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A Generalization of Hiraguchi's: Inequality for Posets

WILLIAM T. TROTTER, JR.

Department of Mathematics and Computer Science, University of South Carolina, Columbia, South Carolina 29208

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For a poset X, $\mathrm{Dim}(X)$ is the smallest positive integer t for which X is isomorphic to a subposet of the cartesian product of t chains. Hiraguchi proved that if $|X| \geqslant 4$, then $\mathrm{Dim}(X) \leqslant [|X|/2]$. For each $k \leqslant 2$, we define $\mathrm{Dim}_k(X)$ as the smallest positive integer t for which X is isomorphic to a subposet of the cartesian product of t chains, each of length k. We then prove that if $|X| \geqslant 5$, $\mathrm{Dim}_3(X) \leqslant \{|X|/2\}$ and if $|X| \geqslant 6$, then $\mathrm{Dim}_4(X) \leqslant [|X|/2]$.

1. INTRODUCTION

A partially ordered set or poset is a set X equipped with a reflexive, antisymmetric, and transitive relation $\leqslant$. Dushnik and Miller [3] defined the dimension of a poset X, denoted $\mathrm{Dim}(X)$, as the smallest positive integer t for which there exist linear extensions $L_1, L_2, \ldots, L_t$ of X such that $x \leqslant y$ in X iff $x \leqslant y$ in each L_i. Equivalently, Ore [7] defined $\mathrm{Dim}(X)$ as the smallest positive integer t for which X is isomorphic to a subposet of a cartesian product $C_1 \times C_2 \times \cdots \times C_t$, where each C_i is a chain. For each $k \geqslant 2$, we define $\mathrm{Dim}_k(X)$ as the smallest positive integer t for which X is isomorphic to a subposet of a cartesian product $C_1 \times C_2 \times \cdots \times C_t$ where each C_i is a chain and $|C_i| = k$. For a real number x, we let $[x]$ denote the largest integer among those which are less than or equal to x; similarly, $\{x\}$ denotes the smallest integer among those which are greater than or equal to x.

Hiraguchi [4] proved that if $|X| \geqslant 4$, then $\mathrm{Dim}(X) \leqslant [|X|/2]$. In [9], the author proved that $\mathrm{Dim}_2(X) \leqslant |X|$ for all X. In this paper, we show that if $|X| \geqslant 5$, then $\mathrm{Dim}_3(X) \leqslant \{|X|/2\}$, and if $|X| \geqslant 6$, then $\mathrm{Dim}_4(X) \leqslant [|X|/2]$. We establish the first inequality by an argument based on the graph theoretic concept of a matching; the second inequality will be proved by applying a sequence of removal theorems.

We refer the reader to [1, 2, 8, 10] for additional material on the dimension theory for posets. We also refer the reader to [9] for special results on $\mathrm{Dim}_2(X)$ and to [11] where a formula for $\mathrm{Dim}_k(X)$ is given when X is a distributive lattice.

2. Preliminary Development

We denote an n element chain by $\underset{\sim}{n}$ and an n element antichain by $\bar{n}$. We will find it convenient to use the labeling $0 < 1 < 2 < \cdots < n - 1$ for $\underset{\sim}{n}$. We also denote the cartesian product of n copies of a poset X by X^n; with this notation $\mathrm{Dim}_k(X)$ is the smallest positive integer t for which X is isomorphic to a subposet of $\underset{\sim}{k}^t$. A map $F\colon X \to Y$ between posets X and Y is called an embedding when $x_1 \leqslant x_2$ in X iff $F(x_1) \leqslant F(x_2)$ in Y. An embedding $F\colon X \to \underset{\sim}{k}^t$ assigns to each $x \in X$ a sequence $F(x)(1)$, $F(x)(2),\ldots, F(x)(t)$ of numbers from $\underset{\sim}{k}$ with $x \leqslant y$ in X iff $F(x)(i) \leqslant F(y)(i)$ for all $i \leqslant t$.

