{A}_1 < \mathcal{A}_2 < \cdots < \mathcal{A}_n$ in L. For each i, j with $1 \le i < j \le n$, let $B(i,j) \in \mathcal{A}_j$ be a set so that there is no set $A \in \mathcal{A}_i$ with $A \subseteq B(i,j)$. Then for each i, j, k with $1 \le i < j < k \le n$, choose an element $\alpha \in B(j,k) - B(i,j)$, and set $\phi(\{i,j,k\}) = \alpha$. Then ϕ is a coloring of $\mathbf{S}(3,n)$.

Conversely, if the chromatic number of $\mathbf{S}(3,n)$ is at most t and ϕ: $\binom{[n]}{3} \to [t]$ is a coloring, we define for each i, j with $1 \le i < j \le n$, the set $B(i,j) = \{\phi(\{i,j,k\}) : j < k \le n\}$. Then for each $i \in [n]$, set $\mathcal{B}_i = \{B(i,j) : i < j \le n\}$. Partial order each $\mathbf{B}_i$ by inclusion and let $\mathcal{A}_i$ be the maximal elements. Then each $\mathcal{A}_i$ is an antichain in $\mathbf{2^t}$ and $\mathcal{A}_{i_1} \ne \mathcal{A}_{i_2}$ when $i_1 \ne i_2$. ∎

Although no closed form solution to Dedekind's problem has been found, relatively tight estimates are known (see [77], e.g.). For our purposes, we may use the estimate which results as follows. There are $\binom{t}{\lceil t/2 \rceil}$ subsets of size $\lceil t/2 \rceil$. Any subset of these sets forms an antichain in $\mathbf{2^t}$.

Theorem 9.4 *The chromatic number of the double shift graph satisfies:*

$$\chi(\mathbf{S}(3,n)) = \lg\lg n + (1/2 + o(1)) \lg\lg\lg n. \quad ∎ \qquad (6)$$

Hopefully, the reader has noticed the fo
problem for canonical interval orders. The l
use of repeated end points, a phenomenon w
the representation. After the modification, i
problem is much harder. However, this turn

Theorem 9.5 *Let n and t be positive inte*

$$n \le 2^{\binom{t}{\lceil t/2 \rceil}}$$

then the dimension of the canonical interva

Proof We let M_1 and M_2 be the following
$[i_1, j_1]$ and $[i_2, j_2]$ be distinct elements of
$i_1 < i_2$, or if both $i_1 = i_2$ and $j_1 > j_2$. D
$j_1 < j_2$, or if both $j_1 = j_2$ and $i_1 > i_2$. It ren
extensions $L_1, L_2, \ldots, L_{t+1}$ so that when i -
one L_α so that $[i_1, j_1] > [i_2, j_2]$ in L_α.

Let $s = \binom{t}{\lceil t/2 \rceil}$ and let $S_1, S_2, \ldots, S_s$ be
subsets of $[t]$. Note that $A = \{S_1, S_2, \ldots,$
lattice $\mathbf{2^t}$. Also let X denote the set of all
elements of X are functions from $[s]$ to $\{0,$
order on X. By this, we mean that if f, g
which $f(j) \ne g(j)$, then $f < g$ in L if and
$f(j) = 0$ and $g(j) = 1$. Now let $L' = [f_1, f_2$
n distinct elements of X.

Now let $[i_1, j_1]$ and $[i_2, j_2]$ be elements o
that we allow the possibility that $i_2 = j_1$. S
either 3 or 4 elements. Choose the least integ
Note that $f_{i_1}(k_1) = 0$ and $f_{j_2}(k_1) = 1$. Furt
ing statements holds:

1. For every $i \in E$ with $i \ne i_1$, $f_i(j_1) = 1$
2. For every $i \in E$ with $i \ne j_2$, $f_i(j_1) = 0$
3. $f_{i_2}(k_1) = 0$ and $f_{j_1}(k_1) = 1$.

When the third of these statements holds, w
two may occur with either $|E| = 3$ or $|E| =$
holds, we will require that $[i_1, j_1] > [i_2, j_2]$ in

When the first statement holds, set $E' =$
statement holds, set $E' = E - \{j_2\}$. In eithe
where $|\{f_i(k_1) : i \in E'\}| > 1$. If the first stat
and require $[i_1, j_1] > [i_2, j_2]$ in L_α. If the
$\alpha \in S_{k_2} - S_{k_1}$ and require $[i_1, j_1] > [i_2, j_2]$ in

We leave it as an exercise that such line

graph theory is the following formula for
graph (throughout this paper, we use the
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$\mathcal{A}_n$ in L. For each i, j with $1 \leq i < j \leq n$,
ere is no set $A \in \mathcal{A}_i$ with $A \subseteq B(i,j)$. Then
$\leq n$, choose an element $\alpha \in B(j,k) - B(i,j)$,
a coloring of $\mathbf{S}(3,n)$.
mber of $\mathbf{S}(3,n)$ is at most t and $\phi\colon \binom{[n]}{3} \to [t]$
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antichain in $\mathbf{2^t}$.

mber of the double shift graph satisfies:

$$n + (1/2 + o(1)) \lg\lg\lg n. \quad ■ \qquad (6)$$

Hopefully, the reader has noticed the following subtlety to the dimension problem for canonical interval orders. The lower bound depends heavily on the use of repeated end points, a phenomenon which we can eliminate by modifying the representation. After the modification, it is conceivable that the dimension problem is much harder. However, this turns out not to be the case.

Theorem 9.5 *Let n and t be positive integers with $n \geq 2$. If*

$$n \leq 2^{\binom{t}{\lceil t/2 \rceil}}, \qquad (7)$$

then the dimension of the canonical interval order $\mathbf{I}_n$ is at most $t+3$.

Proof We let M_1 and M_2 be the following two linear extensions of $\mathbf{I}_n$. Let $[i_1, j_1]$ and $[i_2, j_2]$ be distinct elements of $\mathbf{I}_n$. Set $[i_1, j_1] < [i_2, j_2]$ in M_1 if $i_1 < i_2$, or if both $i_1 = i_2$ and $j_1 > j_2$. Dually, set $[i_1, j_1] < [i_2, j_2]$ in M_2 if $j_1 < j_2$, or if both $j_1 = j_2$ and $i_1 > i_2$. It remains to find $t+1$ additional linear extensions $L_1, L_2, \ldots, L_{t+1}$ so that when $i - 1 < i_2 \leq j_1 < j_2$, there is at least one L_α so that $[i_1, j_1] > [i_2, j_2]$ in L_α.

Let $s = \binom{t}{\lceil t/2 \rceil}$ and let $S_1, S_2, \ldots, S_s$ be a listing of all the $\lceil t/2 \rceil$-element subsets of $[t]$. Note that $A = \{S_1, S_2, \ldots, S_s\}$ is an antichain in the subset lattice $\mathbf{2^t}$. Also let X denote the set of all 0-1 sequences of length s, i.e., the elements of X are functions from $[s]$ to $\{0,1\}$. We let L be the lexicographic order on X. By this, we mean that if $f, g \in X$ and j is the least integer for which $f(j) \neq g(j)$, then $f < g$ in L if and only if $f(j) < g(j)$. This implies $f(j) = 0$ and $g(j) = 1$. Now let $L' = [f_1, f_2, \ldots, f_n]$ be the restriction of L to n distinct elements of X.

Now let $[i_1, j_1]$ and $[i_2, j_2]$ be elements of $\mathbf{I}_n$ with $i_1 < i_2 \leq j_1 < j_2$. Note that we allow the possibility that $i_2 = j_1$. Set $E = \{i_1, j_2, j_1, j_2\}$ so that E has either 3 or 4 elements. Choose the least integer k_1 so that $|\{f_i(k_1) : i \in E\}| > 1$. Note that $f_{i_1}(k_1) = 0$ and $f_{j_2}(k_1) = 1$. Furthermore, exactly one of the following statements holds:

1. For every $i \in E$ with $i \neq i_1$, $f_i(j_1) = 1$.
2. For every $i \in E$ with $i \neq j_2$, $f_i(j_1) = 0$.
3. $f_{i_2}(k_1) = 0$ and $f_{j_1}(k_1) = 1$.

When the third of these statements holds, we must have $|E| = 4$, but the first two may occur with either $|E| = 3$ or $|E| = 4$. Also, when the third statement holds, we will require that $[i_1, j_1] > [i_2, j_2]$ in L_{t+1}.

When the first statement holds, set $E' = E - \{i_1\}$, and when the second statement holds, set $E' = E - \{j_2\}$. In either case, let k_2 be the least element where $|\{f_i(k_1) : i \in E'\}| > 1$. If the first statement holds, choose $\alpha \in S_{k_1} - S_{k_2}$ and require $[i_1, j_1] > [i_2, j_2]$ in L_α. If the second statement holds, choose $\alpha \in S_{k_2} - S_{k_1}$ and require $[i_1, j_1] > [i_2, j_2]$ in L_α.

We leave it as an exercise that such linear extensions exist. ■

The preceding theorem shows that the lower bound provided in inequality (6) is also an upper bound. With a little more work, the same kind of estimate works for arbitrary interval orders (see [44] for details).

Theorem 9.6 *The maximum dimension $d(h)$ of an interval order of height h satisfies:*

$$d(h) = \lg\lg h + (1/2 + o(1)) \lg\lg\lg h. \quad \blacksquare \tag{8}$$

Before closing this section, we comment that the dimension problem for interval orders is closely related to the problem of determining the dimension of the poset consisting of all 1-element and 2-element subsets of $\{1, 2, \ldots, n\}$, partially ordered by inclusion. Spencer [107] was the first to establish the connection between this problem and the classic result of Erdős and Szekeres concerning monotonic subsequences of a sequence of integers. In recent years, there has been rapid progress in estimating the dimension of posets consisting of layers of the subset lattice. A summary of this work together with additional references is provided by Trotter in [121].

10 Interval orders and overlap graphs

A graph $\mathbf{G} = (V, E)$ is called an *overlap graph* when there exists a function I assigning to each vertex $x \in V$ a closed interval $I(x) = [a_x, b_x]$ of $\mathbb{R}$ so that for all $x, y \in V$, $\{x, y\} \in E$ if and only if $I(x) \cap I(y) \neq \emptyset$, $I(x) \not\subseteq I(y)$ and $I(y) \not\subseteq I(x)$, i.e., the intervals intersect, but neither is contained in the other. Again, we call the function I a *representation* of the overlap graph $\mathbf{G}$. If required, we may assume that a representation of an overlap graph is *distinguishing.*

In general, overlap graphs need not be perfect, e.g. a cycle on 5 vertices is an overlap graph. However, when all the intervals used in the representation intersect, then the graph is perfect, and it is easy to color such graphs.

Proposition 10.1 *Let I be a distinguishing representation for an overlap graph $\mathbf{G} = (V, E)$. If $I(x) \cap I(y) \neq \emptyset$, for all $x, y \in V$, then $\mathbf{G}$ is the comparability graph of a poset $\mathbf{P} = (V, P)$ with $\dim(\mathbf{P}) \leq 2$, so $\mathbf{G}$ is perfect. Furthermore, the First Fit algorithm will provide an optimal coloring of $\mathbf{G}$ if the vertices are colored in the order determined by the left end points.*

Proof Let L_1 and L_2 be the linear orders on V determined by left and right end points, respectively. Then let $P = L_1 \cap L_2$. Clearly, $\{x, y\}$ is an edge of $\mathbf{G}$ if and only if x and y are comparable in the the poset $\mathbf{P} = (V, P)$. From its definition, we know that $\dim(\mathbf{P}) \leq 2$. The First Fit algorithm applied to the vertices in the order of their left end points is the minimum antichain partition described in the proof of Theorem 4.1. $\blacksquare$

When the intervals used in the representation do not share a common point, it is not immediately clear that there is any bound on the chromatic number of

an overlap graph in terms of the maximur
fact is due to Gyárfás [55] and the best b
and Kratochvíl [78]. Recall that the no
there exists a positive constant c and an
all $k > k_0$.

Theorem 10.2 *Let* $m(k) = \max\{\chi(\mathbf{G})$
imum clique size k\}. Then

1. $m(k) = \Omega(k \log k)$.
2. $m(k) = O(2^k)$. $\blacksquare$

Quite recently, the concepts used in th
applied by Kierstead and Trotter [75] to
dimension theory for interval orders. We c
for the best possible constants.

Theorem 10.3 *For every interval order* $\mathbf{Q}$
so that if $\mathbf{P} = (X, P)$ is any interval order
a subposet isomorphic to $\mathbf{Q}$. $\blacksquare$

Since every interval order $\mathbf{Q}$ is isomor
interval order $\mathbf{I}_n$, provided n is sufficiently
showing that for every integer $n \geq 2$, ther
an interval order with $\dim(\mathbf{P}) > t_n$, then
the canonical interval order $\mathbf{I}_n$.

Let $\mathbf{P} = (X, P)$ be an interval order.
subposet $\mathbf{T}$ of $\mathbf{P}$ an *m–tower* $\mathbf{T}$ when

1. $\mathbf{T}$ contains an m–element chain $Z =$
2. For every pair i, j with $1 \leq i < j$
   which is incomparable with $z_i, z_{i+1}, .$
   elements of Z.

It is an easy exercise to show that if $\mathbf{P}$
subposet isomorphic to $\mathbf{I}_n$. So, Theorem 1(
somewhat more technical result.

Theorem 10.4 *For every integer $m \geq 2$*
if $\mathbf{P}$ is an interval order with $\dim(\mathbf{P})$
isomorphic to an m–tower.

