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Split semiorders

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Abstract

A poset $P = (X, \prec)$ is a *split semiorder* if there are maps $a, f : X \to \mathbb{R}$ with $a(x) \leqslant f(x) \leqslant a(x) + 1$ for every $x \in X$ such that $x \prec y$ if and only if $f(x) < a(y)$ and $a(x) + 1 < f(y)$. A *split interval order* is defined similarly with $a(x) + 1$ replaced by $b(x)$, $a(x) \leqslant f(x) \leqslant b(x)$, such that $x \prec y$ if and only if $f(x) < a(y)$ and $b(x) < f(y)$. We investigate these generalizations of semiorders and interval orders through aspects of their numerical representations, three notions of poset dimensionality, minimal forbidden posets, and inclusion relationships to other classes of posets, including several types of tolerance orders. © 1999 Elsevier Science B.V. All rights reserved

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1. Introduction

Let $\mathscr{P}$ denote the class of all finite posets of the form $P = (X, \prec)$ with nonempty finite ground set X partially ordered by $\prec$. We assume that $\prec$ is asymmetric and transitive and denote its symmetric complement by $\sim$ so that $x \sim y$ if neither $x \prec y$ nor $y \prec x$. We write $x \| y$ and say that x and y are *incomparable* in P if $x \sim y$ and $x \neq y$.

Our main purpose is to investigate subclasses of $\mathscr{P}$ referred to as split semiorders and split interval orders. These subclasses will be considered on their own and in relation to other subclasses defined by real interval representations or by exclusion of forbidden posets. The others include the intensively studied linear orders, weak orders, semiorders and interval orders, plus versions of tolerance orders in Bogart [1,2], Bogart and Isaak [4], Bogart and Trenk [5], Fishburn [9] and Langley [12], among others.

In the present formulation, $P = (X, \prec)$ is a *linear order* or *chain* if $x \| y$ for no $x, y \in X$. For positive integers m and n let $\mathbf{m} + \mathbf{n}$ denote a poset on $m + n$ points that

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consists of two disjoint chains on m points and n points with $x||y$ whenever x and y are in different chains. A linear order has no $\mathbf{1}+\mathbf{1}$ as an induced subposet. Other minimal forbidden induced subposet definitions are: P is a *weak order* if it has no $\mathbf{1}+\mathbf{2}$; P is a *semiorder* if it has no $\mathbf{1}+\mathbf{3}$ and no $\mathbf{2}+\mathbf{2}$; P is an *interval order* if it has no $\mathbf{2}+\mathbf{2}$.

We recall also that $P=(X,\prec)$ is a semiorder [8,13,15] if there is a function U that assigns a closed unit interval $U(x)=[a(x),a(x)+1]$ to each $x\in X$ such that $x\prec y$ if and only if $a(x)+1<a(y)$, and an interval order [7,8,18] if there is a function I that assigns a closed interval $I(x)=[a(x),b(x)]$ to each $x\in X$ such that $x\prec y$ if and only if $b(x)<a(y)$. Split semiorders and split interval orders generalize these representations with the addition of a splitting point in each interval. We define $P=(X,\prec)$ as a *split semiorder* if there a function U that assigns a closed unit interval $U(x)=[a(x),a(x)+1]$ to each $x\in X$ and a set $F=\{f(x):x\in X\}$ of real numbers such that:

1. For all $x\in X$, $a(x)\leqslant f(x)\leqslant a(x)+1$, and
2. For all $x,y\in X$, $x\prec y$ if and only if $f(x)<a(y)$ and $a(x)+1<f(y)$.

Similarly, $P=(X,\prec)$ is a *split interval order* if there is a function I that assigns a closed interval $I(x)=[a(x),b(x)]$ to each $x\in X$ and a set $F=\{f(x):x\in X\}$ of real numbers such that:

1. For all $x\in X$, $a(x)\leqslant f(x)\leqslant b(x)$, and
2. For all $x,y\in X$, $x\prec y$ if and only if $f(x)<a(y)$ and $b(x)<f(y)$.

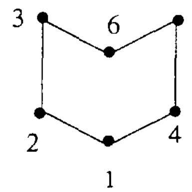
We refer to (U,F) as a *split semiorder representation* of P and to (I,F) as a *split interval order representation* of P. When the type of order is clear, we may simply refer to (U,F) or (I,F) as a representation.

Langley [12] and Bogart and Isaak [4] prove that P is a split interval order if and only if it is a proper bitolerance order. The latter order has a different representational definition that we present in the next section, but its representation can be mapped into a split interval order representation, and conversely, without disturbing $\prec$. Split semiorders as well as split interval orders are central to [9,10]. Their results are integrated into the present study. We rely also on Bogart and Trenk [5] as well as results included in Trotter [18] when we discuss bipartite orders later in the paper.

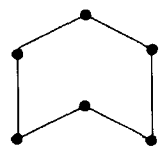
Several definitions relevant throughout are noted before we outline the rest of the paper. The *dual* of $P=(X,\prec)$ is $P^d=(X,\prec^d)$ with $x\prec^d y$ if $y\prec x$. The *height* $H(P)$ of P is the number of points in a cardinally maximum chain in P. An *antichain* is a height-1 poset. We say that P is *bipartite* if $H(P)\leqslant 2$ and denote by $\mathscr{P}_2$ the class of bipartite posets in $\mathscr{P}$. *Crown* C_n for $n\geqslant 2$ is the $2n$-point bipartite poset $(\{x_1,\dots,x_n\}\cup\{y_1,\dots,y_n\},\prec)$ with $\prec$ completely specified by $\{x_i\prec y_i, x_i\prec y_{i+1}\}$, $i=1,\dots,n-1$, and $\{x_n\prec y_n, x_n\prec y_1\}$.