For a poset X, the dual of X, denoted $\hat{X}$ is the poset defined by $x \leqslant y$ in $\hat{X}$ iff $y \leqslant x$ in X. It is clear that $\mathrm{Dim}_k(X) = \mathrm{Dim}_k(\hat{X})$ for all $k \geqslant 2$ and we frequently employ this observation to shorten arguments appearing in this paper.

The free sum of posets X and Y is denoted $X + Y$; the poset obtained from $X + Y$ by adding all comparabilities of the form $x < y$, where $x \in X$ and $y \in Y$ is called the lexicographic sum of X and Y, is denoted $X \oplus Y$.

Since the length of the longest chain in $\underset{\sim}{k}^t$ is $t(k - 1) + 1$, it follows that $\mathrm{Dim}_k(\underset{\sim}{n}) = \{(n - 1)/(k - 1)\}$. Although there is no simple formula for $\mathrm{Dim}_k(\bar{n})$, we note that the computation can be made for specific values of k and n using the generalizations of Sperner's theorem compiled by Katona [5]. In particular, $\mathrm{Dim}_3(\bar{2}) = \mathrm{Dim}_3(\bar{3}) = \mathrm{Dim}_4(\bar{4}) = 2$ and $\mathrm{Dim}_3(\bar{4}) = \mathrm{Dim}_4(\bar{5}) = 3$. Furthermore, it is easy to establish the following inequalities. $\mathrm{Dim}_3(\bar{n}) \leqslant \{(n + 1)/2\}$ for all n and $\mathrm{Dim}_3(\bar{n}) \leqslant [n/2]$ when $n \geqslant 6$. These inequalities are quite generous for large values of n.

The width of a poset X, denoted $W(X)$, is the number of points in a maximum antichain in X. Hiraguchi [4] proved that $\mathrm{Dim}(X) \leqslant W(X)$. The analogous result for $\mathrm{Dim}_k(X)$ is:

Theorem 1. *Let $k \geqslant 2$ and $X = C_1 \cup C_2 \cup \cdots \cup C_n$ be a set decomposition of X into chains, where $|C_i| \leqslant k - 1$ for each $i \leqslant n$. Then* $\mathrm{Dim}_k(X) \leqslant n$.

Proof. For each $i \leqslant n$, let $C_i = \{x_{i0} < x_{i1} < x_{i2} < \cdots < x_{im_i}\}$, where $m_i \leqslant k - 2$. Now for each $i \leqslant n$, we define a function $f_i\colon X \to \underline{k}$ as follows. Let $x \in X$; if x is not less than or equal to any element of C_i, then $f_i(x) = k - 1$. If the least element of C_i which is greater than or equal to x is x_{ij}, define $f_i(x) = j$. We will call f_i an upper extension of C_i in $\underline{k}$ (lower extensions are defined similarly).

The function $F\colon X \to \underline{k}^n$ defined by $F(x)(i) = f_i(x)$ is easily seen to be an embedding of X in $\underline{k}^n$ and we conclude that $\mathrm{Dim}_k(X) \leqslant n$.

3. An Application of Matching Theory to Posets

In this section, we use the graph theoretic concept of a matching for the comparability graph of a poset to obtain the inequality $\mathrm{Dim}_3(X) \leqslant \{(|X| + 1)/2\}$ for all X.

For a poset X, a matching $\mathscr{M}$ is a collection of pairwise disjoint two-element chains from X. If $\bigcup \mathscr{M} = X$, then $\mathscr{M}$ is called a perfect matching; a matching $\mathscr{M}$ is called a maximum matching if $|\mathscr{M}|$ is maximum among all the matchings for X. We note that if a poset X has a perfect matching, then it follows from Theorem 1 that $\mathrm{Dim}_4(X) \leqslant \mathrm{Dim}_3(X) \leqslant [|X|/2]$. Thus, we will be concerned primarily with posets which do not have perfect matchings.