Proof We proceed by induction on m.
contain a 2–tower is a weak order and has
to take $t_2 = 3$. Now consider a value of n
an integer t_{m-1} so that any interval orde

that the lower bound provided in inequal
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$P = L_1 \cap L_2$. Clearly, $\{x, y\}$ is an edge of $\mathbf{G}$
ble in the the poset $\mathbf{P} = (V, P)$. From its
2. The First Fit algorithm applied to the
l points is the minimum antichain partition
4.1. ■

epresentation do not share a common point,
e is any bound on the chromatic number of

an overlap graph in terms of the maximum clique size. The first proof of this fact is due to Gyárfás [55] and the best bounds to date are due to Kostochka and Kratochvíl [78]. Recall that the notation $f(k) = \Omega(g(k))$ means that there exists a positive constant c and an integer k_0 so that $f(k) \geq cg(k)$, for all $k > k_0$.

Theorem 10.2 *Let $m(k) = \max\{\chi(\mathbf{G}) : \mathbf{G}$ is an overlap graph with maximum clique size $k\}$. Then*

1. *$m(k) = \Omega(k \log k)$.*
2. *$m(k) = O(2^k)$.* ■

Quite recently, the concepts used in the proof of Theorem 10.2 have been applied by Kierstead and Trotter [75] to solve a long standing problem in dimension theory for interval orders. We outline this work, but we do not aim for the best possible constants.

Theorem 10.3 *For every interval order $\mathbf{Q} = (Y, Q)$, there exists an integer t_0 so that if $\mathbf{P} = (X, P)$ is any interval order with $\dim(\mathbf{P}) > t_0$, then $\mathbf{P}$ contains a subposet isomorphic to $\mathbf{Q}$.* ■

Since every interval order $\mathbf{Q}$ is isomorphic to a subposet of the canonical interval order $\mathbf{I}_n$, provided n is sufficiently large, Theorem 10.3 is equivalent to showing that for every integer $n \geq 2$, there exists an integer t_n so that if $\mathbf{P}$ is an interval order with $\dim(\mathbf{P}) > t_n$, then $\mathbf{P}$ contains a subposet isomorphic to the canonical interval order $\mathbf{I}_n$.

Let $\mathbf{P} = (X, P)$ be an interval order. For an integer $m \geq 2$, we call a subposet $\mathbf{T}$ of $\mathbf{P}$ an *m–tower* $\mathbf{T}$ when

1. $\mathbf{T}$ contains an m–element chain $Z = \{z_1 < z_2 < \cdots < z_m\}$, and
2. For every pair i, j with $1 \leq i < j \leq m$, $\mathbf{T}$ contains an element $w(i, j)$ which is incomparable with $z_i, z_{i+1}, \ldots, z_j$ and comparable with all other elements of Z.

It is an easy exercise to show that if $\mathbf{P}$ contains a $3n$–tower, it contains a subposet isomorphic to $\mathbf{I}_n$. So, Theorem 10.3 is also equivalent to the following somewhat more technical result.

Theorem 10.4 *For every integer $m \geq 2$, there exists an integer t_m so that if $\mathbf{P}$ is an interval order with $\dim(\mathbf{P}) \geq t_m$, then $\mathbf{P}$ contains a subposet isomorphic to an m–tower.*

Proof We proceed by induction on m. An interval order which does not contain a 2–tower is a weak order and has dimension at most 2. So it suffices to take $t_2 = 3$. Now consider a value of $m \geq 3$ and assume that there exists an integer t_{m-1} so that any interval order whose dimension is at least t_{m-1}

contains an $(m-1)$–tower. Now let $\mathbf{P} = (X, P)$ be any interval order whose dimension t is at least $t_{m-1}+9$. We show that $\mathbf{P}$ contains an m–tower.

The key idea in the remainder of the proof is the notion of distance in the overlap graph. Let I be a distinguishing representation of $\mathbf{P} = (X, P)$. We proceed to build a realizer of P, starting with the two linear extensions M_1 and M_2, the orderings determined by the left and right end points respectively in the representation I. The important thing to notice is that it only remains to reverse critical pairs of the form (x, y), where $a_x < a_y < b_x < b_y$. In particular, x and y are adjacent in the overlap graph.

Let $\mathbf{G}$ denote the overlap graph determined by I, and let $\mathbf{G}_1, \mathbf{G}_2, \dots, \mathbf{G}_s$ denote the components of $\mathbf{G}$. Then let $\mathbf{X}_i = (X_i, P_i)$ be the subposet determined by the vertex set of $\mathbf{G}_i$. For each $i \in [s]$, define the *root* of $\mathbf{G}_i$ to be the unique vertex in $\mathbf{G}_i$ whose left end point is minimal. We denote the root of $\mathbf{G}_i$ by $\mathbf{r}_i$. For every vertex $x \in \mathbf{G}_i$, let $d(x, \mathbf{r}_i)$ denote the distance from x to $\mathbf{r}_i$ in $\mathbf{G}_i$. The following key lemma is due to Gyárfás [55]. We leave the proof as an exercise.

Lemma 10.5 *Let $i \in [s]$ and let $j \geq 0$. Then let x, y and z each be at distance j from the root $\mathbf{r}_i$ of a component $\mathbf{G}_i$ of $\mathbf{G}$. If $I(z) \subset I(x) \cap I(y)$, and $\mathbf{r}_i = u_0, u_1, \dots, u_i = z$ is a shortest path from $\mathbf{r}_i$ to z in $\mathbf{H}$, then $I(u_{i-1}) \not\subset I(x) \cup I(y)$.* ■

Next we classify all vertices of $\mathbf{G}$ as either *left* or *right*, and we denote the set of all left vertices by $\mathcal{L}$ and the set of all right vertices by $\mathcal{R}$. A vertex x belongs to $\mathcal{L}$ if and only if there exists a shortest path $\mathbf{r}_i = u_0, u_1, \dots, u_i = x$ from the root of the component to which x belongs to x so that the left end point of $I(u_{i-1})$ is less than the left end point of $I(x)$. Then set $\mathcal{R} = X - \mathcal{L}$.

Similary, we classify all vertices as either *even* or *odd* and denote these two sets by $\mathcal{E}$ and $\mathcal{O}$, respectively. A vertex x belongs to $\mathcal{E}$ if and only if the distance from x to the root of the component to which it belongs is even. Then set $\mathcal{O} = X - \mathcal{E}$.

Then let M_3, M_4, M_5 and M_6 be linear extensions of P with

1. $\mathcal{L}$ over $\mathcal{R}$ in M_3;
2. $\mathcal{R}$ over $\mathcal{L}$ in M_4;
3. $\mathcal{E}$ over $\mathcal{O}$ in M_5; and
4. $\mathcal{O}$ over $\mathcal{E}$ in M_5.

It follows that we may assume that there is a pair i, j with $i \in [s]$ and $j \geq 2$ so that the subposet $\mathbf{Q} = (Y, Q)$ determined by all left vertices at distance j from the root $\mathbf{r}_i$ of component $\mathbf{G}_i$ has dimension at least $t_{m-1}+3$.

Consider the following recursive definition. Set $Y_0 = Y$. If Y_k has already been defined for some $k \geq 0$ and the dimension of the subposet $\mathbf{Y}_k$ is less

than $t_{m-1}+1$, set $Z_{k+1} = Y_k$ and halt.
sion of $\mathbf{Y}_k$ is at least $t_{m-1}+1$, let y_{k+1}
$\dim(W(y_{k+1}, Y_i)) \geq t_{m-1}+1$ whose left
Then set $B_{k+1} = \{y \in Y_{k+1} : b_{y_{k+1}} \in I$
and $Y_{i+1} = Y_i - W(y_{k+1}, Y_k)$. It follows
$\dim(\mathbf{Z}_{i+1}) = t_{m-1}$.
Suppose this recursive definition halts
$Z = Z_1 \cup Z_2 \cup Z_s$ and $B = B_1 \cup B_2 \cup B_s$
$\dim(\mathbf{B}) \geq t_{m-1}+1$. Also, note that for ea
hypothesis implies that $\mathbf{Z}_i$ contains an $(m$
Since the dimension of $\mathbf{B}$ is large, it follo
are integers k_1 and k_2 with $1 < k_1 < k_2-3$ a
that $b_{k_1} \| b_{k_2}$ in P. It follows that the interva
from two disjoint $(m-1)$–towers, one fro
Choose a vertex x from Z_{k_1+1}, and conside
For each vertex, $y \in T$, the interval corres
on the shortest path from $\mathbf{r}_i$ to y proper
the lemma, this interval also has a left end
point of b_{k_1}. Thus this interval also interse
m–tower. ■

11 Semi-orders and balancing p

Let $\mathbf{P} = (X, P)$ be a poset and let $\mathcal{F}$
linear extensions of P. Consider the linea
uniform sample space. For a distinct pair
over y in $\mathcal{F}$, denoted $\text{Prob}_{\mathcal{F}}[x > y]$, is defin

$$\text{Prob}_{\mathcal{F}}[x > y] = \frac{1}{t}|\{i : 1 \leq i$$

In this section, we are concerned with
linear extensions of P. We let $\lambda(P) = |\Lambda($
subscript and just write $\text{Prob}[x > y]$. Note
in P; $\text{Prob}[x > y] = 1$, when $x > y$ in P
$x \| y$ in P. In 1969, S. S. Kislitsyn [76] ma
remains one of the most intriguing proble
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Conjecture 11.1 *If $\mathbf{P} = (X, P)$ is a finit
there exists an incomparable pair $x, y \in X$*

$$1/3 \leq \text{Prob}(x > y)$$

This conjecture was made independently b
and many papers on this subject attribute

than $t_{m-1}+1$, set $Z_{k+1} = Y_k$ and halt. If on the other hand, the dimension of $\mathbf{Y}_k$ is at least $t_{m-1}+1$, let y_{k+1} be the unique element of Y_k with $\dim(W(y_{k+1}, Y_i)) \geq t_{m-1}+1$ whose left end point is as small as possible. Then set $B_{k+1} = \{y \in Y_{k+1} : b_{y_{k+1}} \in I(y)\}$, $Z_{k+1} = W(y_{k+1}, Y_k) - B_{k+1}$ and $Y_{i+1} = Y_i - W(y_{k+1}, Y_k)$. It follows that $\dim(\mathbf{Y}_{i+1}) = t_{m-1}+1$ and $\dim(\mathbf{Z}_{i+1}) = t_{m-1}$.

Suppose this recursive definition halts in a partition $Y = Z \cup B$, where $Z = Z_1 \cup Z_2 \cup Z_s$ and $B = B_1 \cup B_2 \cup B_{s-1}$. Then $\dim(\mathbf{Z}) = t_{m-1}$, so that $\dim(\mathbf{B}) \geq t_{m-1}+1$. Also, note that for each $i = 1, 2, \ldots, s-1$, the inductive hypothesis implies that $\mathbf{Z}_i$ contains an $(m-1)$–tower.

Since the dimension of $\mathbf{B}$ is large, it follows (being very generous) that there are integers k_1 and k_2 with $1 < k_1 < k_2 - 3$ and elements $b_{k_1} \in B_{k_1}, b_{k_2} \in B_{k_2}$ so that $b_{k_1} \| b_{k_2}$ in P. It follows that the interval for b_{k_1} properly contains intervals from two disjoint $(m-1)$–towers, one from Z_{k_1+1} and the other from Z_{k_1+2}. Choose a vertex x from Z_{k_1+1}, and consider the $(m-1)$–tower T from Z_{k_2+2}. For each vertex, $y \in T$, the interval corresponding to the vertex just before y on the shortest path from $\mathbf{r}_i$ to y properly overlaps the interval for y. By the lemma, this interval also has a left end point which precedes the left end point of b_{k_1}. Thus this interval also intersects x. It follows that P contains an m–tower. ■

11 Semi-orders and balancing pairs

Let $\mathbf{P} = (X, P)$ be a poset and let $\mathcal{F} = \{M_1, \ldots, M_t\}$ be a multiset of linear extensions of P. Consider the linear extensions of $\mathcal{F}$ as outcomes in a uniform sample space. For a distinct pair $x, y \in X$, the *probability* that x is over y in $\mathcal{F}$, denoted $\mathrm{Prob}_{\mathcal{F}}[x > y]$, is defined by

$$\mathrm{Prob}_{\mathcal{F}}[x > y] = \frac{1}{t}|\{i : 1 \leq i \leq t,\ x > y \text{ in } M_i\}|. \tag{9}$$

In this section, we are concerned with the family $\Lambda(P)$ consisting of all linear extensions of P. We let $\lambda(P) = |\Lambda(P)|$. For this family, we drop the subscript and just write $\mathrm{Prob}[x > y]$. Note that $\mathrm{Prob}[x > y] = 0$, when $x < y$ in P; $\mathrm{Prob}[x > y] = 1$, when $x > y$ in P and $0 < \mathrm{Prob}[x > y] < 1$, when $x \| y$ in P. In 1969, S. S. Kislitsyn [76] made the following conjecture, which remains one of the most intriguing problems in the combinatorial theory of posets.

Conjecture 11.1 *If $\mathbf{P} = (X, P)$ is a finite poset which is not a chain, then there exists an incomparable pair $x, y \in X$ so that*

$$1/3 \leq \mathrm{Prob}(x > y) \leq 2/3. \quad ■ \tag{10}$$

This conjecture was made independently by both M. Fredman and N. Linial, and many papers on this subject attribute the conjecture to them. It is now known as the 1/3–2/3 conjecture. If true, the conjecture would be best possible, as shown by $\mathbf{2}+\mathbf{1}$.

The first major breakthrough in this area came in 1984, when Kahn and Saks [62] used the Alexandrov/Fenchel inequalities for mixed volumes to prove the following result.

