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Dimensions of Split Semiorders

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Abstract. A poset $\mathbf{P} = (X, P)$ is a *split semiorder* when there exists a function I that assigns to each $x \in X$ a closed interval $I(x) = [a_x, a_x + 1]$ of the real line $\mathbb{R}$ and a set $F = \{f_x : x \in X\}$ of real numbers, with $a_x \leq f_x \leq a_x + 1$, such that $x < y$ in P if and only if $f_x < a_y$ and $a_x + 1 < f_y$ in $\mathbb{R}$. Every semiorder is a split semiorder, and there are split semiorders which are not interval orders. It is well known that the dimension of a semiorder is at most 3. We prove that the dimension of a split semiorder is at most 6. We note also that some split semiorders have semiorder dimension at least 3, and that every split semiorder has interval dimension at most 2.

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1. Introduction

We assume throughout that $\mathbf{P} = (X, P)$ is a poset (partially ordered set) with finite ground set X and order relation P on X that is reflexive, antisymmetric and transitive. The notations $x \leq y$ in P, $y \geq x$ in P and $(x, y) \in P$ are used interchangeably. We write $x < y$ in P and $y > x$ in P when $x \leq y$ in P and $x \neq y$. When $x, y \in X$, $(x, y) \notin P$ and $(y, x) \notin P$, we say x and y are *incomparable* and denote this by $x \| y$ in P. The set of all incomparable ordered pairs is $\text{inc}(\mathbf{P}) = \{(x, y) \in X \times X : x \| y \text{ in } P\}$.

Our main result is that a split semiorder has dimension at most 6. This result has the same flavor as Rabinovitch's theorem [18] that a semiorder has dimension at most 3. Our proof is preceded by constructions and lemmas that begin in the next section with definitions of split orders and their representations. We then comment on dimensionality in Section 3 and incomparable pairs in Section 4. Section 5 approaches the main result with lemmas on split semiorders and linear extensions, then proves the result. Section 6 discusses semiorder dimensions and interval order dimensions of split semiorders and includes some open problems.

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2. Intervals and Split Orders

We recall that $\mathbf{P} = (X, P)$ is an *interval order* if there is a function I that assigns a closed real interval $I(x) = [a_x, b_x]$ to each $x \in X$ such that $x < y$ in P if and only if $b_x < a_y$ in $\mathbb{R}$. We call I an *interval representation* of $\mathbf{P}$. An interval order is a *semiorder* if it has an interval representation in which all assigned intervals have length 1. End points of intervals assigned by a representation need not be distinct, but any interval representation can be modified to make all intervals non-degenerate and all end points distinct. We refer to such an I as a *distinguishing* interval representation. Additional background material on interval orders and semiorders is given in Fishburn [11] and Trotter [20, 21].

A poset $\mathbf{P} = (X, P)$ is a *split interval order* if there exists a function I that assigns a closed real interval $I(x) = [a_x, b_x]$ to each $x \in X$ and a set $F = \{f_x : x \in X\}$ of real numbers such that:

1. For all $x \in X$, $a_x \leq f_x \leq b_x$, and
2. For all $x, y \in X$, $x < y$ in P if and only if $f_x < a_y$ and $b_x < f_y$ in $\mathbb{R}$.

We call (I, F) a *representation* of $\mathbf{P}$ in this case. When $\mathbf{P} = (X, P)$ is a split interval order, it has a *distinguishing* representation (I, F) for which $|\{a_x : x \in X\} \cup \{b_x : x \in X\} \cup \{f_x : x \in X\}| = 3|X|$ (see Fishburn and Trotter [14]).

Bogart and Isaak [4] refer to a representation (I, F) as a *Fishburn representation* following informal correspondence with that author in 1989. They prove that the class of split interval orders equals the class of proper unit bitolerance orders discussed previously in Langley [16] and Bogart and Trenk [8]. We say that $\mathbf{P} = (X, P)$ is a *proper unit bitolerance order* if there is a function U that assigns a closed unit interval $U(x) = [a_x, a_x + 1]$ to each $x \in X$ and sets $G = \{g_x : x \in X\}$ and $H = \{h_x : x \in X\}$ of real numbers so that:

1. For all $x \in X$, $a_x \leq g_x, h_x \leq a_x + 1$, and
2. For all $x, y \in X$, $x < y$ in P if and only if $h_x < a_y$ and $a_x + 1 < g_y$ in $\mathbb{R}$.

The Bogart–Isaak equivalence proof shows how this unit interval representation with two special points in each interval can be transformed into an (I, F) representation with general length intervals and one special point in each interval, and conversely, while preserving $\mathbf{P}$. The following representation for split semiorders is an obvious specialization of the (I, F) and (U, G, H) representations.