If $\mathscr{M}$ is a maximum matching for a poset X, we let $A_{\mathscr{M}} = X - \bigcup \mathscr{M}$. If $A_{\mathscr{M}} \neq \varnothing$, then it is an antichain. Among the maximum matchings for X, we wish to identify those for which $A_{\mathscr{M}}$ is as "low as possible" in X. We begin by saying that all perfect matchings satisfy property P. Then we say that a nonperfect maximum matching $\mathscr{M}$ satisfies property P if there does not exist a maximum matching $\mathscr{M}'$ such that $A_{\mathscr{M}} - A_{\mathscr{M}'} = \{a\}$, $A_{\mathscr{M}'} - A_{\mathscr{M}} = \{a'\}$ and $a' < a$. Clearly every poset has maximum matching satisfying property P.

Theorem 2. *$\mathrm{Dim}_3(X) \leqslant \{(|X| + 1)/2\}$ for all X.*

Proof. Let $\mathscr{M}$ be a maximum matching for X which satisfies property P. If $\mathscr{M}$ is perfect, our conclusion follows; we assume then that $\mathscr{M}$ is not perfect.

Let $L_0(\mathscr{M}) = \{C \in \mathscr{M}\colon$ There exists $x \in C$ and $a \in A_{\mathscr{M}}$ such that $x < a\}$. If $L_k(\mathscr{M})$ has been defined, we then define $L_{k+1}(\mathscr{M}) = L_k(\mathscr{M}) \cup \{C \in \mathscr{M}\colon$ There exist $D \in L_k(\mathscr{M})$, $y \in C$ and $x \in D$ such that $x > y\}$. Then let $L(\mathscr{M}) = \bigcup \{L_n(\mathscr{M})\colon n \geqslant 0\}$ and $U(\mathscr{M}) = \mathscr{M} - L(\mathscr{M})$. Next we define subsets U and L of X by $U = \bigcup U(\mathscr{M})$ and $L = \bigcup L(\mathscr{M})$.

We note that $X = U \cup A \cup L$ is a partition. We now prove that this partition satisfies the following three properties:

(i) $x \in U$ and $a \in A$ imply $x \not< a$.
(ii) $x \in U$ and $y \in L$ imply $x \not< y$.
(iii) $x \in A$ and $y \in L$ imply $a \not< y$.

We note that statements (i) and (ii) follow from the definitions given above. Now suppose that $y \in L$, $a \in A$, and $y > a$. Then it follows that there exist an integer $n \geqslant 1$, a collection $\mathscr{C} = \{C_i = \{x_i > y_i\}: 1 \leqslant i \leqslant n\}$ of chains from $L(\mathscr{M})$, and an element $a' \in A_{\mathscr{M}}$ such that $y_1 < a'$, $x_i > y_{i+1}$ for $1 \leqslant i \leqslant n - 1$, and $x_n > a$. If a and a' are distinct, then

$$\mathscr{M}' = \mathscr{M} - \mathscr{C} \cup \{\{a' > y_1\}, \{x_1 > y_2\}, \{x_2 > y_3\}, \ldots, \{x_{n-1} > y_n\}, \{x_n > a\}\}$$

is a matching and $|\mathscr{M}'| = |\mathscr{M}| + 1$. We then conclude that $a = a'$. In this case, the matching $\mathscr{M}'' = \mathscr{M} - \mathscr{C} \cup \{\{x_1 > y_2\}, \{x_2 > y_3\}, \ldots, \{x_{n-1} > y_n\}, \{x_n > a\}\}$ is also a maximum matching for X with $A_{\mathscr{M}''} - A_{\mathscr{M}} = \{y_1\}$, $A_{\mathscr{M}} - A_{\mathscr{M}''} = \{a\}$, and $y_1 < a$. The contradiction completes the proof of statement iii.