The *dimension* $\dim(P)$ of $P=(X,\prec)$ is the minimum k for which there are linear orders $(X,\prec_1),\dots,(X,\prec_k)$ with $\prec\ =\ \cap_{j=1}^k \prec_j$. The *interval dimension* $\mathrm{Idim}(P)$ of P is the minimum k for which there are interval orders $(X,\prec_1),\dots,(X,\prec_k)$ with $\prec\ =\ \cap_{j=1}^k \prec_j$. The *semiorder dimension* $\mathrm{Sdim}(P)$ of P is the minimum k for which there are semiorders $(X,\prec_1),\dots,(X,\prec_k)$ with $\prec\ =\ \cap_{j=1}^k \prec_j$. The definitions imply

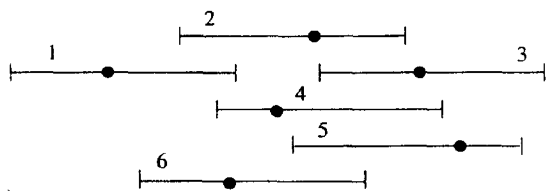
$\mathrm{Idim}(P) \leqslant \mathrm{Sdim}(P) \leqslant \dim(P)$ for all $P \in \mathscr{P}$. We have $\mathrm{Idim}(C_3) = \dim(C_3) = 3$ and $\mathrm{Idim}(P) = \mathrm{Sdim}(P) = 2$ but $\dim(P) = 3$ for the chevron.



and its dual



Up to duality, the chevron is the uniquely smallest 3-dimensional poset that is a split semiorder, with (U,F) representation



in which the solid dot in the interval for i denotes $f(i)$. Important dimensionality results used later are Rabinovitch's theorem [14] that every semiorder has dim $\leqslant 3$, and the main theorem in [10] which says that every split semiorder has dim $\leqslant 6$.

The next section of the paper defines an array of special classes in $\mathscr{P}$ and presents their inclusion diagram from [9] along with maximum values or bounds on Idim, Sdim and dim within each class.

Sections 3 and 4 discuss basic aspects of split semiorders and split interval orders. Section 3 proves that their representations need never use the same real number more than once for an end point or splitting point. Section 4 focuses on minimal forbidden posets. It is motivated by the fact that there is no finite set $\mathscr{P}^*$ of posets such that $P \in \mathscr{P}$ is a split semiorder (or split interval order) if and only if no induced subposet of $\mathscr{P}$ is in $\mathscr{P}^*$. For example, no crown C_n for $n \geqslant 3$ is a split interval order, but this class of minimal forbidden posets is a very small segment of the minimal forbidden posets for either split semiorders or split interval orders. Other notable minimal forbidden posets for split semiorders are $\mathbf{2} + \mathbf{3}$ and $\mathbf{1} + \mathbf{4}$, but neither is forbidden for split interval orders.

Section 5 focuses on $\mathscr{P}_2$. It shows the reconfiguration of the poset classes of Section 2 in the bipartite domain. Nonidentical bipartite classes from Section 2 are linearly ordered by proper inclusion and, with one exception, there is a uniquely smallest poset between adjacent classes that is in the upper class but not in the lower class.

In the bipartite context, split semiorders are equivalent to split interval orders and other classes, including the class of all bipartite posets with $\mathrm{Idim} \leqslant 2$. It follows that the family of 3-interval irreducible posets of height two described in Trotter and Moore [20, pp. 375–377] (see also [17,18, pp. 81–85]) characterizes the family of minimal forbidden bipartite posets for split semiorders as well as for the other classes in the next section whose bipartite restrictions are also characterized by $\mathrm{Idim} \leqslant 2$.

The paper concludes with a brief summary and open problems.

2. Poset classes

We arrange further definitions of classes in $\mathscr{P}$ into three groups based on exclusions, dimensionality, and enhanced interval representations. The exclusionary mode defines P as a *semitransitive order* if it has no induced $\mathbf{1}+\mathbf{3}$, and as a *subsemiorder* if it has no induced $\mathbf{2}+\mathbf{3}$ and no induced $\mathbf{1}+\mathbf{4}$. Subsemiorders appear in Trenk [16] as part of a more extensive system of classes in $\mathscr{P}$. Unlike interval orders (no $\mathbf{2}+\mathbf{2}$) and semiorders (no $\mathbf{2}+\mathbf{2}$, no $\mathbf{1}+\mathbf{3}$), the semitransitive order class and the subsemiorder class include $\mathscr{P}_2$.

The second group defines P as a *bilinear order* if $\dim(P)\leqslant 2$, a *bisemiorder* if $\mathrm{Sdim}(P)\leqslant 2$, and a *bi-interval order* if $\mathrm{Idim}(P)\leqslant 2$. Dushnik and Miller [6] describe alternative characterizations of bilinear orders: see also [8]. Clearly, $P=(X,\prec)$ is a bisemiorder if there are $a,c: X\to\mathbb{R}$ such that $x\prec y$ if and only if $a(x)+1<a(y)$ and $c(x)+1<c(y)$, and is a bi-interval order if there are $a,b,c,d: X\to\mathbb{R}$ such that $a\leqslant b, c\leqslant d$, and $x\prec y$ if and only if $b(x)<a(y)$ and $d(x)<c(y)$. The interval configurations in the latter representation have given rise to the name *trapezoid order* as an alternative to bi-interval order.

The most general poset considered for the third group is known as a bitolerance order. We say that $P=(X,\prec)$ is a *bitolerance order* if there exist $a,b,f_1,f_2: X\to\mathbb{R}$ such that:

1. For all $x\in X$, $a(x)\leqslant f_i(x)\leqslant b(x)$ for $i=1,2$, and
2. For all $x,y\in X$, $x\prec y$ if and only if $f_1(x)<a(y)$ and $b(x)<f_2(y)$.

Bogart and Trenk [5] prove that P is a bitolerance order if and only if it is a trapezoid order. Hence the bitolerance, trapezoid, and bi-interval designations denote the same thing.

We note five subclasses of bitolerance orders defined by restrictions on the preceding bitolerance representation. A functional equality such as $f_1+f_2=a+b$ in the following list means that the equality holds for all $x\in X$, e.g. $f_1(x)+f_2(x)=a(x)+b(x)$ for all $x\in X$. The five restricted classes are:

proper bitolerance order: $a(x)<a(y)\Leftrightarrow b(x)<b(y)$,
unit bitolerance order: $b=a+1$,
tolerance order: $f_1+f_2=a+b$,
unit tolerance order: $b=a+1$ and $f_1+f_2=a+b$,
50% tolerance order: $f_1=f_2$ and $f_1+f_2=a+b$.

As mentioned earlier, the classes of proper bitolerance orders and split interval orders are identical [4,12]. Both are also identical to the class of unit bitolerance orders [4], so the split interval, proper bitolerance and unit bitolerance designations are equivalent. In addition, Bogart et al. [3] prove equivalence between unit tolerance orders and 50% tolerance orders.