Theorem 11.2 *If $\mathbf{P}=(X,P)$ is a finite poset which is not a chain, then there exists an incomparable pair $x,y\in X$ so that*

$$\frac{3}{11} < \mathrm{Prob}[x>y] < \frac{8}{11}. \quad \blacksquare \tag{11}$$

Recently, there has been a slight improvement in this result using a special case of a conjecture called the Cross-product conjecture. The result is due to Brightwell, Felsner and Trotter [18].

Theorem 11.3 *If $\mathbf{P}=(X,P)$ is a finite poset which is not a chain, then there exists an incomparable pair $x,y\in X$ so that*

$$\frac{5-\sqrt{5}}{10} < \mathrm{Prob}[x>y] < \frac{5+\sqrt{5}}{10}. \quad \blacksquare \tag{12}$$

As pointed out in [18], there is an infinite semi-order for which the inequality in Theorem 11.3 is best possible, so that the 1/3–2/3 conjecture is *false* if one attempts to extend it to infinite posets. However, for finite semi-orders, we can do even better. For a poset $\mathbf{P}=(X,P)$, we say x *covers* y and write $x :> y$ in P when $x>y$ in P and if $x\geq z\geq y$ in P, then either $x=z$ or $y=z$. The next result is due to Brightwell [16].

Theorem 11.4 *If $\mathbf{P}=(X,P)$ is a finite semi-order which is not a chain, then there exists an incomparable pair $x,y\in X$ so that*

$$\frac{1}{3} \leq \mathrm{Prob}[x>y] \leq \frac{2}{3}. \tag{13}$$

Proof Suppose that the theorem is false. Choose a counterexample $\mathbf{P}=(X,P)$ with $|X|=n$ minimum. Then let I be a distinguishing representation. Label the points of X as $x_1,x_2,\dots,x_n$ in the order determined by left end points. Define a linear order L on P by setting $x<y$ in L if and only if $\mathrm{Prob}[x>y]<1/3$. Clearly, L is a linear extension of P. Furthermore L orders X as $x_1<x_2<\dots<x_n$.

We claim that $x_i\|x_{i+1}$, for all $i=1,2,\dots,n-1$. To the contrary, suppose $x_i<x_{i+1}$ in P. Then $\mathbf{P}$ is the lexicographic sum over a two-element chain of the subposets determined by $\{x_1,x_2,\dots,x_i\}$ and $\{x_{i+1},x_{i+2},\dots,x_n\}$. One of these posets is not a chain, and we immediately contradict our choice of $\mathbf{P}=(X,P)$ as a minimum counterexample.

We say that a point x_j *separates x_i and x_{i+1} from above* if $x_j :> x_i$ and $x_j\|x_{i+1}$ in P. Dually, we say x_j *separates x_i and x_{i+1} from below* if $x_{i+1} :> x_j$

and $x_i\|x_j$ in P. Finally, we say x_j *separat*
from above or from below. Note that if x_j
then $x_k<x_j$ in P, for all $k=1,2,\dots,i$. I
from below, then $x_j<x_k$ in P, for all $k=i+$
at most two pairs, one from above and one fr
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$h\colon\Lambda_1\to\Lambda_3$ defined as follows. For a linea
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$|\Lambda_1|\leq|\Lambda_3|$. Furthermore, $|\Lambda_3|/t=\mathrm{Prob}[x_i$
particular, there exists a unique element x
x_j separates from above, $1/3<\mathrm{Prob}[x_{i+1}$
below, then $1/3<\mathrm{Prob}[x_j>x_i]<2/3$. $\blacksquare$

There are some other special classes of
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orders having dimension at most two.

12 Interval orders and extremal

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with $0\leq k\leq\binom{n}{2}$, let $Q(n,k)$ denote the fan
k comparable pairs. Then set $e(n,k)=\max$

Theorem 12.1 *Every poset $\mathbf{P}=(X,P)\in$
semi-order.*

Proof Suppose that $\mathbf{P}=(X,P)\in Q(n,k)$,
a semi-order. Suppose further that $\mathbf{P}$ is not
a subposet isomorphic to $\mathbf{2}+\mathbf{2}$. Label the
$\{x,y,u,v\}$, so that $u\in D(x)-D(y)$ and

and $x_i \| x_j$ in P. Finally, we say x_j *separates* x_i and x_{i+1} if it separates them from above or from below. Note that if x_j separates x_i and x_{i+1} from above, then $x_k < x_j$ in P, for all $k = 1, 2, \ldots, i$. Dually, if x_k separates x_i and x_{i+1} from below, then $x_j < x_k$ in P, for all $k = i{+}1, i{+}2, \ldots, n$. So each x_j separates at most two pairs, one from above and one from below. Furthermore, x_1 and x_2 do not separate pairs from below, while x_{n-1} and x_n do not separate pairs from above. It follows that there at most $2(n-4)+4 = 2n-4$ pairs (i, j) so that x_j separates x_i and x_{i+1}. From this we conclude that there is an integer i (in fact, there are at least two such values) for which there is at most one integer j so that x_j separates x_i and x_{i+1}. We show that $1/3 \leq \text{Prob}[x_i > x_{i+1}] \leq 2/3$.

Let $\Lambda(P)$ be the set of all linear extensions of P, and let $|\Lambda(P)| = t$. Set $\Lambda_1 = \{L \in \lambda : x_i < x_{i+1}$ in L, but there is no element of X which separates x_i and x_{i+1} between them in $L\}$; $\Lambda_2 = \{L \in \Lambda - \Lambda_1 : x_i < x_{i+1}$ in $L\}$; and $\Lambda_3 = \Lambda - (\Lambda_1 \cup \Lambda_2)$. Then $|\Lambda_3|/t = \text{Prob}[x_i > x_{i+1}] < 1/3$. Consider the map $h\colon \Lambda_1 \to \Lambda_3$ defined as follows. For a linear extension $L \in \Lambda_1$, form $h(L)$ by exchanging x_i and x_{i+1}. Clearly, the map h is an injection. It follows that $|\Lambda_1| \leq |\Lambda_3|$. Furthermore, $|\Lambda_3|/t = \text{Prob}[x_i > x_{i+1}] < 1/3$, so $|\Lambda_2| > t/3$. In particular, there exists a unique element x_j which separates x_i and x_{i+1}. If x_j separates from above, $1/3 < \text{Prob}[x_{i+1} > x_j] < 2/3$. If x_j separates from below, then $1/3 < \text{Prob}[x_j > x_i] < 2/3$. ∎

There are some other special classes of posets for which the 1/3–2/3 conjecture is known to be true. For example, Fishburn, Gehrlein and Trotter [39] showed that it is valid for all posets of height 2.

In a poset $\mathbf{P} = (X, P)$, a sequence $(x_1, x_2, \ldots, x_n)$ of length $n \geq 3$ is called a *linear extension majority cycle*, or just an LEM cycle for short, when $\text{Prob}[x_i > x_{i+1}] > \frac{1}{2}$, for all $i \in [n]$. It is an easy exercise to show that semi-orders do not contain LEM cycles, but Brightwell, Fishburn and Winkler [19] show that LEM cycles can exist in interval orders—in fact, even in interval orders having dimension at most two.

12 Interval orders and extremal problems

Here are two interesting extremal problems involving semi-orders. The first problem is investigated by Fishburn and Trotter in [41]. For integers n and k with $0 \leq k \leq \binom{n}{2}$, let $Q(n,k)$ denote the family of all posets with n points and k comparable pairs. Then set $e(n,k) = \max\{|\Lambda(P)| : \mathbf{P} = (X, P) \in Q(n,k)\}$.

Theorem 12.1 *Every poset* $\mathbf{P} = (X, P) \in Q(n,k)$ *with* $|\Lambda(P)| = e(n,k)$ *is a semi-order.*

Proof Suppose that $\mathbf{P} = (X, P) \in Q(n,k)$, $|\Lambda(P)| = e(n,k)$, but that $\mathbf{P}$ is not a semi-order. Suppose further that $\mathbf{P}$ is not an interval order. Then $\mathbf{P}$ contains a subposet isomorphic to $\mathbf{2}+\mathbf{2}$. Label the 4 points in the copy of $\mathbf{2}+\mathbf{2}$ as $\{x, y, u, v\}$, so that $u \in D(x) - D(y)$ and $v \in D(y) - D(x)$. Of all copies of

$\mathbf{2}+\mathbf{2}$ in $\mathbf{P}$, we may assume that we have chosen one so that $|U(x)|+|U(y)|$ is minimum. It follows that one of $U(x)$ and $U(y)$ is a subset of the other. Without loss of generality, we assume that $U(x) \subseteq U(y)$. Let $\mathbf{P}' = (X, P')$ be the poset obtained from $\mathbf{P}$ by replacing the relations $z < y$ by $z < x$ for all $z \in D(y) - D(x)$. Then $\mathbf{P}' \in Q(n,k)$.

Interchanging the points x and y transforms a linear extension from $\Lambda(P) - \Lambda(P')$ into a linear extension from $\Lambda(P') - \Lambda(P)$. Furthermore, this map is an injection. It is not a surjection, because any linear extension with $y < u < v < x$ is not in the image of the map. The contradiction shows that $\mathbf{P}$ is an interval order.

Now assume that $\mathbf{P}$ contains a subposet isomorphic to $\mathbf{3}+\mathbf{1}$, and label the elements in the 3-element chain so that $x < y < z$ in P. Label the element incomparable to these three points as w. Now form a poset $\mathbf{P}'' = (X, P'') \in Q(n,k)$ by replacing relations $t < y$ by $t < w$ for all $t \in D(y) - D(w)$. Then $\mathbf{P}' \in Q(n,k)$. As before, $\mathbf{P}''$ has more linear extensions than $\mathbf{P}$. ∎

The second problem sounds similar. It was posed to me by Peter Winkler. Define the *flexibility* of a poset $\mathbf{P} = (X, P)$, denoted $\operatorname{flex}(\mathbf{P})$, by

$$\operatorname{flex}(\mathbf{P}) = \sum_{x \in X} |U(x) + D(x)|^2. \qquad (14)$$

Then the same kind of argument used to prove Theorem 12.1 can be used to show that among all posets with n points and k comparable pairs, those with maximum flexibility are semi-orders. Despite our knowledge about the structure of the extremal posets, little progress has been made in solving either of these problems in full generality.

Now here is an interesting extremal problem for posets on which significant results have been obtained for interval orders. When L is a linear extension of $\mathbf{P} = (X, P)$, let $j(L, P)$ count the number of consecutive pairs of elements in L which are incomparable in P. The *jump number* of $\mathbf{P}$ is then the minimum value of $j(L,P)$ taken over all linear extensions of $\mathbf{P}$. In [89], Mitas shows that determining the jump number of an interval order is NP-complete. However, Mitas [89], Felsner [29] and Syslo [108] have (independently) given a polynomial algorithm for approximating the jump number within a ratio of 3/2.

Bogart and Stellpflug [11] define the *representation length* of a semi-order as the least positive integer k for which it has a a representation using intervals of length k with integer endpoints. For each $k \geq 1$, they provide a forbidden subposet characterization of semi-orders with representation length k.

For interval orders, we have the following natural extremal problem posed by Peter Fishburn in [36]. Given an interval order $\mathbf{P}$, find the least positive integer k for which $\mathbf{P}$ has a representation using intervals having k distinct lengths. This parameter is called the *interval count* of $\mathbf{P}$. Two interesting questions are immediate. First, what is the maximum value of the interval count of an interval order on n points? Second, can the removal of a single point drop the interval count by an arbitrarily large amount?

13 Interval orders and hamilton

Considered as a graph, the diagram c
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of sets. Felsner and Trotter [34] called
$S_j - (S_i \cup S_{i-1}) \neq \emptyset$, for all i, j with $1 \leq$
maximum h for which there exists an α–s
S_i a subset of $[t]$. For example, $\alpha(3) = 5$ a
$(\emptyset, \{1\}, \{2\}, \{3\}, \{1,3\}, \{1,2,3\})$. Note th
terms from an α–sequence is again an α–s

Let $D(h)$ denote the maximum chrom
height h. Clearly, $D(1) = 1$ and $D(2) = 2$.

Theorem 13.1 *For each $h \geq 2$, $D(h)$ is t*

Proof We first show that if $\alpha(t) > h$, ther
be an α–sequence of subsets of $[t]$, and let
Then let I be a distinguishing representat
ally least maximum chain be $C = \{y_1 < y$
partition into antichains be $X = A_1 \cup A$
$r_i = b_{y_i}$. Then let r_0 be any real number v

For each $x \in X$, set $i(x) = \max\{i$
$\max\{j : 1 \leq j \leq h,\ r_j \in I(x)\}$. Note that
define a coloring $\phi\colon X \to [t]$ as follows. If
$i_x > 1$, choose $\phi(x) \in S_{j_x} - (S_{i_x} \cup S_{i_x - 1})$.

We claim that ϕ is a proper coloring
$x <: y$ in $\mathbf{P}$. Then either $i_y = j_x$ or i_y
$\phi(x) \neq \phi(y)$.

We now sketch the proof that if $D(h$
$h, m \geq 2$, let $\mathbf{P}(h,m)$ denote the interval
all closed intervals with length at least m
$\{1, 2, \ldots, m(h+1) - 1\}$. Note that the h
that for each $h \geq 2$, there exists an inte
chromatic number of the diagram of $\mathbf{P}(h,$
show that the choice $m_0 = 2^{h^2}$ works.