A poset $\mathbf{P} = (X, P)$ is a *split semiorder* when there exists a function U that assigns a closed real interval $U(x) = [a_x, a_x + 1]$ to each $x \in X$ and a set $F = \{f_x : x \in X\}$ of real numbers so that:

1. For all $x \in X$, $a_x \leq f_x \leq a_x + 1$, and
2. For all $x, y \in X$, $x < y$ in P if and only if $f_x < a_y$ and $a_x + 1 < f_y$ in $\mathbb{R}$.

We call (U, F) a *representation* of $\mathbf{P}$ in this case. When $|\{a_x : x \in X\} \cup \{a_x + 1 : x \in X\} \cup F| = 3|X|$, the representation is *distinguishing*. It is again an easy exercise to show that a split semiorder has a distinguishing representation.

It is easily seen that the class of all semiorders is properly included in the class of all split semiorders, the classes of interval orders and split semiorders are properly included in the class of split interval orders, and the class of split semiorders is neither included in nor includes the class of interval orders. These and many other comparisons between special posets are discussed in Bogart and Trenk [8] and Fishburn [12]. See also Bogart [1, 2], Bogart, Fishburn, Isaak and Langley [3], Doignon, Monjardet, Roubens and Vincke [9] and Roubens and Vincke [19].

3. Dimensions and Linear Extensions

A *linear order* L on X is a complete order relation on X, so that for all $x, y \in X$, $x \leq y$ in L or $y \leq x$ in L. Let $\mathcal{C}(X)$ denote a class of order relations on X. We may then define the $\mathcal{C}$*-dimension* of $\mathbf{P} = (X, P)$ as the minimum cardinality of a subset of $\mathcal{C}(X)$ whose members have P as their intersection. Some posets may not have a representation as an intersection of posets from $\mathcal{C}$, and in this case we say that their $\mathcal{C}$-dimension is infinite. To discuss alternative formulations of dimension, we refer to the original Dushnik and Miller [10] definition of dimension as "linear dimension". When $\mathcal{C}$ is the class of interval orders, we have *interval dimension*, and when $\mathcal{C}$ is the class of semiorders, we have *semiorder dimension*. We denote the interval dimension of a poset $\mathbf{P}$ by Idim($\mathbf{P}$) and the semiorder dimension of $\mathbf{P}$ by Sdim($\mathbf{P}$). These two parameters were first introduced in [6], and the concept of interval dimension has been studied extensively in the interim (see [15] and [7], for example). Other families which have been studied include angle order dimension [13] and series parallel dimension [15].

For now, we concentrate on linear dimension, returning to the other concepts in Section 6. A *linear extension* of P is a linear order L on X for which $P \subseteq L$. A set $\mathcal{R}$ of linear extensions of P is called a *realizer* of $\mathbf{P}$ when $P = \bigcap \mathcal{R}$. Thus the linear dimensionality of $\mathbf{P}$, denoted by dim($\mathbf{P}$), is the minimum cardinality of a realizer of $\mathbf{P}$. Although interval orders can have large dimensions [5, 20], this is not true for semiorders [17, 18].

THEOREM 3.1. *If* $\mathbf{P} = (X, P)$ *is a semiorder, then* $\dim(\mathbf{P}) \leq 3$.

Because the class of split semiorders is neither included in nor includes the class of interval orders, it is conceivable that some split semiorders have large dimensions. However, our main result says otherwise. We continue towards its proof by recalling a result on reversals of incomparable pairs.

4. Reversals of Incomparable Pairs

A family $\mathcal{R}$ of linear extensions of P is obviously a realizer of P if and only if for every $(x, y) \in \mathrm{inc}(\mathbf{P})$, there is an $L \in \mathcal{R}$ for which $x > y$ in L. We say L *reverses* the incomparable pair (x, y) when $x > y$ in L. Similarly, we say that $\mathcal{L}$ *reverses* a set $S \subseteq \mathrm{inc}(\mathbf{P})$ if for every $(x, y) \in S$, $x > y$ in L for at least one $L \in \mathcal{L}$. So dim(**P**) is simply the minimum cardinality of a family of linear extensions of P which reverses inc(**P**).

We find it useful to have a convenient test which tells when there is a linear extension which reverses a given set of incomparable pairs. Given an integer $k \geq 2$, a subset $S = \{(x_i, y_i) : 1 \leq i \leq k\} \subseteq \text{inc}(\mathbf{P})$ is called an *alternating cycle* when $x_i \leq y_{i+1}$ in P, for all $i = 1, 2, \ldots, k$. In this definition, and in arguments to follow, the subscripts on elements of an alternating cycle are interpreted cyclically, so that the condition $x_k \leq y_{k+1}$ in P means $x_k \leq y_1$ in P. The following elementary result is due to Trotter and Moore [22]. A short proof and applications appear in [20].