Let $U(\mathscr{M}) = \{C_1, C_2, \ldots, C_s\}$ and $L(\mathscr{M}) = \{C_{s+1}, C_{s+2}, \ldots, C_{s+t}\}$. For each $i \leqslant s$, let f_i be an upper extension of C_i in $\underline{3}$; for each $i \leqslant t$, let f_{s+i} be a lower extension of C_{s+i} in $\underline{3}$.

Now let $|A_{\mathscr{M}}| = n$. It follows that there is an embedding F of $A_{\mathscr{M}}$ in $\underline{3}^q$ where $q = \{(n+1)/2\}$. We now define a mapping $G: X \to \underline{k}^{s+t+q}$ by $G(x)(i) = f_i(x)$ if $1 \leqslant i \leqslant s$, $G(x)(s+i) = f_{s+i}(x)$ if $1 \leqslant i \leqslant t$, $G(x)(s+t+i) = 2$ if $x \in U$ and $1 \leqslant i \leqslant q$, $G(x)(s+t+i) = 0$ if $x \in L$ and $1 \leqslant i \leqslant q$, $G(x)(s+t+i) = F(x)(i)$ if $x \in A_{\mathscr{M}}$ and $1 \leqslant i \leqslant q$.

It is straightforward to verify that G is an embedding of X in $\underline{3}^{s+t+q}$ and since $s + t + q = \{(|X| + 1)/2\}$, the proof of our theorem is complete.

Since the maximum length of a chain in $\underline{k}^t$ which does not contain either of the universal bounds is $t(k-1) - 1$, it follows that $\text{Dim}_k(\underline{t}(\underline{k} - \underline{1}) + \underline{1}) = t + 1$ and thus $\text{Dim}_3(\underline{2n} + \underline{1}) = n + 1$ for all $n \geqslant 1$. We conclude that the inequality given in Theorem 2 is best possible when $|X|$ is odd.

4. An Improved Bound for $\text{Dim}_3(X)$.

In this section, we develop some removal theorems which will allow us to improve the bound for $\text{Dim}_3(X)$ given in Theorem 2 when $|X|$ is even. We begin with the following statements.

Fact 1: If $|X| = 4$, then $\text{Dim}_3(X) = 2$ unless $X = \bar{\underline{4}}$ or $X = \underline{2} + \bar{\underline{2}}$, in which case $\text{Dim}_3(X) = 3$.

Fact 2: If $|X| = 6$, then $\text{Dim}_3(X) \leqslant 3$.

Fact 1 may be verified by examining the Hasse diagram for $\underline{3}^2$ to find the fourteen posets in question as subposets. It is not so trivial to verify Fact 2; although we do not include the details here, an argument can be obtained from the removal theorems developed in this paper.

If x and y are distinct points in a posets X, $x \not\leqslant y$, and $y \not\leqslant x$, then we say x and y are incomparable and write xIy.

LEMMA L. *If a is maximal element of X, b is a minimal element of X, aIb, and $X - \{a, b\}$ has at least two maximal elements and at least two minimal elements, then* $\mathrm{Dim}_k(X) \leqslant 1 + \mathrm{Dim}_k(X - \{a, b\})$ *for every* $k \geqslant 3$.

Proof. Let $F: X - \{a, b\} \to \underline{k}^t$ be an embedding. We define an embedding $G: X \to \underline{k}^{t+1}$ by $G(x)(i) = F(x)(i)$ for every $x \in X - \{a, b\}$ and every $i \leqslant t$, $G(a)(i) = k - 1$ for every $i \leqslant t$, $G(b)(i) = 0$ for every $i \leqslant t$, $G(x)(t + 1) = 0$ if $x \leqslant a$, $G(x)(t + 1) = 2$ if $x \geqslant b$, and $G(x)(t + 1) = 1$ if $x \not\leqslant a$ and $x \not\geqslant b$.

If $a > b$ in a poset X but there does not exist a point $c \in X$ for which $a > c > b$, we say a covers b.