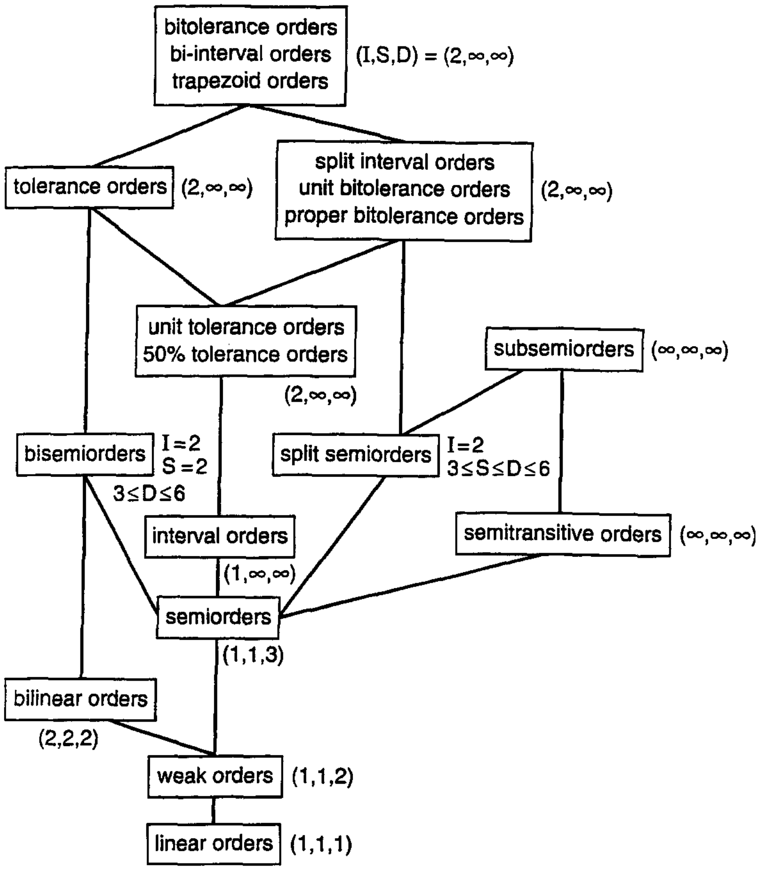


Fig. 1. Proper inclusion diagram for poset classes with equivalences and maximum dimensionalities.

Fig. 1 arranges our $\mathscr{P}$ classes by equivalences and proper inclusions. We note also by (I,S,D) the maximum values of Idim, Sdim and dim for each class. Thus, for class $\mathscr{C}$, $I = \max\{\mathrm{Idim}(P) : P \in \mathscr{C}\}$, $S = \max\{\mathrm{Sdim}(P) : P \in \mathscr{C}\}$, and $D = \max\{\dim(P) : P \in \mathscr{C}\}$, with ∞ denoting no upper bound. The only cases in which D is not precisely known occur for bisemiorders and split semiorders.

Theorem 2.1. *The subclasses of $\mathscr{P}$ defined by each row of Fig. 1 are identical within each box. The boxed classes are partially ordered by proper inclusion from bottom to top according to the lines in the diagram. Maximum dimensionalities are shown by the (I,S,D) triple next to each box, with $I = S = 2$ and $3 \leqslant D \leqslant 6$ for bisemiorders, and $I = 2$ and $3 \leqslant S \leqslant D \leqslant 6$ for split semiorders.*

Proof. All aspects of the theorem except for dimensionalities are established in [9] and in supporting references cited above. Verifications for (I,S,D) follow:

Linear orders: $(1,1,1)$. $D = 1$ by definition. $I = S = 1$ by $1 \leqslant I \leqslant S \leqslant D$.

Weak orders: $(1,1,2)$. $S = 1$ because every weak order is a semiorder. $D = 2$ by duality in $\sim$ classes [8, p. 77].

Bilinear orders: $(2,2,2)$. $D = 2$ by definition. $\mathrm{Idim}(\mathbf{2}+\mathbf{2}) = 2$.

Semiorders: $(1,1,3)$. $S = 1$ by definition. $D = 3$ by [14].

Bisemiorders: $I = S = 2$, $3 \leqslant D \leqslant 6$. $S = 2$ by definition. $I = 2$ because $I \leqslant S$ and $I = 2$ for bilinear orders. $3 \leqslant D \leqslant 6$ because $D = 3$ for semiorders.

Split semiorders: $I = 2$, $3 \leqslant S \leqslant D \leqslant 6$. $3 \leqslant S$ because some split semiorders are not bisemiorders [9]. $D \leqslant 6$ by [10]. Given representation (U,F) for split semiorder $P = (X,\prec)$, define $\prec_1$ and $\prec_2$ on X by $x \prec_1 y$ if $f(x) < ma(y)$, and $x \prec_2 y$ if $a(x) + 1 < f(y)$, so $\prec = (\prec_1 \cap \prec_2)$. It is easily seen that $\prec_1$ and $\prec_2$ are interval orders, so $\mathrm{Idim}(P) \leqslant 2$. Hence $I = 2$.

Interval orders: $(1,\infty,\infty)$. $I = 1$ by definition. Unboundedness of Sdim for interval orders is noted in Trotter and Bogart [19, p. 75].

Unit tolerance orders, tolerance orders, split interval orders, bitolerance orders: $(2,\infty,\infty)$. $I = 2$ because some unit tolerance orders such as $\mathbf{2}+\mathbf{2}$ are not interval orders and all bitolerance orders are bi-interval orders with $\mathrm{Idim} \leqslant 2$. $S = \infty$ by the preceding paragraph.

Semitransitive orders, subsemiorders: (∞,∞,∞). The standard bipartite $2n$-point example of Dushnik and Miller [6,8,18] has $\mathrm{Idim} = \mathrm{Sdim} = \dim = n$. $\square$

3. Distinguishing representations

In this section we prove that representations of split semiorders and split interval orders need never use the same real number for more than one end point or splitting point. We use this fact in our proof [10] that $\dim \leqslant 6$ for every split semiorder.