THEOREM 4.1. *Suppose* $\mathbf{P} = (X, P)$ *and* $S \subseteq \text{inc}(\mathbf{P})$. *Then the following statements are equivalent.*

1. *There is a linear extension L of P which reverses S.*
2. *S does not include an alternating cycle.*

5. The Principal Theorem

We continue towards the proof of our main result with a few lemmas for split interval orders and an observation for arbitrary posets.

Let (I, F) be a representation of a split interval order $\mathbf{P} = (X, P)$. Given $(x, y) \in \text{inc}(\mathbf{P})$, we say that x *captures* y if $b_x \geq f_y$, and y *captures* x if $a_y \leq f_x$. Because $x < y$ in P if and only if $f_x < a_y$ and $b_x < f_y$, exactly one of the following three statements holds for every $(x, y) \in \text{inc}(\mathbf{P})$:

1. x captures y, but y does not capture x.
2. y captures x, but x does not capture y.
3. x captures y, and y captures x.

Accordingly, we partition the pairs in inc(**P**) into Type 1, Type 2 and Type 3, respectively. Note that the classification of incomparable pairs depends on the representation.

LEMMA 5.1. *Suppose (I, F) is a representation of a split interval order* $\mathbf{P} = (X, P)$, *and X_1 and X_2 are disjoint subsets of X. Then there exists a linear extension L of P which reverses all the Type* 1 *and Type* 3 *pairs in* $\text{inc}(\mathbf{P}) \cap (X_1 \times X_2)$.

Proof. Let S denote the set of all Type 1 and Type 3 pairs in $\text{inc}(\mathbf{P}) \cap (X_1 \times X_2)$. Suppose there is no linear extension which reverses S. Then by Theorem 4.1, there is some $k \geq 2$ and an alternating cycle $\{(x_i, y_i) : 1 \leq i \leq k\}$ contained in S. Since x_i captures y_i, for each $i = 1, 2, \ldots, k$, we know that $f_{y_i} \leq b_{x_i}$, for each $i = 1, 2, \ldots, k$. Also, $x_i \leq y_{i+1}$ in P and $X_1 \cap X_2 = \emptyset$ imply $x_i < y_{i+1}$ in P, for $i = 1, 2, \ldots, k$. We conclude that $b_{x_i} < f_{y_{i+1}}$, for each $i = 1, 2, \ldots, k$, hence that $f_{y_i} < f_{y_{i+1}}$, for $i = 1, 2, \ldots, k$. This is impossible, and the contradiction shows the existence of a linear extension reversing all pairs in S. □

The dual version is also valid.

LEMMA 5.2. *Suppose* (I, F) *is a representation of a split interval order* $\mathbf{P} = (X, P)$*, and* X_1 *and* X_2 *are disjoint subsets of* X*. Then there exists a linear extension* L *of* P *which reverses all the Type* 2 *and Type* 3 *pairs in* $\text{inc}(\mathbf{P}) \cap (X_1 \times X_2)$.

The next lemma bounds the height of a split interval order when all the intervals used in the representation have a common point. Recall that the *height* of **P** is the maximum cardinality of a chain in **P**.

LEMMA 5.3. *Suppose* (I, F) *is a representation of a split interval order* $\mathbf{P} = (X, P)$*, and that* $a_x \leq f \leq b_x$*, for all* $x \in X$*, for a fixed* $f \in \mathbb{R}$*. Then* **P** *has height at most* 2*.*

Proof. Given the hypothesis, let $B = \{x \in X : f_x < f\}$ and let $T = X - B$. We claim (1) B and T are antichains, and (2) if $x < y$ in P, then $x \in B$ (Bottom) and $y \in T$ (Top). Suppose that $x < y$ in P. If $x \in T$, then $a_y \leq f \leq f_x < a_y$, which is impossible; if $y \in B$, then $b_x < f_y < f \leq b_x$, which is impossible. □

We conclude our preliminaries with an elementary lemma valid for all posets. Its proof is left as an exercise.

LEMMA 5.4. *Suppose that* $\mathbf{P} = (X, P)$ *and let* $X = A_1 \cup A_2 \cup \cdots \cup A_m$ *be a partition of* X *so that, for all* $x, y \in X$, $[x \in A_i, y \in A_j, x < y \text{ in } P] \Rightarrow i \leq j$*. Let* L_i *be a linear extension of* $P(A_i)$*, for each* $i = 1, 2, \ldots, m$*. Then there is a linear extension* L *of* P *so that* $L_i = L(A_i)$*, for each* $i = 1, 2, \ldots, m$*.*

We now prove our main theorem.