13 Interval orders and hamiltonian paths

Considered as a graph, the diagram of a poset of height h cannot have chromatic number exceeding h. However, the "partite" construction of Nešetřil and Rődl [90] shows that for every integer h, there exists a poset $\mathbf{P}$ of height h so that the chromatic number of the diagram of $\mathbf{P}$ is exactly h.

For interval orders, the situation is completely different, and the chromatic number of the diagram of an interval order of height h is much less than h. The open intervals with integer end points in $\{1, 2, \ldots, h+1\}$ form an interval order of height h. Furthermore, the diagram of this interval order is just the shift graph $S(2, h+1)$, a graph whose chromatic number is exactly $\lceil \lg(h+1) \rceil$. Surprisingly, this is not far from best possible.

Let t be a positive integer, and let $\mathcal{S} = (S_0, S_1, \ldots, S_h)$ be a sequence of sets. Felsner and Trotter [34] called $\mathcal{F}$ an *α-sequence* if $S_1 \not\subseteq S_0$ and $S_j - (S_i \cup S_{i-1}) \neq \emptyset$, for all i, j with $1 \leq i < j \leq h$. Define $\alpha(t)$ to be the maximum h for which there exists an α–sequence $(S_0, S_1, \ldots, S_h)$, with each S_i a subset of $[t]$. For example, $\alpha(3) = 5$ as evidenced by the α-sequence $\mathcal{A} = (\emptyset, \{1\}, \{2\}, \{3\}, \{1,3\}, \{1,2,3\})$. Note that any subsequence of consecutive terms from an α–sequence is again an α–sequence.

Let $D(h)$ denote the maximum chromatic number of an interval order of height h. Clearly, $D(1) = 1$ and $D(2) = 2$.

Theorem 13.1 *For each $h \geq 2$, $D(h)$ is the least t for which $\alpha(t) \geq h$.*

Proof We first show that if $\alpha(t) > h$, then $D(h) \leq t$. Let $\mathcal{S} = (S_0, S_1, \ldots, S_h)$ be an α–sequence of subsets of $[t]$, and let $\mathbf{P}$ be an interval order of height h. Then let I be a distinguishing representation of $\mathbf{P}$. Let be the lexicographically least maximum chain be $C = \{y_1 < y_2 < \cdots < y_h\}$, and let the canonical partition into antichains be $X = A_1 \cup A_2 \cup \cdots \cup A_h$. For each $i \in [h]$, let $r_i = b_{y_i}$. Then let r_0 be any real number with $r_0 < a_x$ in $\mathbb{R}$, for every $x \in X$.

For each $x \in X$, set $i(x) = \max\{i : 0 \leq i \leq h,\ r_i < a_x\}$ and $j_x = \max\{j : 1 \leq j \leq h,\ r_j \in I(x)\}$. Note that $i_x < j_x$, for every $x \in X$. We then define a coloring $\phi\colon X \to [t]$ as follows. If $i_x = 0$, choose $\phi(x) \in S_{j_x} - S_0$. If $i_x > 1$, choose $\phi(x) \in S_{j_x} - (S_{i_x} \cup S_{i_x - 1})$.

We claim that ϕ is a proper coloring of the diagram of $\mathbf{P}$. Suppose that $x <: y$ in $\mathbf{P}$. Then either $i_y = j_x$ or $i_y = 1 + j_x$. In either case, note that $\phi(x) \neq \phi(y)$.

We now sketch the proof that if $D(h) \leq t$, then $\alpha(t) \geq h$. For integers $h, m \geq 2$, let $\mathbf{P}(h,m)$ denote the interval order determined by the family of all closed intervals with length at least $m - 1$ having integer end points from $\{1, 2, \ldots, m(h+1) - 1\}$. Note that the height of $\mathbf{P}(h,m)$ is h. We now show that for each $h \geq 2$, there exists an integer m_0 so that if $m > m_0$ and the chromatic number of the diagram of $\mathbf{P}(h,m)$ is t, then $\alpha(t) \geq h$. In fact, we show that the choice $m_0 = 2^{h^2}$ works.

Fix $h \geq 2$ and then let m be any integer with $m > m_0$. Suppose that the chromatic number of the diagram of $\mathbf{P}(h,m)$ is t. Note $t \leq h$. Now suppose that ϕ is a coloring of the diagram of $\mathbf{P}(m,h)$ using colors from $[t]$. For each $j = 1, 2, \dots, m(h+1) - 1$, let $A_j = \{\phi([i,j]) : 1 \leq i \leq j - m + 1\}$. Then for each $i = 0, 1, \dots, m-1$, let $V_i = (A_{m+i}, A_{2m+i}, \dots, A_{hm+i})$. Each V_i is a vector of length h with each entry a subset of $[t]$. Since there are at most 2^{h^2} such vectors, it follows that there exist integers i_1, i_2 with $0 \leq i_1 < i_2 \leq m-1$ for which $V_{i_1} = V_{i_2}$.

Set $S_0 = \emptyset$ and $S_k = A_{mk+i_1}$, for $k = 1, 3, \dots, h$. We claim that the sequence $(S_0, S_1, \dots, S_h)$ is an α–sequence. Clearly, $S_1 \neq \emptyset$ as S_1 contains the color ϕ assigns to the interval $[1+i_1, m+i_1]$. Thus $S_1 \not\subseteq S_0$. Now suppose that $1 < i < j \leq h+1$ and $S_j \subseteq S_{i-1} \cup S_i$. Suppose that ϕ assigns color $\beta \in [t]$ to the interval $x = [1 + mi + i_1, mj + i_2]$. Then there is an interval $y \in S_{i-1} \cup S_i$ with $\phi(y) = \beta$. Since $S_i = A_{mi+i_1} = A_{mi+i_2}$, and $S_{i-1} = A_{m(i-1)+i_1} = A_{m(i-1)+i_2}$, there is an interval y with $b_y \in \{mi+i_1, m(i-1)+i_2\}$ so that $\phi(y) = \phi(x) = \beta$. This is a contradiction, since $x :> y$. ■

Felsner and Trotter [34] conjecture that

$$\alpha(t) = 2^{t-1} + \left\lfloor \frac{t-1}{2} \right\rfloor . \qquad (15)$$

If this conjecture is true, then an α–sequence $\mathcal{S}$ of subsets of $[t]$ of maximum size has the following property. If we form a new sequence $\mathcal{H}$ from $\mathcal{S}$ by inserting between two consecutive sets in $\mathcal{S}$ their union, when the first set is not a subset of the second, then we get a listing of all 2^t subsets of $[t]$. For example, from the 6 term α–sequence of subsets of [3] given above, this listing is $(\emptyset, \{1\}, \{1,2\}, \{2\}, \{2,3\}, \{3\}, \{1,3\}, \{1,2,3\})$. This listing is a special kind of hamiltonian path in the t–cube. Whenever a set appears in the list, all of its subsets, with at most a single exception, appear previously. If there is an exception, it is listed next.

We call such a path an *order-preserving hamiltonian path* in the t–cube. This is a slight abuse of the concept of order-preserving, but it is the strongest notion that makes sense. It is known that there are order-preserving hamiltonian paths in the t–cube for $1 \leq t \leq 8$, but the general question is open.

We should point out that attempts to settle whether equation (15) is always valid have produced the best known partial result on the well known "middle two levels" problem. The origins of the problem are a bit unclear, but it was first told to me by Ivan Havel during a visit to Prague.

Problem 13.2 *Is the diagram of the poset consisting of all k–element and $(k+1)$–element subsets of a $(2k+1)$–element set, partially ordered by inclusion, a hamiltonian graph?* ■

We refer the reader to [34] and [99] for details.

14 On-line and un-cooperative c

An on-line optimization problem, such
considered as a two-person game involving a
is played in a series of rounds with the playe
of on-line coloring involves two parameters:
has n vertices, the game lasts at most n rou
Builder presents the vertex v_i of $\mathbf{G}$ and c
vertices in $\{v_j : 1 \leq j < i\}$. This inform
particular, if the game lasts all n rounds,
specified the entire graph.

After receiving the information for the
assign to v_i a color from the set $\{1, 2, \dots, t$
those previously assigned to neighbors of v_i.

The $(t, \mathbf{G})$ game ends at Round i and E
no legitimate choice of a color for the new
Colorer is able to respond with a legitima
of $\mathbf{G}$, then Colorer is the winner. The *on-li*
is then the least t for which Colorer has
game—regardless of the strategy employed

In [71], Kierstead and Trotter prove the

Theorem 14.1 *The on-line chromatic nu*
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Proof Here's the winning strategy for C
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that First Fit will color S_i with the 3 colors f
We accomplish this by showing that for $i \geq$

Fix $i \geq 2$. When a vertex u is presented
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Now let u and v be adjacent vertices fro
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mon with the First Fit algorithm discussed
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14 On-line and un-cooperative coloring

An on-line optimization problem, such as on-line graph coloring, can be considered as a two-person game involving a *Builder* and a *Colorer*. The game is played in a series of rounds with the players alternating turns. Each instance of on-line coloring involves two parameters: an integer t and a graph $\mathbf{G}$. If $\mathbf{G}$ has n vertices, the game lasts at most n rounds. In Round i, where $1 \leq i \leq n$, Builder presents the vertex v_i of $\mathbf{G}$ and describes all edges joining v_i with vertices in $\{v_j : 1 \leq j < i\}$. This information is complete and correct. In particular, if the game lasts all n rounds, then Builder must have correctly specified the entire graph.

After receiving the information for the new vertex v_i, Colorer must then assign to v_i a color from the set $\{1, 2, \ldots, t\}$ so that this color is distinct from those previously assigned to neighbors of v_i. These assignments are permanent.

The $(t, \mathbf{G})$ game ends at Round i and Builder is the winner if Colorer has no legitimate choice of a color for the new vertex v_i. If on the other hand, Colorer is able to respond with a legitimate color for each of the n vertices of $\mathbf{G}$, then Colorer is the winner. The *on-line chromatic number* of a graph $\mathbf{G}$ is then the least t for which Colorer has a winning strategy for the $(t, \mathbf{G})$ game—regardless of the strategy employed by Builder.

In [71], Kierstead and Trotter prove the following foundational result.

Theorem 14.1 *The on-line chromatic number of an interval graph of maximum clique size k is at most $3k - 2$.*

Proof Here's the winning strategy for Colorer. Given a new vertex x by Builder, Colorer assigns x to a set S_i where i is the least positive integer for which there is no complete subgraph of size $i + 1$ containing x and i other vertices previously assigned to $S_1 \cup S_2 \cup \cdots \cup S_i$. Note that S_1 is just an independent set, so it can be colored with a single color. For each $i \geq 2$, we show that First Fit will color S_i with the 3 colors from the set $\{3i - 4, 3i - 3, 3i - 2\}$. We accomplish this by showing that for $i \geq 2$, S_i is the disjoint sum of paths.

Fix $i \geq 2$. When a vertex u is presented by Builder and assigned by Colorer to S_i, as opposed to S_{i-1}, there is a clique K_u consisting of u and $i - 1$ vertices from $S_1 \cup S_2 \cup \cdots \cup S_{i-1}$. Then the intersection of the intervals from K_u is a nonempty interval I_u, which is contained in the interval corresponding to u. Now let u and v be adjacent vertices from S_i. Then it is easy to see that I_u does not intersect the interval corresponding to v and I_v does not intersect the interval corresponding to u. From this, it follows easily that S_i is triangle free and that each vertex from S_i has at most 2 neighbors in S_i. ■

The algorithm presented in the preceding theorem has one feature in common with the First Fit algorithm discussed in Theorem 4.2: it is not necessary to know the maximum clique size in advance. If First Fit is used to color an interval graph, and the vertices are not processed in the order of left end points, then it is not clear how many colors will be used. In [65], Kierstead showed that First Fit will use at most $40k$ colors on an interval graph with maximum clique size k, regardless of the order in which the vertices are processed. Subsequently, Kierstead and Qin [70] improved this upper bound to $26k$. From below, Chrobak and Slusarek [23] showed that no on-line algorithm can color all interval graphs with maximum clique size k with fewer than $4.4k$ colors.

Kierstead's analysis of the performance of First Fit in coloring interval graphs provided a solution to an important long standing problem in computer science called the *Dynamic Storage Allocation* problem. The standard two dimensional bin packing problem is to pack a family of rectangles in $\mathbb{R}^2$, with sides parallel to the coordinate axes, into a region of minimum area. The Dynamic Storage Allocation problem is to pack the rectangles into a region of minimum height—when the projections of the rectangles onto the horizontal coordinate axis form a fixed interval graph. Of course, by "pack," we mean that the rectangles are to be placed so that their interiors are disjoint. So if the maximum sum of the heights of rectangles whose projections have a common point is t, then t is a lower bound on the height required for a packing, and it was conjectured that a height of $O(t)$ would suffice.

One proposed approach to finding a reasonably good packing was to assume all rectangles had height a power of 2. This assumption would at most double the optimal height required for a packing. These rectangles would then be partitioned into subrectangles of height one. Finally, First Fit would be used to color the rectangles (intervals) with all intervals formed from the same rectangle colored consecutively. The number of colors used by First Fit would then be an upper bound on the minimum height required for a packing. Accordingly, Kierstead and Qin's bound implies that the rectangles can in fact be packed into a region of height $52t$.

We refer the reader to [66] for a full discussion. As an added bonus, this paper provides an alternative approach to the dynamic storage allocation problem which stands as the best solution to date. This approach uses the same partition of rectangles, but colors them with a modified version of the on-line algorithm used in Theorem 14.1 rather than with First Fit. The end result is to show that the rectangles can be packed into a height of $6t - 4$.