Let $a(X) = \{a(x) : x \in X\}$, and similarly for other functions. A representation (I,F) for a split interval order is *distinguishing* if $|a(X) \cup b(X) \cup f(X)| = 3|X|$. A representation (U,F) for a split semiorder is *distinguishing* if $|a(X) \cup \{a(x)+1 : x \in X\} \cup f(X)| = 3|X|$. Our proof for distinguishing representations uses the relation $\approx$ on X defined for poset $P = (X,\prec)$ by

$$x \approx y \quad \text{if, for all } z \in X,\ x \prec z \Leftrightarrow y \prec z \text{ and } z \prec x \Leftrightarrow z \prec y\ .$$

It is easily seen that $\approx$ is an equivalence relation. We denote by $X/\approx$ the family of equivalence classes in X determined by $\approx$.

Theorem 3.1. *Every split semiorder and split interval order has a distinguishing representation.*

Proof. We begin with the split semiorder case. Let $(Y,\prec)$ be a split semiorder. Define $\approx$ on Y as above, choose a representative from each class of $Y/\approx$, and denote by X the system of representatives. It suffices to prove for the split semiorder part of Theorem 3.1 that $(X,\prec)$ has a distinguishing split semiorder representation, for we can

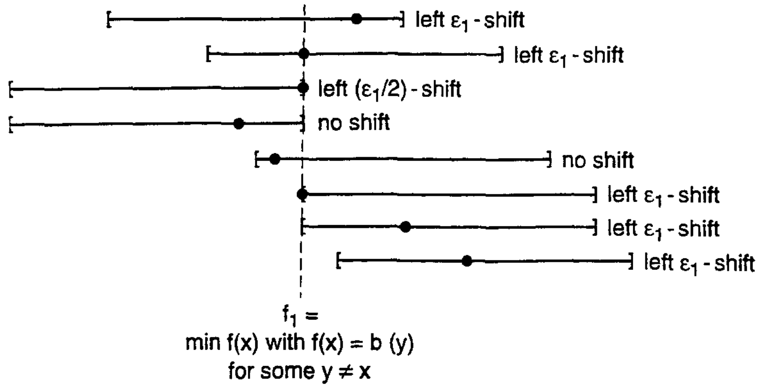


Fig. 2. Part of (U,F) prior to left shifts.

then define $\varepsilon : Y\backslash X \to \mathbb{R}$ and take $(a(y), f(y), a(y)+1) = (a(x)+\varepsilon(y), f(x)+\varepsilon(y), a(x)+1+\varepsilon(y))$ when $y \in Y\backslash X$, $x \in X$ and $y \approx x$, in such a way that $(Y,\prec)$ has a distinguishing representation. If $|X| = 1$, the desired result is obvious, so assume henceforth that $|X| > 1$.

Let (U,F) be a split semiorder representation for $(X,\prec)$, and for convenience denote $a(x)+1$ by $b(x)$. Also let $N = |X|$ and assume that $|a(X) \cup b(X) \cup f(X)| < 3N$, else there is nothing to prove. We need to undo equalities like $f(x) = b(y)$ and $f(x) = a(y)$ between splitting points and interval end points, and will translate intervals for this purpose. We refer to the translation of $(a(x), f(x), b(x))$ to $(a(x)-\varepsilon, f(x)-\varepsilon, b(x)-\varepsilon)$ as a *left ε-shift* of x, and to the translation of $(a(x), f(x), b(x))$ to $(a(x)+\varepsilon, f(x)+\varepsilon, b(x)+\varepsilon)$ as a *right ε-shift* of x.

Suppose there are distinct $x, y \in X$ with $f(x) = b(y)$. Let $X_1 = \{x \in X : f(x) = b(y)$ for some $y \neq x$ in $X\}$ with

$$f(X_1) = \{f_1 < f_2 < \cdots < f_K\}.$$

Let Δ_1 be the minimum distance between distinct points in $a(X) \cup b(X) \cup f(X)$, and fix ε_1 so that $0 < \varepsilon_1 < \Delta_1/N$. Let $Z = \{z \in X : f_1 \leqslant f(z)\}$. We modify (U,F) by a left ε_1-shift of z for every $z \in Z$, with one exception: if $f_1 = f(z) = b(z)$, this unique z undergoes a left $(\varepsilon_1/2)$-shift. Fig. 2 illustrates the procedure.

Because the shift magnitudes are much smaller than Δ_1, the left shifts for Z preserve all strict inequalities among the original points in $a(X) \cup b(X) \cup f(X)$. Hence every $\prec$ instance is preserved by the shifts. If $p \in X\backslash Z$, $q \in Z$ and $p \sim q$, then the left shift of q will not induce $q \prec p$ for the modified representation because $a(p) \leqslant f(p) < f_1 \leqslant f(q)$ and hence $a(p) < f(q) - \varepsilon_1$. Likewise, when the exceptional $f_1 = f(z) = b(z)$ occurs, the left $(\varepsilon_1/2)$-shift of z coupled with the left ε_1-shift of $w \in Z\backslash\{z\}$ for which $w \sim z$ will not induce $w \prec z$ because $a(z) - \varepsilon_1/2 < f(w) - \varepsilon_1$. Hence the modified (U,F) after the shifts produces the same $\prec$ on X as the original (U,F) according to the split semiorder representational definition. In the process, we undo all cases of $f(x) = b(y) = f_1$, shift f_2 through f_K leftward by ε_1, and have $f_2 - \varepsilon_1$ as the new minimum for $f(x) = b(y) = f_2 - \varepsilon_1$ in the modified representation unless no $b(y)$ in this case was shifted due to $f(y) < f_1$. In any event, if there remain distinct x and y in the modified (U,F) such that $f(x) = b(y)$, their common value must be in $\{f_2 - \varepsilon_1, \dots, f_K - \varepsilon_1\}$. If so, we take the smallest applicable value in this set and repeat the shift process with f_1 replaced by $f_k - \varepsilon_1$. The process continues in the obvious way, but fewer than N times, until all cases of $f(x) = b(y)$ have been changed into $f(x) < b(y)$ in the final modification. Points in X whose intervals are near the right end of the original representation can be left ε_1-shifted many times, but never more than $K < N$ times.