THEOREM 5.5. *If* $\mathbf{P} = (X, P)$ *is a split semiorder, then* $\dim(\mathbf{P}) \leq 6$*.*

Proof. Let (U, F) be a distinguishing representation of a split semiorder $\mathbf{P} = (X, P)$ with

$$U(x) = [a_x, a_x + 1] \quad \text{and} \quad f_x \in U(x) \cap F.$$

We define a partition $X = A_1 \cup A_2 \cup \cdots \cup A_m$ recursively left to right along $\mathbb{R}$. Set $X_1 = X$. Whenever $X_i \neq \emptyset$, let z_i be the unique element of X_i that minimizes a_z, for $z \in X_i$, and let

$$g_i = a_{z_i} + 1.$$

Define $A_i = \{x \in X_i : a_x \leq g_i \leq a_x + 1\}$ and $X_{i+1} = X_i - A_i$ (see Figure 1). Suppose that the procedure halts with a partition $X = A_1 \cup A_2 \cup \cdots \cup A_m$. Then for all $x, y \in X$, $[x \in A_i, y \in A_j, x < y \text{ in } P] \Rightarrow i \leq j$. Also, note that for each $i = 1, 2, \ldots, m - 1$, $g_i + 1 < g_{i+1}$.

For each $i = 1, 2, \ldots, m$, let $\mathbf{A}_i$ be the subposet of **P** induced by A_i. Also, let $B_i = \{x \in A_i : f_x < g_i\}$ and $T_i = A_i - B_i$. We know from the proof of Lemma 5.3

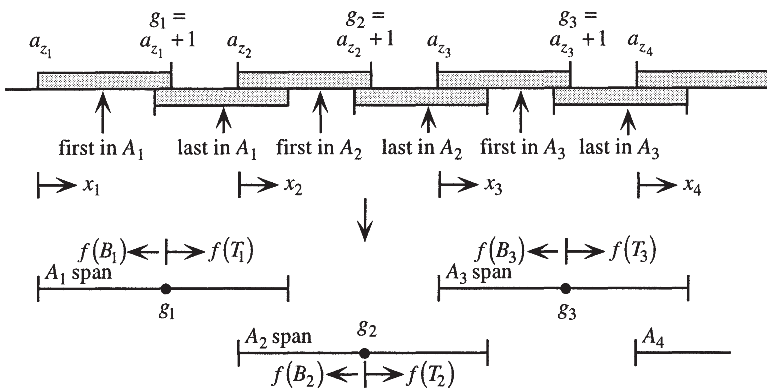

Figure 1.

that $\mathbf{A}_i$ has height at most 2, the members of B_i are minimal in $\mathbf{A}_i$, and those in T_i are maximal in $\mathbf{A}_i$. Note that for each $i = 1, 2, \ldots, m$, $f_{z_i} < a_{z_i} + 1 = g_i \leq a_{z_{i+1}}$, so $z_i \in B_i$ and $B_i \neq \emptyset$. However, it may happen that $T_i = \emptyset$.

CLAIM 1. *If $i, j \in \{1, 2, \ldots, m\}$, $x \in A_i$, $y \in A_j$ and $j \geq i + 2$, then $x < y$ in P.*

Proof. The hypotheses imply $a_x \leq g_i, g_j \leq a_y + 1$ and $g_i + 2 < g_j$ in $\mathbb{R}$, so $a_x + 1 < a_y$ in $\mathbb{R}$. Thus $x < y$ in P. □

CLAIM 2. *If $i \in \{1, 2, \ldots, m - 1\}$, $x \in B_i$ and $y \in T_{i+1}$, then $x < y$ in P.*

Proof. The hypotheses imply $f_x < g_i$ and $g_{i+1} \leq f_y$. Since $g_i + 1 < g_{i+1}$, we conclude that $x < y$ in P. □

The preceding claims coupled with a left-to-right perspective in the lower part of Figure 1 show the crucial reversals for inc($\mathbf{P}$) in constructing a realizer $\mathcal{R}$ of $\mathbf{P}$ are those within each $\mathbf{A}_i$, namely $B_i \times B_i$, $T_i \times T_i$ and $B_i \times T_i$, and those between consecutive $\mathbf{A}_i$, namely $B_i \times B_{i+1}$, $T_i \times T_{i+1}$ and $T_i \times B_{i+1}$. The latter are also the incomparable pairs in $T_i \times A_{i+1}$, where Type 1 and Type 3 can occur, and in $A_i \times B_{i+1}$, where Type 2 and Type 3 can occur. Reversals in the dual sense, apart from $B_i \times B_i$ and $T_i \times T_i$, i.e., those for which $<$ adheres to the natural left-to-right sense of Figure 1, are accounted for in piecing together the parts of realizing linear extensions that we construct for the crucial reversals. We therefore focus on the crucial reversals.