LEMMA 2. *If a is a maximal element of X, a covers b, a is the only maximal element which is greater than b, and $X - \{a, b\}$ has at least two maximal elements, then* $\mathrm{Dim}_k X \leqslant 1 + \mathrm{Dim}_k(X - \{a, b\})$ *for every* $k \geqslant 3$.

Proof. Let $F: X - \{a, b\} \to \underline{k}^t$ be an embedding and let f be an upper extension of the chain $a > b$ in $\underline{k}$. We define an embedding $G: X \to \underline{k}^{t+1}$ by $G(x)(i) = F(x)(i)$ for every $x \in X$ and every $i \leqslant t$, $G(a)(i) = G(b)(i) = k - 1$ for every $i \leqslant t$, and $G(x)(t + 1) = f(x)$ for every $x \in X$.

Distinct points x, y are said to have the same holdings in X if for every $z \in X - \{x, y\}$, $z > x$ iff $z > y$ and $z < x$ iff $z < y$.

THEOREM 3. *If* $|X| \geqslant 6$, *then* $\mathrm{Dim}_3(X) \leqslant \{|X|/2\}$.

Proof. We show that $n \geqslant 3$ and $|X| = 2n$, then $\mathrm{Dim}_3(X) \leqslant n$; we assume validity for $n \leqslant m$ where $m \geqslant 3$ and then suppose X is a poset with $|X| = 2m + 2$ and $\mathrm{Dim}_3(X) > m + 1$.

Let $\mathscr{M}$ be a maximum matching which satisfies Property P. If $\mathscr{M}$ is a perfect matching, then $\mathrm{Dim}_3(X) \leqslant |\mathscr{M}| = m + 1$.

If $U(\mathscr{M}) \neq \phi \neq L(\mathscr{M})$, then it follows that at least one of the posets, $U \cup A$ and $A \cup L$ has at least six points. Suppose $|U \cup A| \geqslant 6$; then $\mathrm{Dim}_3(U \cup A) \leqslant |U \cup A|/2 = s$. We choose an embedding F of $U \cup A$ in $\underline{3}^s$ and extend F to X by defining $F(x)(i) = 0$ for every $x \in L$ and for every $i \leqslant s$. Now let $L(\mathscr{M}) = \{D_1, D_2, \ldots, D_t\}$ where $t \geqslant 1$; then for each $i \leqslant t$, let f_i be a lower extension of D_i in $\underline{3}$.

Finally we define $G: X \to \underline{3}^{s+t}$ by $G(x)(i) = F(x)(i)$ for every $x \in X$

and for every $i \leqslant s$, $F(x)(s+i) = f_i(x)$ for every $x \in X$ and for every $i \leqslant t$. It is straightforward to verify that G is an embedding and thus $\mathrm{Dim}_3(X) \leqslant s + t = m + 1$. The argument when $| L \cup A | \geqslant 6$ is dual.

Thus we may assume without loss of generality that $L(\mathscr{M}) = \phi$, i.e. the elements of $A_{\mathscr{M}}$ are minimal elements of X. We also note that if $| A_{\mathscr{M}} | \geqslant 6$, then the construction used in the proof of Theorem 2 shows that $\mathrm{Dim}_3(X) \leqslant m + 1$. Now suppose $| A_{\mathscr{M}} | = 4$.

Suppose there exists a distinct pair $a, a' \in A_{\mathscr{M}}$ which do not have the same holdings in X. We first assume that there exists $x \in U$ such that $x > a_1, x > a_2$, and $x > a_3$, but xIa_4. Now let $U(\mathscr{M}) = \{C_1, C_2, ..., C_s\}$ and for each $i \leqslant s$, let f_i be an upper extension of C_i in $\underline{3}$. Now define an embedding $F; X \to \underline{3}^{s+2}$ by $F(x)(i) = f_i(x)$ for every $x \in X$ and every $i \leqslant s$, $F(x)(s+1) = F(x)(s+2) = 2$ for every $x \in U$, $F(a_4)(s+2) = 0$, $F(a_1)(s+1) = F(a_3)(s+2) = 2$, $F(a_3)(s+1) = F(a_1)(s+2) = 0$, and $F(a_2)(s+1) = F(a_2)(s+2) = 1$.