Ironically, the research which led to the proof of Theorem 14.1 was motivated, not by the Dynamic Storage Allocation problem, but by the on-line version of Dilworth's theorem. In [64], Kierstead proved that there is an on-line algorithm which will partition a poset built one point at a time into $(5^w - 1)/4$ chains, where w denotes the width of the poset. When the poset is known to be an interval order, then the preceding theorem asserts that $3w - 2$ chains suffice.

Kierstead's on-line chain partitioning algorithm requires knowledge of the order. Just knowing whether points are comparable is not enough. However, for interval graphs, our algorithm only makes use of the comparability graph. For many years, it remained an open problem to determine whether a com-

parability graph of independence number
bounded number of complete subgraphs.
by Kierstead, Penrice and Trotter in [69].

In [68], Kierstead, McNulty and Trot
Here the game is between a Realizer and
a family $\mathcal{R}$ of linear extensions of a poset
point at a time. They show that the on-li
posets is infinite. However, the posets in thi
crown $\mathbf{S}_3^0$. They then proceed to show that
bounded width is well defined, provided th
not contain any 3-dimensional crown $\mathbf{S}_3^k$.

Theorem 14.2 *The on-line dimension of*
most $t!$, where $t = (5^{k+1} - 1)/4$. ■

On the surface, this result has nothing
proof makes use of an auxiliary order at a
structure turns out to be an interval order,
this structure gains from being an interval

Other sources of information about on
more recent survey by Kierstead [67]. In
a concise treatment of the recent breakthr
showing that for all $k \geq 3$, there exists an
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argument shows that $\epsilon = O(1/k!)$. Probabl
Another good source of problems (some of
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In [27], Faigle, Kern, Kierstead and T
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possible. In [74], Kierstead and Trotter sh
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with game chromatic number at least 8.

For interval graphs, the following resul
known bound on the game chromatic numb

Theorem 14.3 *The game chromatic numb*
with maximum clique size k is at most $3k$

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parability graph of independence number k can be partitioned on-line into a bounded number of complete subgraphs. An affirmative answer was provided by Kierstead, Penrice and Trotter in [69].

In [68], Kierstead, McNulty and Trotter investigate *on-line dimension*. Here the game is between a Realizer and a Builder, with Realizer building a family $\mathcal{R}$ of linear extensions of a poset $\mathbf{P}$ which Builder is constructing one point at a time. They show that the on-line dimension of a class of width 4 posets is infinite. However, the posets in this class all contain the 3-dimensional crown $\mathbf{S}_3^0$. They then proceed to show that the on-line dimension of posets of bounded width is well defined, provided that the posets are *crown-free*, i.e., do not contain any 3-dimensional crown $\mathbf{S}_3^k$.

Theorem 14.2 *The on-line dimension of a crown-free poset of width k is at most $t!$, where $t=(5^{k+1}-1)/4$.* ∎

On the surface, this result has nothing to do with interval orders, but the proof makes use of an auxiliary order at a critical point in the argument. This structure turns out to be an interval order, and the order-theoretic properties this structure gains from being an interval order are key elements of the proof.

Other sources of information about on-line coloring include [72] and the more recent survey by Kierstead [67]. In particular, this last paper contains a concise treatment of the recent breakthrough where Kierstead succeeded in showing that for all $k \geq 3$, there exists an $\epsilon > 0$ so that the on-line chromatic number of any k-colorable graph on n vertices is at most $n^{1-\epsilon}$. Kierstead's argument shows that $\epsilon = O(1/k!)$. Probably, this can be improved to $O(1/k)$. Another good source of problems (some of which are on-line) concerning interval graphs and other classes of perfect graphs in Gyárfás' survey paper [54].

In [27], Faigle, Kern, Kierstead and Trotter consider the following game theoretic problem for graphs. Two players, Alice and Bob, color a graph $\mathbf{G}$ using elements of the set $[t]$ as colors. They alternate turns with Alice having the first move. Alice wins if the graph is eventually colored and Bob (an uncooperative partner) wins if at some step before the graph is colored, there is no legitimate move. The *game chromatic number* of $\mathbf{G}$ is the least t for which Alice has a winning strategy. For example, it is shown in [27] that the game chromatic number of a tree is at most 4; furthermore, this result is best possible. In [74], Kierstead and Trotter show that a planar graph has game chromatic number at most 33; they also show that there exists a planar graph with game chromatic number at least 8.

For interval graphs, the following result, given in [27], provides the best known bound on the game chromatic number of an interval graph.

Theorem 14.3 *The game chromatic number of an interval graph $\mathbf{G}=(V,E)$ with maximum clique size k is at most $3k-2$.*

Proof Let I be a distinguishing representation of an interval graph $\mathbf{G}$ with maximum clique size k. When it is her turn to color, Alice prefers to color a vertex x adjacent to the vertex just colored by Bob. Among such, she prefers those whose intervals intersect the interval corresponding to the vertex just colored by Bob. Finally among such vertices, Alice prefers the one whose interval has right end point as large as possible. She then colors this vertex by First Fit.

We claim that Alice and Bob can never reach an impasse if the number of colors is $3k - 2$. It suffices to show that the strategy given for Alice can be used by either player. Let x be the vertex to be colored. It suffices to show that x has at most $3k - 3$ colored neighbors. Split the colored neighbors into three sets $N_1 \cup N_2 \cup N_3$, where

1. N_1 is the set of colored neigbors of x whose intervals contain the right end point of $I(x)$;
2. N_2 is the set of colored neighbors of x whose intervals are properly contained in $I(x)$; and
3. N_3 is the set of colored neighbors of x whose intervals contain the left end point of $I(x)$ but not the right.

Clearly, $|N_1| \leq k - 1$ and $|N_3| \leq k - 1$, so our claim follows if we can show that $|N_2| \leq k - 1$. Now our strategy for Alice insures that she will not have colored any of the vertices in N_2, since she will always prefer to color x. So all vertices in N_2 are colored by Bob, and at every turn—except possibly the last one—Alice has selected a vertex other than x to color. Such a vertex must have an interval containing the interval corresponding to the vertex in N_2 just colored by Bob, and its right end point is greater than the right end point of x. Therefore Alice's response was to color a vertex from N_1. It follows that $|N_2| \leq k$. Now suppose that $|N_2| = k$. Among the vertices in N_2, let y be the unique vertex whose right end point is as large as possible. Then y, x and the $k - 1$ vertices in N_2 form a clique of size $k + 1$. ■

The reader should note that it is just a coincidence that the expression $3k - 2$ appears in both the preceding two theorems. In the first case, we know that it is best possible, but in the second, we believe it is not. We leave it as an exercise to show that, for each $k \geq 2$, there exists an interval graph **G** whose maximum clique size is k and whose game chromatic number is at least $2k$.

15 Fractional dimension and ramsey theory for probability spaces

It is often useful to consider a fractional version of an integer valued combinatorial parameter as, in many cases, the resulting LP relaxation sheds light on the original problem. In [20], Brightwell and Scheinerman proposed to investigate fractional dimension for posets.

Let $\mathbf{P} = (X, P)$ be a poset, and let
linear extensions of P. Brightwell and Sche
of P if, for each incomparable pair (x, y),
in $\mathcal{F}$ which reverse the pair (x, y), i.e., $|\{i$
fractional dimension of $\mathbf{P}$, denoted by fd
real number $q \geq 1$ for which there exists
of P so that $k/t \geq 1/q$ (it is easily verified
real numbers q is indeed attained and is t
this terminology, the *dimension* of $\mathbf{P}$ is jus
1–fold realizer of P. It follows immediatel
poset $\mathbf{P}$.

The dimension or fractional dimension
the least upper bound of $\dim(\mathbf{P})$ (respect
the class. We have seen that $\dim(\mathcal{I}) = \infty$ f
Brightwell and Scheinerman showed that
that if $\mathbf{P} = (X, P)$ is an interval order and
of P with $x > y$ in L for any incomparabl
Building a realizer from one such L for e
gives $\text{fdim}(\mathbf{P}) < 4$.

Brightwell and Scheinerman conjectur
though no example of an interval order of fr
was then known. In the remainder of this se
by Trotter and Winkler in [126] to settle t

First, the following preliminary result
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true and events being false. See [126] for t

Theorem 15.1 *For every $\epsilon > 0$, there exis*
and $\{U_i : 1 \leq i \leq m\}$ is any sequence of
there exist integers i and j with $1 \leq i < j \leq$

With this result in hand, we can now sk
argument makes extensive use of ramsey
about sets hold in a uniform manner. To b
small errors, and the argument takes some
kept under control. In this sketch, we igno

Theorem 15.2 *For every $\epsilon > 0$, there exi*
the fractional dimension of the canonical i

Proof Let $\epsilon > 0$, and suppose that fdim(
of n. We argue to a contradiction, provide

Let $S = \{s_1, s_2, \ldots, s_{2m}\}$ be a $2m$–elen
$\cdots < s_{2m}$. Then let $U(S)$ denote the even

Let $\mathbf{P} = (X, P)$ be a poset, and let $\mathcal{F} = \{M_1, \ldots, M_t\}$ be a multiset of linear extensions of P. Brightwell and Scheinerman [20] call $\mathcal{F}$ a *k–fold realizer* of P if, for each incomparable pair (x, y), there are at least k linear extensions in $\mathcal{F}$ which reverse the pair (x, y), i.e., $|\{i : 1 \leq i \leq t,\ x > y \text{ in } M_i\}| \geq k$. The *fractional dimension* of $\mathbf{P}$, denoted by $\mathrm{fdim}(\mathbf{P})$, is then defined as the least real number $q \geq 1$ for which there exists a k–fold realizer $\mathcal{F} = \{M_1, \ldots, M_t\}$ of P so that $k/t \geq 1/q$ (it is easily verified that the least upper bound of such real numbers q is indeed attained and is therefore a rational number). Using this terminology, the *dimension* of $\mathbf{P}$ is just the least t for which there exists a 1–fold realizer of P. It follows immediately that $\mathrm{fdim}(\mathbf{P}) \leq \dim(\mathbf{P})$, for every poset $\mathbf{P}$.

The dimension or fractional dimension of a class of posets is defined to be the least upper bound of $\dim(\mathbf{P})$ (respectively $\mathrm{fdim}(\mathbf{P})$) over all posets $\mathbf{P}$ in the class. We have seen that $\dim(\mathcal{I}) = \infty$ for the class $\mathcal{I}$ of interval orders, but Brightwell and Scheinerman showed that $\mathrm{fdim}(\mathcal{I}) \leq 4$. To see this, observe that if $\mathbf{P} = (X, P)$ is an interval order and $A \subset X$, there is a linear extension L of P with $x > y$ in L for any incomparable pair (x, y) with $x \in A$ and $y \notin A$. Building a realizer from one such L for each subset A of X of size $\lfloor |X|/2 \rfloor$ gives $\mathrm{fdim}(\mathbf{P}) < 4$.

Brightwell and Scheinerman conjectured in [20] that $\mathrm{fdim}(\mathcal{I}) = 4$, even though no example of an interval order of fractional dimension even as high as 3 was then known. In the remainder of this section, we sketch the approach taken by Trotter and Winkler in [126] to settle this conjecture in the affirmative.

First, the following preliminary result is required. Intuitively, this theorem asserts that in a sufficiently long sequence of events, one cannot do substantially better than toss a fair coin in trying to balance between events being true and events being false. See [126] for the proof.

Theorem 15.1 *For every $\epsilon > 0$, there exists an integer m_0 so that if $m \geq m_0$ and $\{U_i : 1 \leq i \leq m\}$ is any sequence of events in a probability space, then there exist integers i and j with $1 \leq i < j \leq m$ so that* $\mathrm{Prob}[U_i\overline{U_j}] < \frac{1}{4} + \epsilon$. ∎

With this result in hand, we can now sketch the proof of the solution. The argument makes extensive use of ramsey theory to make certain statements about sets hold in a uniform manner. To be precise, these statements involve small errors, and the argument takes some care to show that the errors can be kept under control. In this sketch, we ignore these errors.

Theorem 15.2 *For every $\epsilon > 0$, there exists an integer n_0 so that if $n > n_0$, the fractional dimension of the canonical interval order $\mathbf{I}_n$ is at least $4 - \epsilon$.*

Proof Let $\epsilon > 0$, and suppose that $\mathrm{fdim}(\mathbf{I}_n) < 4 - \epsilon$, regardless of the size of n. We argue to a contradiction, provided n is sufficiently large.

Let $S = \{s_1, s_2, \ldots, s_{2m}\}$ be a $2m$–element subset of $[n]$, with $s_1 < s_2 < \cdots < s_{2m}$. Then let $U(S)$ denote the event that for some i with $1 < i \leq m$,

$[s_1, s_{m+1}] > [s_i, s_{m+i}]$. Using ramsey theory, it is relatively easy to see that for fixed m, if n is sufficiently large, we may assume that the probability of $U(S)$ is constant, for all $2m$–element subsets of $[n]$. But Trotter and Winkler show more. They show that one may also assume that the event $U(S)$ depends only on s_1 and s_{m+1}. We denote this event by $U(x, y)$, where $x = s_1$ and $y = s_{m+1}$.

Dually, let $D(S)$ denote the event that for some j with $1 \le j < m$, $[s_j, s_{m+j}] > [s_m, s_{2m}]$. This time, the event $D(S)$ depends only on s_m and s_{2m}. So we can just write $D(x, y)$, where $x = s_m$ and $y = s_{2m}$.