At this juncture we have a modified (U,F) in which there are no distinct x and y with $f(x) = b(y)$. Given the new (U,F), suppose there are distinct $x, y \in X$ with $f(x) = a(y)$. Let $X_2 = \{x \in X : f(x) = a(y)$ for some $y \neq x$ in $X\}$ with

$$f(X_2) = \{g_1 > g_2 > \dots > g_J\} .$$

Let Δ_2 be the minimum distance between distinct points in the new $a(X) \cup b(X) \cup f(X)$, and fix ε_2 so that $0 < \varepsilon_2 < \Delta_2/N$. Let $Z = \{z \in X : f(z) \leqslant g_1\}$ and modify the new (U,F) by a right ε_2-shift of z for every $z \in Z$, with one exception: if $g_1 = f(z) = a(z)$, this unique z undergoes a right $(\varepsilon_2/2)$-shift. The process continues in a symmetric manner to that for X_1: once g_1 has been resolved, we look for the largest of $\{g_2 + \varepsilon_2, \dots, g_J + \varepsilon_2\}$ at which $f(x) = a(y)$, resolve that case, and repeat the process until all instances of $f(x) = a(y)$ have been converted to $a(y) < f(x)$. The definition of ε_2 ensures that an instance of $f(x) = b(y)$ will not reappear.

We now have a split semiorder representation (U,F) in which neither $f(x) = a(y)$ nor $f(x) = b(y)$ occurs for *distinct* x and y. It could still happen that some x have $f(x) = a(x)$ or $f(x) = b(x)$, but in every such case there is no $a(y), f(y)$ or $b(y)$ with the same value for $y \neq x$. Hence we can increase $f(x)$ slightly for each $f(x) = a(x)$ case, decrease $f(x)$ slightly for each $f(x) = b(x)$ case, and not disturb $\prec$ while ensuring that there are no $x, y \in X$ for which $f(x) = a(y)$ or $f(x) = b(y)$. In the process, $a(x) < f(x) < b(x)$ for all x.

Next, if $f(x) = f(y)$ for distinct x and y, slight changes in f will undo these equalities without disturbing $\prec$ or the properties just noted. Finally, it could still be true that $a(x) = a(y)$, hence also $b(x) = b(y)$, for distinct x and y. Then slight changes in endpoints that preserve the unit-interval feature will yield a (U,F) representation with $|a(X) \cup b(X) \cup f(X)| = 3N$.

This completes our proof for split semiorders. The proof for split interval orders has a few differences. Assume that $(Y, \prec)$ is a split interval order with system of representatives X for $Y/\approx$. It suffices to show that $(X, \prec)$ has a distinguishing split interval order representation.

Let (I,F) be a representation for $(X, \prec)$ with $N = |X| > 1$, and assume that $|a(X) \cup b(X) \cup f(X)| < 3N$. Let $X_0 = \{x \in X\colon a(x) = b(x)\}$ for degenerate intervals. If X_0 is not empty, let Δ_0 be the minimum distance between distinct points in $a(X) \cup b(X) \cup f(X)$, fix ε_0 so that $0 < \varepsilon_0 < \Delta_0/2$, and replace $(a(x), f(x), b(x))$ for each $x \in X_0$ by

$(f(x)-\varepsilon_0, f(x), f(x)+\varepsilon_0)$. The representation thus modified is easily seen to a split interval order representation for $(X, \prec)$.

Assume henceforth that (I,F) has $a(x)<b(x)$ for all $x \in X$. Suppose there are distinct $x, y \in X$ with $f(x)=b(y)$. Define $X_1, f(X_1)$, Δ_1, ε_1 and Z as before. In modifying (I,F) on Z, we left ε_1-shift all $z \in Z$ except those for which $f_1=f(z)=b(z)$. There can be many such z with different $a(z)$ values. Each is left ε-shifted with a different ε value between 0 and ε_1. The resultant shifts fully preserve $\prec$ and give the next case for $f(x)=b(y)$, $x \neq y$, at $f_2-\varepsilon_1$ or to its right. The process continues until all instances in which a right endpoint of one interval equals the splitting point of another have been changed to inequalities.

We then continue as in the X_2 part of the split semiorder proof with the obvious change for multiple instances of $f(z)=a(z)$. The rest of the split semiorder proof applies without change except for the unit-interval aspect. If desired, the lengths of all intervals in the first (I,F) representation for which $a(x)<b(x)$ for all $x \in X$ can be preserved throughout the process. □

4. Forbidden posets

A poset Q is *minimally forbidden* for the class of split semiorders if Q is not a split semiorder and every proper induced subposet of Q is a split semiorder. A similar definition applies to the class of split interval orders. Our purpose in this section is to verify some of the simpler minimal forbidden posets for split semiorders and split interval orders. We consider crowns and $\mathbf{m}+\mathbf{n}$ posets, then conclude with a minimal forbidden poset for split semiorders that has eight points and height 3.

The following theorem is a consequence of Theorem 5.5 and the classification of 3-interval irreducible posets in [20]. We include a proof here that may be instructive.

Theorem 4.1. *Every C_n for $n \geqslant 3$ is minimally forbidden for split semiorders and split interval orders.*

Proof. Let $C_n=(\{x_1,\ldots,x_n\} \cup \{y_1,\ldots,y_n\}, \prec)$ with $\{x_i \prec y_i,\ x_i \prec y_{i+1}\}$ for $i=1,\ldots,n-1$ and $\{x_n \prec y_n, x_n \prec y_1\}$, $n \geqslant 3$. When y_1 is removed, what remains is a split semiorder: see Fig. 3, where all intervals have unit length. It follows from symmetry and duality that every proper induced subposet of C_n is a split semiorder, hence also a split interval order.