We denote our realizer of $\mathbf{P}$ by $\mathcal{R} = \{L_1, L_2, \ldots, L_6\}$. The construction of the L_j begins with reversals for $B_i \times T_i$, $A_i \times B_{i+1}$ and $T_i \times A_{i+1}$ according to the following prescriptions.

S1. If $1 \leq i \leq m$ and $i \equiv j \bmod 6$, then L_j reverses all Type 1 and Type 3 pairs in inc($\mathbf{P}$) $\cap$ ($B_i \times T_i$).

S2. If $1 \le i < m$ and $i \equiv j + 1 \bmod 6$, then L_j reverses Type 2 and Type 3 pairs in $\text{inc}(\mathbf{P}) \cap (A_i \times B_{i+1})$.
S3. If $1 \le i \le m$ and $i \equiv j + 3 \bmod 6$, then L_j reverses Type 2 and Type 3 pairs in $\text{inc}(\mathbf{P}) \cap (B_i \times T_i)$.
S4. If $1 \le i < m$ and $i \equiv j + 4 \bmod 6$, then L_j reverses Type 1 and Type 3 pairs in $\text{inc}(\mathbf{P}) \cap (T_i \times A_{i+1})$.

The reversals in each case are validated by Lemmas 5.1 and 5.2. Note that these four rules do not specify orders on the antichains in $\{T_i : i \equiv j + 2 \mod 6\}$ and $\{B_i : i \equiv j + 4 \mod 6\}$. We use two more prescriptions for this purpose, which are applied after S1–S4 because these can affect the dual L^d parts of S5 and S6.

S5. If $1 \le i \le m$ and $i \equiv j + 2 \bmod 6$, then $L_j(T_i) = L^d_{j+1}(T_i)$.
S6. If $1 \le i \le m$ and $i \equiv j + 4 \bmod 6$, then $L_j(B_i) = L^d_{j+1}(B_i)$.

In applying S5 and S6, the subscripts on the linear extensions in $\mathcal{R}$ are interpreted cyclically.

We now apply Lemma 5.4 for each L_j in our six prescriptions. For example, the parts of the partition for L_1 are the nonempty sets A_1, $A_2 \cup B_3$, T_3, A_4, B_5, $T_5 \cup A_6$, A_7, $A_8 \cup B_9$, $T_9, \ldots,$ and these are taken in the listed order to specify L_1 entirely. Because each L_j is a linear extension of P and $\mathcal{R}$ reverses $\text{inc}(\mathbf{P})$, we conclude that $\mathcal{R}$ is a realizer and that $\dim(\mathbf{P}) \le 6$.

6. Discussion

It is obvious from our definitions in Section 3 that

$$\text{Idim}(\mathbf{P}) \le \text{Sdim}(\mathbf{P}) \le \dim(\mathbf{P}),$$

for every poset $\mathbf{P}$. When $\mathbf{P}$ is a split semiorder, $\text{Idim}(\mathbf{P}) \le 2$. Indeed, $\text{Idim}(\mathbf{P}) \le 2$ for every split interval order, as is evident from $x < y$ in P if and only if $f_x < a_y$ and $b_x < f_y$, and the observation that P_1 and P_2 are interval orders when we take $x < y$ in P_1 if and only if $f_x < a_y$ and $x < y$ in P_2 if and only if $b_x < f_y$. On the other hand, the semiorder dimension of a split interval order can be arbitrarily large [6], whereas

$$3 \le \max\{\text{Sdim}(\mathbf{P}) : \mathbf{P} \text{ is a split semiorder}\} \le 6.$$

The upper bound is by Theorem 5.5, and the lower bound is demonstrated by the fact [12] that there exists a split semiorder that is not the intersection of any two semiorders.

Rabinovitch [17, 18] gives examples of semiorders of linear dimension 3, but we do not know whether there are split semiorders of linear dimension 6. The maximum value of the linear dimension of a split semiorder and the maximum value of the interval dimension of a split semiorder remain open.

A related issue is whether there are "interesting" classes of posets in which the maximum value of interval dimension is less than the maximum value of semiorder dimension which is in turn less than the maximum value of linear dimension.

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