It follows that one can find a large homogeneous subset H so that $U(x, y) \cap D(x, y) = \emptyset$, for every $x, y \in H$ with $x < y$ in $\mathbb{R}$. Furthermore, if $x < y < z < w$ in H, then $U(x, z) \cap U(y, w) = \emptyset$. If the homogenous set H has more than $2m_0$ elements, the result follows from Theorem 15.1. ■

The dimension problem for interval orders is closely related to the graph coloring problem for shift graphs, a subject of independent interest. Similarly, the research on the fractional dimension of interval orders has led to many new and interesting concepts. We give hints to one of these in the sketch of the proof of Theorem 15.2, namely the development of a general Ramsey theory for probability spaces. However, there are several concrete combinatorial problems which are also quite attractive.

Fix integers n and k with $1 \le k < n$. Suppose we have a probability space containing an event E_S for every k–element subset S of $[n]$. We abuse notation and just refer to this event as S. Now consider the minimum probability $\text{Prob}(A\overline{B})$ taken over all (k, n)–shift pairs. In turn, take the maximum value of this probability over all probability spaces and let n go to infinity. The resulting value is called $f(k)$. For example, from Theorem 15.1, it follows that $f(1) = \frac{1}{4}$. In [126], Trotter and Winkler prove that $f(2) = \frac{1}{3}$, $f(3) \ge \frac{3}{8}$ and $f(4) \ge \frac{2}{5}$. In general, they prove that $f(k)$ is strictly increasing and converges to $\frac{1}{2}$.

The relaxation of dimension to fractional dimension is an appealing concept. In [33], Felsner and Trotter show that the fractional dimension of a poset in which each point is comparable with at most k others is at most $k + 1$. They also prove several other inequalities linking fractional dimension with width and cardinality. Nevertheless, there are many challenging open questions in this area.

16 Higher-dimensional analogues for graphs

In recent years, there has been a steady stream of results providing higher dimension analogues of interval graphs. Perhaps the first of these is due to Roberts [96] who defined the *boxicity* of a graph $\mathbf{G} = (V, E)$ as the least t for which there exists a function B assigning to each vertex $x \in V$ a sequence $(I_x(1), I_x(2), \ldots, I_x(t))$ of closed intervals of $\mathbb{R}$ so that $\{x, y\} \in E$ if and only if $I_x(i) \cap I_y(i) \ne \emptyset$, for all $i \in [t]$. Equivalently, the boxicity of a graph is just

the least t so that the graph is the inters
graphs are graphs with boxicity one. Rob
graph on n vertices is at most $\lfloor n/2 \rfloor$, when
obtained by taking the complement of a m
and boxicity n. In [127], Wittenshausen sh
only graph with $2n$ vertices and boxicity n.

However, when a graph has $2n + 1$ ver
is modestly more complicated. For examp
boxicity 2. Also, the graph $\mathbf{W}_3$ with vertex
i to $i + 2$, $i + 3$, $i + 4$ and $i + 5$ (cyclically
showed that a graph $\mathbf{G}$ on $2n + 1$ vertices
the following conditions holds:

1. $\mathbf{G}$ contains $\mathbf{H}_n$.
2. $\mathbf{G}_n$ contains the join of $\mathbf{C}_5$ and $\mathbf{H}_{n-2}$
3. $\mathbf{G}_n$ contains the join of $\mathbf{W}_3$ and $\mathbf{H}_{n-}$

In [109], Thomassen showed that the box
in fact, the boxes corresponding to adjacent
sect on a face.

Many of the basic concepts for interval
for digraphs. In [3], Beineke and Zamfiresc
ferent terminology) of an *interval digraph*
vertex x in a digraph $\mathbf{D}$, there are two inte
that $\mathbf{D}$ contains a directed arc from x to y if
questions for interval digraphs are studied i

Define the *interval number* of a graph
$\mathbf{G}$ is the intersection graph of a family of s
of t pairwise disjoint closed intervals of $\mathbb{R}$.
if the maximum degree of $\mathbf{G}$ is d, then th
$\lceil (d + 1)/n \rceil$. This inequality is tight if $\mathbf{G}$
also showed that there exists an absolute c
number of a graph with q edges is at most
West showed that the interval number of
Scheinerman [100] showed that there exists
the interval number of a graph of genus γ i

In [122], Trotter and Harary show that
bipartite graph $\mathbf{K}(m, n)$ is $\lceil (mn + 1)/(m$
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and q edges is at least $\lceil (q + 1)/n \rceil$. This
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representation form the intersection graph
least q edges. This requires $q \le nt - 1$.

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alogues for graphs

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tervals of $\mathbb{R}$ so that $\{x, y\} \in E$ if and only
Equivalently, the boxicity of a graph is just

the least t so that the graph is the intersection of boxes in $\mathbb{R}^t$. So interval graphs are graphs with boxicity one. Roberts showed that the boxicity of a graph on n vertices is at most $\lfloor n/2 \rfloor$, when $n \geq 2$. For example the graph $\mathbf{H}_n$, obtained by taking the complement of a matching on n edges has $2n$ vertices and boxicity n. In [127], Wittenshausen showed that for all $n \geq 1$, $\mathbf{H}_n$ is the only graph with $2n$ vertices and boxicity n.

However, when a graph has $2n+1$ vertices and boxicity n, the situation is modestly more complicated. For example, the cycle $\mathbf{C}_5$ on 5 vertices has boxicity 2. Also, the graph $\mathbf{W}_3$ with vertex set $\{1, 2, \ldots, 7\}$ and edges joining i to $i+2$, $i+3$, $i+4$ and $i+5$ (cyclically) has boxicity 3. In [113], Trotter showed that a graph $\mathbf{G}$ on $2n+1$ vertices has boxicity n if and only if one of the following conditions holds:

1. $\mathbf{G}$ contains $\mathbf{H}_n$.
2. $\mathbf{G}_n$ contains the join of $\mathbf{C}_5$ and $\mathbf{H}_{n-2}$.
3. $\mathbf{G}_n$ contains the join of $\mathbf{W}_3$ and $\mathbf{H}_{n-3}$.

In [109], Thomassen showed that the boxicity of a planar graph is at most 3; in fact, the boxes corresponding to adjacent vertices may be required to intersect on a face.

Many of the basic concepts for interval graphs have natural interpretation for digraphs. In [3], Beineke and Zamfirescu introduced the notion (with different terminology) of an *interval digraph.* By this we mean that for each vertex x in a digraph $\mathbf{D}$, there are two intervals of the real line R_x and S_x, so that $\mathbf{D}$ contains a directed arc from x to y if and only if $R_x \cap S_y \neq \emptyset$. Structural questions for interval digraphs are studied in [84], [85], [105] and [106].

Define the *interval number* of a graph $\mathbf{G} = (V, E)$ as the least t for which $\mathbf{G}$ is the intersection graph of a family of sets, with each set being the union of t pairwise disjoint closed intervals of $\mathbb{R}$. In [53], Griggs and West show that if the maximum degree of $\mathbf{G}$ is d, then the interval number of $\mathbf{G}$ is at most $\lceil (d+1)/n \rceil$. This inequality is tight if $\mathbf{G}$ is triangle-free. Griggs and West also showed that there exists an absolute constant $c > 0$ so that the interval number of a graph with q edges is at most $c\sqrt{q}$. In [103], Scheinerman and West showed that the interval number of a planar graph is at most 3, and Scheinerman [100] showed that there exists an absolute constant $c' > 0$ so that the interval number of a graph of genus γ is at most $c'\sqrt{\gamma}$.

In [122], Trotter and Harary show that the interval number of a complete bipartite graph $\mathbf{K}(m,n)$ is $\lceil (mn+1)/(m+n) \rceil$. The fact that the interval number of $\mathbf{K}(m,n)$ is at least this large follows from the following elementary observation. The interval number of a triangle-free graph $\mathbf{G}$ with n vertices and q edges is at least $\lceil (q+1)/n \rceil$. This inequality follows from the fact that if the interval number of the graph is t, then the nt intervals used in a representation form the intersection graph of a forest on nt vertices and at least q edges. This requires $q \leq nt - 1$.

Somewhat surprisingly, the determination of the interval numbers for complete multipartite graphs proved to be more challenging. The interval number of the complete multipartite graph $\mathbf{K}(n_1, n_2, \ldots, n_s)$, with $n_1 \geq n_2 \geq \cdots \geq n_t \geq 2$, is at least as large as the interval number of $\mathbf{K}(n_1, n_2)$. Call this quantity t_0. In [59], Hopkins, Trotter and West show the interval number of $\mathbf{K}(n_1, n_2, \ldots, n_s)$ is at most $t_0 + 1$. Furthermore, they show that it is equal to t_0, except possibly for the two cases $(n_1, n_2) = (7, 5)$ and $n_1 = n_2^2 - n_2 - 1$. In both these exceptional cases, the interval number of $\mathbf{K}(n_1, n_2, \ldots, n_t)$ may equal $t_0 + 1$, provided there are enough other parts of appropriate size.

Motivated by the formula for complete bipartite graphs, Trotter and Harary [122] conjectured that the maximum interval number of a graph on n vertices is $\lceil (n+1)/4 \rceil$. This conjecture was proved by Griggs in [52].

In [22], Chang and West introduced the concept of *interval number* for digraphs. For a digraph $\mathbf{D}$, the interval number of $\mathbf{D}$ is just the least positive integer i for which there exists a function F assigning to each vertex x two subsets R_x, S_x of the real numbers so that

1. For each node x in $\mathbf{D}$, R_x and S_x are each the union of at most t pairwise disjoint intervals of $\mathbb{R}$, and

2. $\mathbf{D}$ contains an arc from x to y if and only if $R_x \cap S_y \neq \emptyset$.

Chang and West showed that the maximum interval number of a digraph on n nodes is $\Theta(n/\log n)$. They also defined the concept of *boxicity* for digraphs and showed that the maximum boxicity of a digraph on n nodes is $\lceil n/2 \rceil$.

Aigner and Andreae [1] introduced an interesting variation of interval number. For an graph $\mathbf{G} = (V, E)$, they defined the *total interval number* of $\mathbf{G}$ as the least positive integer t for which there exists a function F assigning to each vertex x of $\mathbf{G}$ a set $F(x)$ which is the union of t_x pairwise disjoint closed intervals of $\mathbb{R}$ so that:

1. For every $x, y \in V$, $\{x, y\} \in E$ if and only if $F(x) \cap F(y) \neq \emptyset$, and

2. $\sum_{x \in X} t_x = t$.

Aigner and Andreae [1] produced upper bounds on total interval number for several classes of graphs. For example, they showed that the maximum total interval number of a tree on n nodes is $\lfloor (5n-3)/4 \rfloor$. In [80], Kratzke and West showed that the maximum total interval number of an outerplanar graph on n nodes is $\lfloor 3n/2 - 1 \rfloor$ while the maximum total interval number of a general graph on n nodes is $\lceil (n^2+1)/4 \rceil$. These results settled conjectures made by Aigner and Andreae in [1]. Other results on total interval number are given by Kostochka and West in [79]; in particular, they bound the total interval number in terms of the maximum degree, and characterize graphs for which the bound is sharp. The components of these graphs are balanced complete bipartite graphs.

In [81], Kratzke and West provide a li
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17 Higher dimensional analogue

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interval dimension at most 2. This characte
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been provided by several authors, but th
Spinrad [86].
One of the most appealing aspects of
solution of the *removable pair* conjecture
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nts of these graphs are balanced complete

In [81], Kratzke and West provide a linear time algorithm for computing the total interval number of a tree, and they show that it is NP-complete to test whether the total interval number of a graph is exactly one more than the number of edges, even for the class of triangle-free, 3–regular planar graphs.

Given a poset $\mathbf{P} = (X, P)$ and points $x, y \in X$, with $x \leq y$ in P, the *interval* $[x, y]$ is just the set $\{u \in X : x \leq u \leq y \text{ in } P\}$. The *poset boxicity* of a graph $\mathbf{G}$ is the least t for which there exists a t-dimensional poset $\mathbf{P}$ for which $\mathbf{G}$ is the intersection graph of intervals in $\mathbf{P}$. In [125], Trotter and West show that there exists an absolute constant $c > 0$ so that the poset boxicity of a graph on n vertices is at most $c \log \log n$. They also show that there exist graphs with arbitrarily large poset boxicity.

In [56], Gyárfás and West consider the *multitrack interval number* of a graph as the least t for which the graph is the union of t interval orders. We will discuss analogous concepts for posets in Sections 17 and 19.

17 Higher dimensional analogues for orders

The investigation of higher dimensional analogues of interval orders has also produced a steady stream of results. First, let $\mathcal{P}$ be any hereditary class of orders which contains the linear orders. Then we can define the $\mathcal{P}$–dimension of a poset $\mathbf{P} = (X, P)$ as the least t for which P is the intersection of t orders from $\mathcal{P}$. The hereditary property serves to ensure that the $\mathcal{P}$–dimension of $\mathbf{P}$ is at most the $\mathcal{P}$–dimension of $\mathbf{Q}$ when $\mathbf{P}$ is contained in $\mathbf{Q}$. Of course, the $\mathcal{P}$–dimension of $\mathbf{P}$ is at most $\dim(\mathbf{P})$, and to emphasize the distinction between the original definition of dimension and variants discussed in the remainder of this paper, the dimension is also called the *ordinary* dimension.

In [14], Bogart and Trotter defined the *interval dimension* of a poset $\mathbf{P} = (X, P)$ as the least t for which P is the intersection of t interval orders on X. So a poset has interval dimension 1 if and only if it is an interval order. Posets with interval dimension at most 2 have also been studied extensively. In [114], Trotter gave a forbidden subposet characterization of height two posets having interval dimension at most 2. This characterization results in a complete listing of all minimal posets of height 2 having interval dimension 3. Polynomial time recognition algorithms for posets having interval dimension at most 2 have been provided by several authors, but the best to date is due to Ma and Spinrad [86].