To show that C_n is not a split interval order, hence also not a split semiorder, assume to the contrary that (I,F) is a distinguishing split interval order representation of C_n. For definiteness let $a(y_2)=\min\{a(y_1), a(y_2), \ldots, a(y_n)\}$. We have

$$\begin{array}{ll} f(x_2)<a(y_2)<a(y_1) & \text{by } x_2 \prec y_2, \\ f(y_1)<b(x_2) & \text{by } x_2 \sim y_1 \text{ and } f(x_2)<a(y_1), \\ b(x_2)<f(y_3) & \text{by } x_2 \prec y_3, \end{array}$$

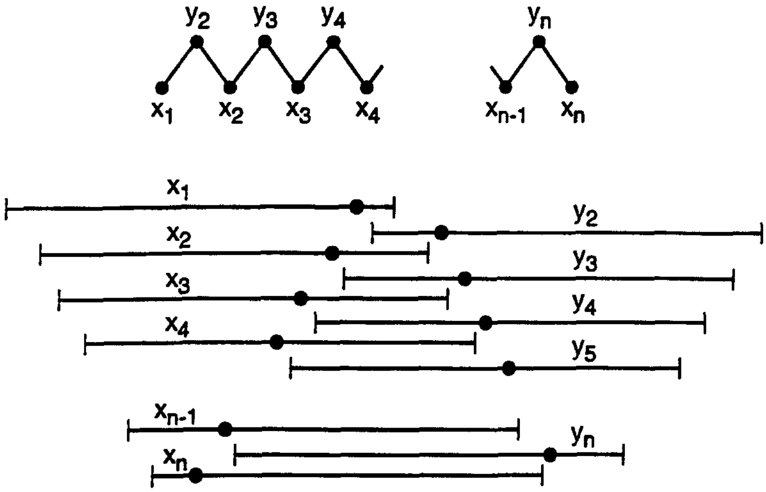


Fig. 3. A split semiorder.

$$\begin{array}{ll} b(x_1) < f(y_1) & \text{by } x_1 \prec y_1, \\ f(x_1) < a(y_2) < a(y_3) & \text{by } x_1 \prec y_2\ . \end{array}$$

The last line gives $f(x_1) < a(y_3)$ and the three before it give $b(x_1) < f(y_3)$, so we obtain $x_1 \prec y_3$, a contradiction to $x_1 \sim y_3$. □

As noted earlier, *all* bipartite posets that are minimally forbidden for split semiorders and split interval orders are described in [18,20].

Theorem 4.2. $\mathbf{m}+\mathbf{n}$ *is minimally forbidden for split semiorders if and only if it is* $\mathbf{2}+\mathbf{3}$ *or* $\mathbf{1}+\mathbf{4}$; $\mathbf{m}+\mathbf{n}$ *is minimally forbidden for split interval orders if and only if it is* $\mathbf{3}+\mathbf{3}$.

Proof. It is easily seen that every $\mathbf{m}+\mathbf{n}$ with $m+n \leqslant 4$ is a split semiorder, and that every $\mathbf{2}+\mathbf{n}$ is a split interval order. The proof is completed by showing that $\mathbf{2}+\mathbf{3}$ and $\mathbf{1}+\mathbf{4}$ are not split semiorders, and $\mathbf{3}+\mathbf{3}$ is not a split interval order. Proposition 4.1 in [5] proves the $\mathbf{3}+\mathbf{3}$ case. For the others set $X = \{x,y,z,p,q\}$, $|X| = 5$.

Suppose $p \prec q$ and $x \prec y \prec z$ along with $p \sim z$. We prove that if $(X,\prec)$ is a split semiorder then $x \prec q$. Consequently, $\mathbf{2}+\mathbf{3}$ is not a split semiorder. Our hypotheses imply for a split semiorder representation (U,F) that

$$\begin{array}{l} f(x) < a(y) \text{ and } a(x)+1 < f(y), \\ f(y) < a(z) \text{ and } a(y)+1 < f(z), \\ f(p) < a(q) \text{ and } a(p)+1 < f(q) \end{array}$$

and either $a(z) \leqslant f(p)$ or $f(z) \leqslant a(p)+1$, for not $(p \prec z)$. Suppose $a(z) \leqslant f(p)$. Then

$$f(x) < a(y) \leqslant f(y) < a(z) \leqslant f(p) < a(q) \leqslant f(q),$$

which in conjunction with $a(x)+1 < f(y)$ implies $f(x) < a(q)$ and $a(x)+1 < f(q)$, so $x \prec q$. Suppose $f(z) \leqslant a(p)+1$. Then

$$a(x)+1 \leqslant f(x)+1 < a(y)+1 < f(z) \leqslant a(p)+1 \leqslant f(p)+1 < a(q)+1 \ ,$$

which in conjunction with $a(p)+1 < f(q)$ implies $f(x) < a(q)$ and $a(x)+1 < f(q)$, so again we have $x \prec q$.

Suppose $x \prec y \prec z \prec p$ and $q \sim p$. We prove that if $(X, \prec)$ is a split semiorder then $x \prec q$. Consequently, $\mathbf{1}+\mathbf{4}$ is not a split semiorder. Our hypotheses imply for a split semiorder representation that

$$\begin{aligned}
&f(x) < a(y) \text{ and } a(x)+1 < f(y),\\
&f(y) < a(z) \text{ and } a(y)+1 < f(z),\\
&f(z) < a(p) \text{ and } a(z)+1 < f(p)
\end{aligned}$$

and either $a(p) \leqslant f(q)$ or $f(p) \leqslant a(q)+1$, for not $(q \prec p)$. Suppose $a(p) \leqslant f(q)$. Then $f(x) < a(q)$ because $f(x) < a(y) < f(z)-1 < a(p)-1 \leqslant f(q)-1 \leqslant a(q)$, and $a(x)+1 < f(q)$ because $a(x)+1 \leqslant f(x)+1 < a(y)+1 < f(z) < a(p) \leqslant f(q)$, so $x \prec q$. Or, if $f(p) \leqslant a(q)+1$, then $f(x) < a(q)$ by $f(x) < a(y) < f(z)-1 < a(p)-1 \leqslant f(p)-1 \leqslant a(q)$, and $a(x)+1 < f(q)$ by $a(x)+1 < f(y) < a(z) < f(p)-1 \leqslant a(q) \leqslant f(q)$, so again we get $x \prec q$. $\square$

We conclude this section with a height-3 poset that is minimally forbidden for split semiorders and does not include $\mathbf{2}+\mathbf{3}$.

Theorem 4.3. *The poset of Fig. 4 is minimally forbidden for split semiorders.*

Proof. Let P denote the eight-point poset of Fig. 4. There are three nonisomorphic seven-point induced subposets of P obtained by deleting x or 1 or 4. Each of the three is a split semiorder, so every proper induced subposet of P is a split semiorder.