One of the most appealing aspects of interval dimension is the positive solution of the *removable pair* conjecture. For ordinary dimension, Trotter conjectured (see [118], for example) that if $\mathbf{P}$ is a poset having three or more points, then there is always a pair of points whose removal decreases the dimension by at most 1. In fact, he conjectured that the removal of a critical pair always decreases the dimension by at most 1. Although the removable pair conjecture remains open, this second conjecture was disproved by Reuter [95], and an infinite family of counterexamples was then constructed by Kierstead and Trotter [73].

However, for interval dimension, we have the following elementary result.

Theorem 17.1 *Let $\mathbf{P} = (X, P)$ be a poset and let $(x, y) \in \text{crit}(\mathbf{P})$. If $\mathbf{Q} = (Y, Q)$ is the subposet determined by $Y = X - \{x, y\}$, then the interval dimension of $\mathbf{P}$ is at most one more than the interval dimension of $\mathbf{Q}$.*

Proof Let $Q_1, Q_2, \ldots, Q_t$ be interval orders on Y whose intersection is Q. For each $i \in [t]$, let P_i be an interval order on X so that $P_i(Y) = Q_i$. Then let L be any linear extension of Y with $D(x) < Y - D(x)$ and $Y - U(y) < U(y)$ in L. Define a partial order P_{t+1} on X by setting $P_{t+1} = P \cup L$. It is easy to see that P_{t+1} is an interval order and that $P = P_1 \cap P_2 \cap \cdots \cap P_{t+1}$. ■

Another appealing aspect of the concept of interval dimension is that there is a relatively simple characterization of posets having maximal dimension for a given number of points (see Bogart and Trotter [13]), while the corresponding problem for ordinary dimension is considerably more difficult. Several other inequalities relating interval dimension to other combinatorial parameters are simpler than the corresponding results for ordinary dimension, e.g., compare the forbidden subposet characterization of the inequality $\dim(P, X) \leq \max\{2, |X - A|\}$, when A is an antichain, for ordinary dimension [111] with the result for interval dimension in [13].

Other aspects of the interplay between dimension and interval dimension are discussed in [30]. In [57], Habib, Kelly and Möhring show that the property of a poset having interval dimension at most 2 is a comparability invariant, i.e., it depends only on the underlying comparability graph and not on the specific order.

Bogart and Trotter also defined the semi-order dimension of a poset and noted that if the semi-order dimension of $\mathbf{P}$ is t, then the ordinary dimension of $\mathbf{P}$ is at most $3t$. This result is tight when $t = 1$, but it is not known whether it is best possible when $t \geq 2$. In [31], Felsner and Möhring show that the property of a poset having semi-order dimension at most 2 is a comparability invariant.

In a somewhat different direction, more closely connected to the concepts discussed in the preceding section, Madej and West [87] define the *interval inclusion number* of a poset $\mathbf{P} = (X, P)$ as the least integer t for which there exist a function F assigning to each $x \in X$ a set $F(x) \subset \mathbb{R}$ so that:

1. For each $x \in X$, $F(x)$ is the union of at most t pairwise disjoint closed intervals of $\mathbb{R}$, and
2. For each $x, y \in X$, $x \leq y$ in P if and only if $F(x) \subseteq F(y)$.

In [88], Madej and West show that "almost all" posets on n points have interval number $o(n)$, but it is still not known whether there exists a positive real number c so that for all n, there exists a poset on n points with interval

inclusion number exceeding cn. It is easy
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18 Intervals, angles and spheres

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Fishburn and Trotter [40] define a pose
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with each set being an angular region deter
a common point. They show that every inte
every poset with dimension at most 4 is an
there exists a 7-dimensional poset which is
several authors showed that there exists a 5
angle order, but the most elegant proof of
"degrees of freedom" developed by Alon an

A *d–sphere* with center $\mathbf{x}$ and radius r
distance to $\mathbf{x}$ is at most r. A 1–sphere is just
$\mathbf{P}$ a *sphere order* if there is some d so that
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union of at most t pairwise disjoint closed

if and only if $F(x) \subseteq F(y)$.

"almost all" posets on n points have interval
nown whether there exists a positive real
exists a poset on n points with interval

inclusion number exceeding cn. It is easy to see that the interval inclusion number of an n-dimensional poset is at most $\lceil n/2 \rceil$. Furthermore, as Madej and West note in [88], the set of all subsets of an n-element set, ordered by inclusion, shows that this last inequality is tight. However, as they point out, the n-dimensional standard example $\mathbf{S}_n$ has interval inclusion number 2, for all $n \geq 2$.

18 Intervals, angles and spheres

Over the past 10 years, there has been a flurry of work on geometric problems which arise when posets are represented by a family of sets (usually some geometrically defined objects) ordered by inclusion. These structures are called *inclusion orders*, and they are the natural order theoretic analogue of intersection graphs. For example, as is well known, a poset has dimension at most 2 if and only if it isomorphic to the inclusion order determined by a family of intervals of the real line. Space limitations do not allow us to discuss the full range of research on inclusion orders, but we will attempt to highlight those which related directly to interval orders.

Fishburn and Trotter [40] define a poset $\mathbf{P} = (X, P)$ to be an *angle order* when $\mathbf{P}$ is the inclusion order of a family of subsets of the euclidean plane, with each set being an angular region determined by two rays emanating from a common point. They show that every interval order is an angle order and that every poset with dimension at most 4 is an angle order. They also showed that there exists a 7-dimensional poset which is not an angle order. Subsequently, several authors showed that there exists a 5-dimensional poset which is not an angle order, but the most elegant proof of this fact results from the theory of "degrees of freedom" developed by Alon and Scheinerman in [2].

A *d-sphere* with center $\mathbf{x}$ and radius r is the set of all points in $\mathbb{R}^d$ whose distance to $\mathbf{x}$ is at most r. A 1-sphere is just a closed interval of $\mathbb{R}$. Call a poset $\mathbf{P}$ a *sphere order* if there is some d so that $\mathbf{P}$ is isormorphic to the inclusion order determined by a family of d-spheres in $\mathbb{R}^d$. When $\mathbf{P}$ is a sphere order, we may define the *sphere dimension* of a poset $\mathbf{P} = (X, P)$ as the least d for which $\mathbf{P}$ is the inclusion order of a family of d-spheres. So a poset has sphere dimension 1 if and only if it has ordinary dimension at most 2.

The problem of determining whether every finite poset is a sphere order is posed by Brightwell and Winkler in [21], and it is widely believed that the answer is negative.

When $d = 2$, there are some interesting results and one especially vexing problem. For historical reasons, posets with sphere dimension at most 2 are called *circle orders*, although it might have been more accurate to call them *disk orders*. The recent article [101] contains a number of interesting perspectives on the problem of representing order by circles in the plane. The range and extent of the connections with other combinatorial problems is most surprising.

In [37], Fishburn shows that every interval order is a circle order. Trivially, every poset with ordinary dimension at most 2 is a circle order—in fact, we can require that the circles all have centers on a fixed line in the plane. By the Alon/Scheinerman theory, there exist 4-dimensional posets which are not circle orders. However, it is not known whether every finite 3-dimensional poset is a circle order. Scheinerman and Weirman [102] showed that the countably infinite 3-dimensional poset $\mathbb{N}^3$ is not a circle order. Subsequently, a somewhat shorter proof of this result was given by Hurlbert [60]. The sharpest result to date is due to Fon-der-Flaass [43] who showed that $\mathbf{2} \times \mathbf{3} \times \mathbb{N}$ is not a sphere order, but that $\mathbf{2} \times \mathbf{2} \times \mathbb{N}$ is a circle order.

On the other hand, it is an easy exercise to show that if $\mathbf{P}$ is a finite poset with ordinary dimension at most 3 and $n \geq 3$, then $\mathbf{P}$ is the inclusion order of a family of regular n–gons in the euclidean plane, and it is easy to suspect that when n is quite large relative to $|X|$, these polygons are extremely close to being circles. However, I would conjecture that there is a finite 3-dimensional poset which is not a sphere order.

In this discussion, the metric used to determine distance plays a critical role. Of course, if $\mathbf{x} = (x_1, x_2, \ldots, x_d)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_d)$, then the ordinary distance from $\mathbf{x}$ to $\mathbf{y}$ is $\sqrt{\sum_{i=1}^{d}(x_i - y_i)^2}$. But if we change this to $\max\{|x_i - y_i| : 1 \leq i \leq d\}$, then a d–sphere is just a *cube*. Furthermore, it is an easy exercise to show that every poset with dimension at most $d+1$ is the inclusion order of a family of cubes in $\mathbb{R}^d$. Again, by the Alon/Scheinerman theory, this is best possible, meaning that there are $(d+2)$-dimensional posets which cannot be represented by cubes in $\mathbb{R}^d$ ordered by inclusion.

19 Tolerances, thresholds and gaps

In the preceding two sections, we discussed higher dimensional analogues for interval graphs and interval orders. In this section, we discuss generalizations which arise when just one interval is assigned but more complex rules are used to determine edges and comparabilities. Here is the basic motivation. If we have an indexed family $\mathcal{F} = \{I(x) : x \in X\}$ of closed intervals with distinct end points, then an interval graph results when we define an edge set E by $\{x, y\} \in X$ if and only if $|I(x) \cap I(y)| > 0$. From an applications standpoint, the problem with this definition is that we take two vertices to be adjacent when their intervals intersect *regardless* of how small this intersection might be. Similarly, an interval order assigns x to be less than y only when $F(x)$ lies entirely to the left on $F(y)$. But there are many scheduling problems where we want to consider one job as preceding another even when there is some overlap in time.

We begin with generalizations of interval graphs. Golumbic and Monma [49] proposed the following definition. Given an indexed family $\mathcal{F} = \{I(x) : x \in X\}$ of closed intervals of $\mathbb{R}$ and a subset $T = \{t_x : x \in X\}$ of the non-negative real numbers $\mathbb{R}_0$, define the *tolerance graph* $\mathbf{G} = \mathbf{G}(\mathcal{F}, T) = (X, E)$ by setting

$E = \{\{x, y\} : x, y \in X,\ x \neq y$ and $|I(x) \cap I($

It is easy to see that an interval graph
a distinguishing representation and give eac
the distance between any two end points
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each $x \in X$, set $t_x = |I(x)|$. A tolerance gr
for all $x \in X$. Golumbic and Monma [49]
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in [50], Golumbic, Monma and Trotter sho
perfect. The proof in the general case follows
of a tolerance graph is perfectly orderable.
graph being perfectly orderable [24] is a we
to show that interval graphs are perfect.

Here is an interesting way in which tole
graphs. Recall that an interval graph is prop
ation using only intervals of length 1. This
In [6], Bogart, Fishburn, Isaak and Langley
erance graphs is strictly larger than the clas
intervals have unit length.

In the last several years, a number of new
graphs have been introduced. Perhaps the
McMorris and Mulder [61] who proposed to s
intervals $\{I_x : x \in X\}$, a subset $T = \{t_x : x \in$
set $\mathbb{R}_0$ of non-negative reals, and a function ϕ
set E to consist of all 2-element sets $\{x, y\}$ f
The original definition of a tolerance graph
$\min\{t_x, t_y\}$.

Now here are some of the new ideas fo
sen (see [8]) propose to study a generaliza
extra conditions are imposed on the gaps
to comparable pairs of points. The defini
$\{I(x) = [a_x, b_x] : x \in X\}$ of closed intervals
of the non-negative reals $\mathbb{R}_0$, and a functi
define a relation P on X by setting (x, y)
(2) $b_y - a_x > \phi(t_x, t_y)$. We call these posets
relation P is defined in terms of the gap bet

In certain cases, P will be a partial or
always the case if ϕ satisfies the triangle i
$\phi(t_x, t_z)$, for all $t_x, t_y, t_z \in \mathbb{R}_0$. In particula
$\phi(t_x, t_y) = \max\{t_x, t_y\}$. On the other hand,
if $\phi(t_x, t_y) = \min\{t_x, t_y\}$. The special case
called a *max-gap* order.

In another direction, Bogart and Trenk

on at most 2 is a circle order—in fact, we
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e euclidean plane, and it is easy to suspect
o $|X|$, these polygons are extremely close to
njecture that there is a finite 3-dimensional

ised to determine distance plays a critical
$\ldots, x_d)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_d)$, then the
$\sqrt{\sum_{i=1}^{d}(x_i - y_i)^2}$. But if we change this to
d–sphere is just a *cube*. Furthermore, it is
poset with dimension at most $d+1$ is the
s in $\mathbb{R}^d$. Again, by the Alon/Scheinerman
ng that there are $(d+2)$-dimensional posets
bes in $\mathbb{R}^d$ ordered by inclusion.

and gaps

we discussed higher dimensional analogues
lers. In this section, we discuss generaliza-
erval is assigned but more complex rules are
arabilities. Here is the basic motivation. If
$x) : x \in X\}$ of closed intervals with distinct
results when we define an edge set E by
$y)| > 0$. From an applications standpoint,
that we take two vertices to be adjacent
rdless of how small this intersection might
gns x to be less than y only when $F(x)$ lies
ere are many scheduling problems where we
ng another even when there is some overlap

interval graphs. Golumbic and Monma [49]
Given an indexed family $\mathcal{F} = \{I(x) : x \in X\}$
set $T = \{t_x : x \in X\}$ of the non-negative
ce graph $\mathbf{G} = \mathbf{G}(\mathcal{F}, T) = (X, E)$ by setting

$E = \{\{x, y\} : x, y \in X,\ x \neq y \text{ and } |I(x) \cap I(y)| \geq \min\{t_x, t_y\}\}$.