Suppose that P itself is a split semiorder with distinguishing representation (U,F): see Theorem 3.1. Assume without loss of generality that $f(6) < f(5) < f(4)$, as in the middle of Fig. 4. We have $a(x)+1 < f(y)$ by $x \prec y$, $f(y) < a(4)$ by $y \prec 4$, and $a(4) < f(2)$ because $2 \prec 5$ gives $a(2)+1 < f(5) < f(4)$, hence $a(2)+1 < f(4)$, so $2 \sim 4$ requires $a(4) < f(2)$. Therefore $a(x)+1 < f(2)$. Because $x \sim 2$, we have $a(2) < f(x)$, and $x \prec y$ implies $f(x) < a(y)$, so $a(2) < a(y)$ and $a(2)+1 < a(y)+1$. Next, $y \prec 6$ implies $a(y)+1 < f(6)$, so $a(2)+1 < f(6)$, and since $2 \sim 6$ we have $a(6) < f(2)$. By $3 \prec 6$, $a(3)+1 < f(6) < f(5)$, so $3 \sim 5$ requires $a(5) < f(3)$. Also, by $2 \prec 5$ and $3 \prec 6$, $f(2) < a(5)$ and $f(3) < a(6)$. Hence $f(2) < a(5) < f(3) < a(6) < f(2)$, a contradiction. We conclude that P is not a split semiorder. $\square$

5. Bipartite orders

This section describes the reconfiguration of classes on Fig. 1 for the bipartite domain $\mathscr{P}_2$. For $P = (X, \prec)$ in $\mathscr{P}_2$, we let $B(P) = \{x \in X : x \prec y \text{ for some } y \in X\}$ and $T(P) =$

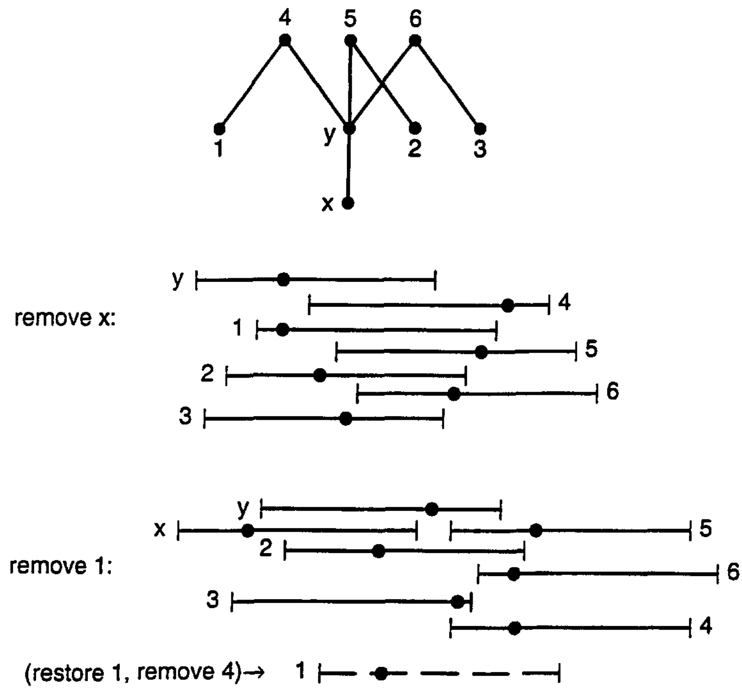
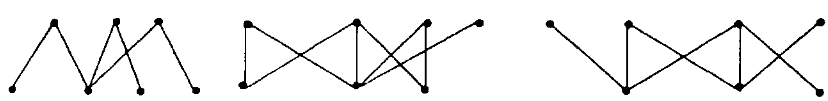


Fig. 4. Split semiorder representations for proper induced subposets.

$\{y \in X : x \prec y$ for some $x \in X\}$. Four lemmas will facilitate the reconfiguration. The first is proved in Trotter [17].

Lemma 5.1. *If* $P \in \mathscr{P}_2$ *then* $\dim(P) \leqslant 1 + \mathrm{Idim}(P)$.

Lemma 5.2. *The only* $P \in \mathscr{P}_2$ *of dimension* 3 *and no more than* 7 *points for which* $\mathrm{Idim}(P) < 3$ *are*



and their duals.

Proof. The characterization of irreducible three-dimensional posets in Trotter and Moore [20], Kelly [11], or Trotter [18] shows that if P has height 2, dimension 3, and 7 points, then either it is one of the three posets in Lemma 5.2, the dual of one of those, or has crown C_3 as an induced poset. The lemma's posets have interval dimension 2, whereas $\mathrm{Idim}(C_3) = 3$. $\square$

The next lemma abbreviates Theorem 2.12 in Bogart and Trenk [5]. Its initial equivalence is central to our reconfiguration. Part 3 of the lemma offers a method of testing for membership in the class.

Lemma 5.3. *Suppose $P \in \mathscr{P}_2$. Then the following are equivalent:*
1. *P is a bitolerance order,*
2. *P is a unit tolerance order,*
3. *If $|B(P)| = m > 0$ and $|T(P)| = n > 0$, then $B(P)$ can be indexed as $x_1, x_2, \dots, x_m$ and $T(P)$ as $y_1, y_2, \dots, y_n$ so that, whenever $x_i \sim y_j$, either $x_i \sim y_k$ for all $k \leqslant j$, or $x_\ell \sim y_j$ for all $\ell \leqslant i$.*

Fig. 1 and the equivalence of unit bitolerance orders and split interval orders [4] show that tolerance orders, unit bitolerance orders, and split interval orders are identical to each other and to the bitolerance orders or unit tolerance orders in the bipartite domain. Moreover, because the general bitolerance class is tantamount to all $P \in \mathscr{P}$ with $\mathrm{Idim}(P) \leqslant 2$, the resultant class in $\mathscr{P}_2$ consists of all bipartite posets with interval dimension at most 2. Lemma 5.1 shows that $\dim(P) \leqslant 3$ for all P in this bipartite class.

Lemma 5.4. *If $P \in \mathscr{P}_2$ is a split interval order then it is a split semiorder.*

Proof. Suppose R is a split interval representation of split interval order $P = (X, \prec)$ in $\mathscr{P}_2$. It is easily seen that the representation's implications for $\prec$ are unchanged when the left endpoints of intervals for $B(P)$ are extended leftward, the right endpoints of intervals for $T(P)$ are extended rightward, and endpoints for isolated points are extended both ways. It follows that R can be modified into R' in which every interval has the same length. □

Our final lemma brings bisemiorders into the picture.

Lemma 5.5. *If $P \in \mathscr{P}_2$ is a split interval order then it is a bisemiorder.*

Proof. Let R with functions a, b and f be a split interval order representation of split interval order $P = (X, \prec)$ in $\mathscr{P}_2$, and assume that $\prec$ is not empty. Let $I(P) = X \setminus [B(P) \cup T(P)]$, the set of isolated points in X. Define $\prec_1$ on X by

$$x \prec_1 y \text{ if } x \in I(P) \quad \text{and} \quad y \in B(P) \cup T(P)\ ,$$

$$x \prec_1 y \text{ if } x \in B(P), y \in T(P) \quad \text{and} \quad f(x) < a(y)\ ;$$

define $\prec_2$ on X by

$$x \prec_2 y \text{ if } x \in B(P) \cup T(P) \quad \text{and} \quad y \in I(P)\ ,$$

$$x \prec_2 y \text{ if } x \in B(P), y \in T(P) \quad \text{and} \quad b(x) < f(y)\ .$$

It follows that $x \prec y$ if and only if $x \prec_1 y$ and $x \prec_2 y$. In $\prec_1$, all isolated points are below all others; in $\prec_2$, all isolated points are above all others. When we delete $I(P)$, the remainders of $\prec_1$ and $\prec_2$ are height-2 interval orders and are therefore semiorders. It follows that $\prec_1$ and $\prec_2$ are semiorders, hence that P is a bisemiorder. □

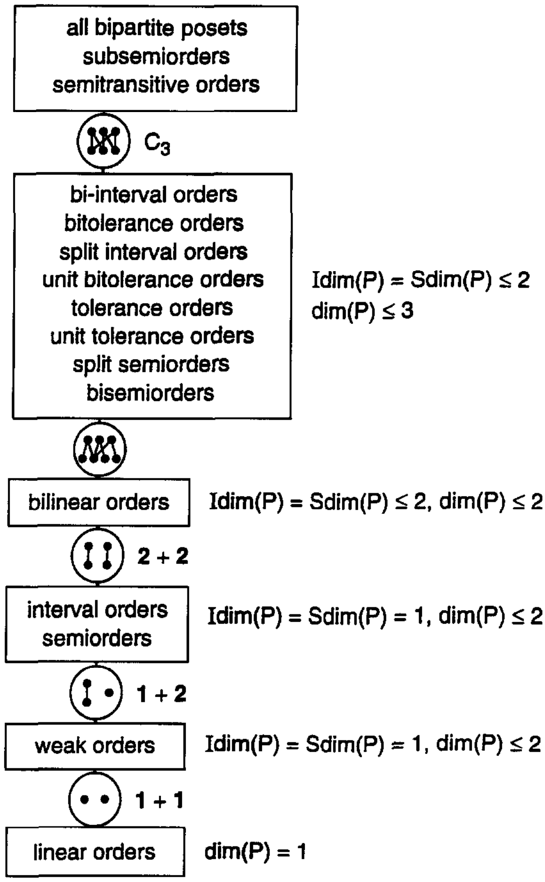


Fig. 5. Bipartite reconfiguration of Fig. 1.

Fig. 5 describes our bipartite reconfiguration of Fig. 1. Each box represents one subclass of $\mathscr{P}_2$. Orders in the top box can have arbitrarily large dimension, and the next three boxes below it are characterized by $\mathrm{Idim}(P)\leqslant 2$, $\dim(P)\leqslant 2$ and $\mathrm{Idim}(P)=1$, respectively.

Theorem 5.6. *The subclasses of $\mathscr{P}_2$ defined by each row of Fig. 5 are identical within each box. The boxed classes are linearly ordered by proper inclusion from bottom to top. The poset shown between adjacent boxes is the uniquely smallest bipartite poset that is contained in the class of the upper box and is not in the class of the lower box, except for the seven-point poset which has the companions of Lemma 5.2 that are split semiorders and not bilinear orders.*

Proof. We comment on within-box equivalences and then note the proper inclusions.

Forbidden $\mathbf{1}+\mathbf{3}$ for semiorders does not occur when $H(P)\leqslant 2$, so the bipartite semiorder and interval order classes are identical. The classes of subsemiorders and semitransitive orders in $\mathscr{P}$ include all bipartite posets, so the three classes in the top box are mutually identical. Lemmas 5.3 and 5.4 along with the comments preceding the latter show that all bipartite classes, except perhaps that of bisemiorders, in the second box down are mutually identical. The bisemiorder class is included in the tolerance order class by Fig. 1, and the tolerance order class is included in the bisemiorder class by Lemma 5.5.

The inclusion of each boxed class on Fig. 5 in the next higher boxed class follows from Fig. 1, Lemma 5.1, the observations on dimensions in the paragraph that precedes Theorem 5.6, and Idim $\leqslant$ dim. The specific posets circled in Fig. 5 show that the inclusions are proper. The minimal cardinality of each poset is obvious for $\mathbf{1}+\mathbf{1}, \mathbf{1}+\mathbf{2}$ and $\mathbf{2}+\mathbf{2}$, and is easily checked for C_3, which is the uniquely smallest poset of height 2 and dimension 3 that is not a split semiorder. Indeed the only other posets of dimension 3 with fewer than seven points are the height-3 chevron and its dual. Lemma 5.2 identifies the minimum-cardinality bipartite posets that are split semiorders and not bilinear orders. $\square$

6. Discussion

Our purpose has been to provide a comprehensive introduction to split semiorders and split interval orders based on prior work in [9,10] and other contributions, especially [5,12,18,20]. Aspects of split orders were considered through their representations, dimensionalities, minimal forbidden posets, and inclusion relationships to other poset classes for the general case and the restricted bipartite case.

Several questions remain open. One is characterizations of split semiorders and split interval orders by minimal forbidden posets. We know the part of this characterization for forbidden height-2 posets, but not for greater heights. Other questions involve dimensionalities. A central problem, emphasized in [10], is to determine the maximum values of Sdim(P) and dim(P) for split semiorders. We wonder whether any split semiorder P has dim(P) − Sdim$(P) \geqslant 3$ or Sdim(P) − Idim$(P) \geqslant 2$.

Another set of questions focuses on representational properties and restrictions. An example is whether there is an interesting characterization of split semiorders that have (U,F) representations in which all splitting points lie in a central range of their intervals' midpoints such as $|f(x) - a(x) - \frac{1}{2}| \leqslant \lambda$ for fixed λ in $[0, 1/2)$.

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