It is easy to see that an interval graph is a tolerance graph. Just take a distinguishing representation and give each vertex a tolerance smaller than the distance between any two end points used in the representation. Also, the complement of an interval graph is a tolerance graph. In this case, for each $x \in X$, set $t_x = |I(x)|$. A tolerance graph is *bounded* if $0 \leq t_x \leq |I(x)|$, for all $x \in X$. Golumbic and Monma [49] showed that a bounded tolerance graph is the complement of a comparability graph and is therefore perfect. This argument does not work for tolerance graphs which are not bounded, but in [50], Golumbic, Monma and Trotter showed that all tolerance graphs are perfect. The proof in the general case follows by showing that the complement of a tolerance graph is perfectly orderable. Note that Chvatál's concept of a graph being perfectly orderable [24] is a weakening of the key property used to show that interval graphs are perfect.

Here is an interesting way in which tolerance graphs differ from interval graphs. Recall that an interval graph is proper if and only if it has a representation using only intervals of length 1. This is not true for tolerance graphs. In [6], Bogart, Fishburn, Isaak and Langley show that the class of proper tolerance graphs is strictly larger than the class of tolerance graphs in which all intervals have unit length.

In the last several years, a number of new concepts for generalizing tolerance graphs have been introduced. Perhaps the most general is due to Jacobson, McMorris and Mulder [61] who proposed to study graphs defined by a family of intervals $\{I_x : x \in X\}$, a subset $T = \{t_x : x \in X\}$ of tolerances drawn from the set $\mathbb{R}_0$ of non-negative reals, and a function $\phi \colon \mathbb{R}_0 \times \mathbb{R}_0 \to \mathbb{R}_0$ by setting the edge set E to consist of all 2-element sets $\{x, y\}$ for which $|I(x) \cap I(y)| > \phi(t_x, t_y)$. The original definition of a tolerance graph is just the function $\phi(t_x, t_y) = \min\{t_x, t_y\}$.

Now here are some of the new ideas for posets. McMorris and Jacobsen (see [8]) propose to study a generalization of interval orders in which extra conditions are imposed on the gaps between intervals corresponding to comparable pairs of points. The definition requires an indexed family $\{I(x) = [a_x, b_x] : x \in X\}$ of closed intervals of $\mathbb{R}$, a subset $T = \{t_x : x \in X\}$ of the non-negative reals $\mathbb{R}_0$, and a function $\phi \colon \mathbb{R}_0 \times \mathbb{R}_0 \to \mathbb{R}_0$. We then define a relation P on X by setting $(x, y) \in P$ if and only if (1) $x = y$ or (2) $b_y - a_x > \phi(t_x, t_y)$. We call these posets *$\phi$–gap* orders to reflect that the relation P is defined in terms of the gap between the two intervals.

In certain cases, P will be a partial order on X. For example, this is always the case if ϕ satisfies the triangle inequality: $\phi(t_x, t_y) + \phi(t_y, t_z) \geq \phi(t_x, t_z)$, for all $t_x, t_y, t_z \in \mathbb{R}_0$. In particular, P is always a partial order if $\phi(t_x, t_y) = \max\{t_x, t_y\}$. On the other hand, we may fail to get a partial order if $\phi(t_x, t_y) = \min\{t_x, t_y\}$. The special case where $\phi(t_x, t_y) = \max\{t_x, t_y\}$ is called a *max-gap* order.

In another direction, Bogart and Trenk [12] call a poset $\mathbf{P} = (X, P)$ a *bi-tolerance order* when there exists a triple (I, F, G) where:

1. I assigns to each $x \in X$ a closed interval $I(x) = [a_x, b_x]$ of $\mathbb{R}$;

2. $F = \{f_x : x \in X\} \subset \mathbb{R}$, $G = \{g_x : x \in X\} \subset \mathbb{R}$;

3. $a_x \leq f_x, g_x \leq b_x$ in $\mathbb{R}$, for every $x \in X$;

4. $x < y$ in P if and only if $b_x < f_y$ and $g_x < a_y$ in $\mathbb{R}$.

For a vertex x, the value $f_x - a_x$ is called the *left tolerance* of x, and the value $b_x - g_x$ is called the *right tolerance* of x. When $f_x - a_x = b_x - g_x$, for all $x \in X$, we call the poset a *tolerance order*. It is an easy exercise to show that the tolerance orders are just the posets which arise from ordering the complement of a bounded tolerance graph. For this reason, bi-tolerance orders were originally called *bounded* bi-tolerance orders, but with time the adjective "bounded" seems to have been discarded.

Every interval order is a bi-tolerance order with $f_x = a_x$ and $g_x = b_x$, for every vertex x, but it is easy to see that there are bi-tolerance orders which are not interval orders. It is an easy exercise to show that if $\mathbf{P} = (X, P)$ is a max-gap order, as evidenced by the intervals $\{[a_x, b_x] : x \in x\}$ and the subset $T = \{t_x : x \in X\} \subset \mathbb{R}_0$, then $\mathbf{P}$ is also a bi-tolerance order, as evidenced by the intervals $\{[a_x - t_x, b_x + t_x] : x \in X\}$ and the families $F = \{a_x : x \in X\}$ and $G = \{b_x : x \in X\}$.

In the definition of a bi-tolerance order, no restriction is placed on the order of f_x and g_x. However, if desired, we can always assume that $g_x \leq f_x$. This follows from the observation that if M is a positive number, then we may modify the representation by setting:

1. $a'_x = f_x - M$, $g'_x = g_x - M$, $b'_x = b_x + M$ and $f'_x = f_x + M$.

When M is sufficiently large, we always have $g'_x \leq f'_x$. We say the representation (I, F, G) of a bi-tolerance order is *separated* if $g_x < f_y$, for every $x, y \in X$. Note that if M is very large, then the new representation is separated. In fact, if desired, one can assume that $\{a_x : x \in X\} = \{1, 2, \ldots, n\}$ and $\{b_x : x \in X\} = \{n+1, n+2, \ldots, 2n\}$, where $n = |X|$. This last observation makes use of the fact that it is only the order on the various values that matters. On the other hand, the question as to which bi-tolerance orders have totally bounded representations is not as well understood, at least not for posets of arbitrary height.

The notion of separation makes it clear that for every $x \in X$, we have two intervals $I_1(x) = [a_x, g_x]$ and $I_2(x) = [f_x, b_x]$. Furthermore, the intervals $\{I_1(x) : x \in X\}$ form an interval order $\mathbf{P}_1$, and the intervals in $\{I_2(x) : x \in X\}$ form another interval order $\mathbf{P}_2$. Since x, y in P if and only if $I_1(x) < I_1(y)$ and $I_2(x) < I_2(y)$, it follows that the bi-tolerance orders are just the posets with interval dimension at most two.

Another interesting translation of the
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The main theorem of [12] asserts that
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Theorem 19.1 *Let* $\mathbf{P}$ *be a poset of heig*
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2. $\mathbf{P}$ *is a unit bi-tolerance order.*

3. $\mathbf{P}$ *is a tolerance order.*

4. $\mathbf{P}$ *is a unit tolerance order.*

5. $\mathbf{P}$ *is a 50%–tolerance order.*

6. $\mathbf{P}$ *is a totally bounded bi-tolerance*

For posets of arbitrary height, the var
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Another interesting translation of the concept of bi-tolerance orders involves the geometric insight gained when the intervals $I_1(x)$ and $I_2(x)$ are selected from two parallel lines in the plane, an interpretation first proposed by Dagan, Golumbic and Pinter [25]. The convex hull in $\mathbb{R}^2$ determined by the two intervals is a trapezoid, and the posets determined by a family of trapezoids is called a *trapezoid order*. So the trapezoid orders are again just the posets with interval dimension at most two. These insights are critical to the argument given by Habib, Kelly and Möhring [57] to show that interval dimension 2 is a comparability invariant. Also, in [9], Bogart, Möhring, and Ryan show that the family of unit area trapezoid orders is strictly contained in the family of proper trapezoid orders. In [32], Felsner, Müller and Wernisch provide fast algorithms for finding optimal partitions of trapezoid orders into chains and antichains. Their results even allow for weights to be assigned to the elements of the poset, and they are able to extend them to posets of large (but fixed) interval dimension.

Returning to Bogart and Trenk's definition of a bi-tolerance order, we should comment that their formulation has prompted the study of a number of interesting families of posets which can be described in terms of restrictions on the triple (I, F, G). As before, we may use the adjectives *proper* and *unit* to describe bi-tolerance orders (also, tolerance orders) in which the intervals are incomparable under inclusion and have unit length, respectively. Following Fishburn [38], we say a bi-tolerance order is *split* when $f_x = g_x$, for all $x \in X$, and we say a split bi-tolerance order is a 50% *tolerance order* when $f_x = g_x = (a_x + b_x)/2$, for all $x \in X$. Finally, we say the bi-tolerance order is *totally bounded* if $f_x \leq g_x$, for all $x \in X$.

The main theorem of [12] asserts that almost all these definitions coincide on posets of height two. Here are some of the equivalencies proved in [12].

Theorem 19.1 *Let* $\mathbf{P}$ *be a poset of height* 2*. Then the following statements are equivalent.*

1. $\mathbf{P}$ *is a proper bi-tolerance order.*

2. $\mathbf{P}$ *is a unit bi-tolerance order.*

3. $\mathbf{P}$ *is a tolerance order.*

4. $\mathbf{P}$ *is a unit tolerance order.*

5. $\mathbf{P}$ *is a* 50%*–tolerance order.*

6. $\mathbf{P}$ *is a totally bounded bi-tolerance order.* ■

For posets of arbitrary height, the various classes begin to separate, and equivalencies become more surprising. Recall that there is a distinction between unit and proper interval graphs. This distinction also holds between unit and proper tolerance orders. However, for bi-tolerance orders, we have the following result of Bogart and Isaak [7].

Theorem 19.2 *Let* **P** *be a poset. Then the following statements are equivalent.*

1. **P** *is a unit bi-tolerance order.*
2. **P** *is a proper bi-tolerance order.*

Proof A distinguishing representation of a unit bi-tolerance order shows that it is also a proper bi-tolerance order. Now let (I, F, G) be a distinguishing representation which evidences that a poset $\mathbf{P} = (X, P)$ is a proper bi-tolerance order. Without loss of generality, we may assume this representation is separated; we may also assume that $\{a_x : x \in X\} = \{1, 2, \ldots, n\}$ and $\{b_x : x \in X\} = \{n+1, n+2, \ldots, 2n\}$, where $|X| = n$. This is now a representation in which each interval has length n. ■

The next equivalency, due to Langley [82], is somewhat more surprising.

Theorem 19.3 *Let* **P** *be a poset. Then the following statements are equivalent.*

1. **P** *is a unit bi-tolerance order.*
2. **P** *is a split interval order.*

Proof Let (I, F, G) be a distinguishing unit representation of a poset $\mathbf{P} = (X, P)$. Modify the representation as follows.

$$a'_x = f_x;\ f'_x = g'_x = b_x \text{ and } b'_x = g_x + 1. \tag{16}$$

It follows easily that

$$b_x < f_y \text{ and } g_x < a_y \text{ if and only if } b'_x < f'_y \text{ and } g'_x < a'_y. \tag{17}$$

This transformation is easily seen to be reversible. ■

Note that an interval order is also a split interval order. To see this, just consider a distinguishing representation of an interval order and set $f_x = g_x = a_x$. On the other hand, there are split interval orders which are not interval orders, e.g. $\mathbf{2} + \mathbf{2}$.

Bogart, Fishburn, Isaak and Langley prove the following equivalency in [6], which is now an immediate corollary to Theorem 19.3.

Corollary 19.4 *Let* **P** *be a poset. Then the following statements are equivalent.*

1. **P** *is a unit tolerance order.*
2. **P** *is a* 50% *tolerance order.* ■

Although the family of tolerance orde
therefore contains posets of arbitrarily la
for many of the families involving restricti
example, here is a very recent result of Fis

Theorem 19.5 *If* **P** *is a split semi-order,*

Bi-tolerance orders must have large heig
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Although the family of tolerance orders includes the interval orders and therefore contains posets of arbitrarily large dimension, this is not the case for many of the families involving restrictions on lengths and tolerances. For example, here is a very recent result of Fishburn and Trotter [42].

Theorem 19.5 *If* $\mathbf{P}$ *is a split semi-order, then* $\dim(\mathbf{P}) \leq 6$. ■

Bi-tolerance orders must have large height in order to have large dimension, and it would be interesting to determine (or estimate) the maximum value of a bi-tolerance order of height h.

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Approximate Counting

Dominic Welsh

Summary I shall survey a range of very basic
which have a unifying geometrical theme. F
#P-hard but there is no obvious obstruction
approximation algorithm. I shall outline the
describe what is currently known. In most cas
a part of the input so there remain many oper

1 Introduction

Consider the following problem.

> How many different 4 × 4 matrices
> entries have row sums
>
> 1000, 9000, 30(
>
> and column sums
>
> 2000, 4000, 200
>
> The answer is approximately 10^{23}, an
>
> 1602 5658 9785 6815
>
> It was found in 2 seconds real time
> However had I posed a similar quest
> would be beyond the scope of everyda

A second problem for which I have thou
too slow) algorithms is the following.

> How does one generate a random plan
> by "random" I mean uniformly at ran
> graphs on n vertices?

Both of these problems are typical of the
ering here. The first is provably hard and w
second is not provably hard and some progr
in [15]. Here I shall discuss a range of pro
common theme in geometric combinatorics.

†The integers were chosen by me on March 22, 1
Mount had running on the machines at MSRI, Berk