Now suppose there exists $u \in U$ such that $x > a$, and $x > a_2$ but xIa_3 and xIa_4. Then modify the function F defined in the preceding paragraph as follows: $F(a_3)(s+1) = F(a_2)(s+1) = 1$, $F(a_1)(s+1) = F(a_4)(s+1) = 0, F(a_1)(s+2) = 2, F(a_2)(s+2) = F(a_4)(s+2) = 1$, and $F(a_3)(s+1) = 0$.

When there exists an element $x \in U$ such that $x > a$, but xIa_2, xIa_3, and xIa_4, we define $F(a_1)(s+1) = F(a_2)(s+1) = F(a_4)(s+2) = 2$, $F(a_3)(s+1) = F(a_1)(s+1) = F(a_3)(s+2) = 1$, and $F(a_4)(s+1) = F(a_2)(s+1) = 0$.

Therefore we assume that all four points in $A_{\mathscr{M}}$ have the same holdings in X.

Now choose an embedding F of $X - \{a_3, a_4\}$ in $\underline{3}^m$; then define an embedding $G: X \to \underline{3}^{m+1}$ by $G(x)(i) = F(x)(i)$ for every $x \in X - \{a_3, a_4\}$ and for every $i \leqslant m$, $G(a_3)(i) = \max\{F(a_1)(i), F(a_2)(i)\}$ for every $i \leqslant m$, $G(a_4)(i) = \min\{F(a_1)(i), F(a_2)(i)\}$, $G(x)(m+1) = 2$ if $x \in X - \{a_1, a_2, a_3, a_4\}$ and $x > a_1$, $G(x)(m+1) = 1$ if $x \in X - \{a_1, a_2, a_3, a_4\}$ and xIa_1, $G(a_4)(m+1) = 2, G(a_1)(m+1) = G(a_2)(m+1) = 1$, and $G(a_3)(m+1) = 0$.

Now suppose $| A_{\mathscr{M}} | = 2$ and let $A_{\mathscr{M}} = \{a_1, a_2\}$. As before we now assume $L = \phi$ i.e. a_1 and a_2 are minimal elements. Also it is easy to see that we may also assume that a_1 and a_2 have the same holdings.

If $X = (X - \{a_1, a_2\}) + \{a_1\} + \{a_2\}$. Then $\mathrm{Dim}(X) \leqslant [| X |/2]$ by Lemma 1. Thus we may assume that there exists a maximal element x_1 such that $x_1 > a_1$ and $x_1 > a_2$. If $\{x_1 > y_1\} \in \mathscr{M}$, then $y_1 I a_1$ and $y_1 I a_2$. Now consider the maximum matching

$$\mathscr{M}_1 = \mathscr{M} - \{x_1 > y_1\} \cup \{x_1 > a_1\}$$

for which $A_{\mathscr{M}'} = \{y_1, a_2\}$. Although $\mathscr{M}'$ may not satisfy property P, it is easy to see that we may obtain from $\mathscr{M}'$, a maximum matching $\mathscr{M}''$ satisfying property P where $A_{\mathscr{M}''} = \{a_3, a_2\}$ with $a_3 I a_1$ and $a_3 I a_2$. Hence we may assume that a_1, a_2, and a_3 all have the same holdings.

Let x_0 be the element of U for which $\{x_0 > a_3\} \in \mathscr{M}$. If x_0 is the only element of X which is greater than a_1, then $\mathrm{Dim}_3(X) \leqslant m + 1$ Lemma 2. Therefore we may choose elements $x_2, y_3 \in X - \{x_0, a_1, a_2, a_3\}$ such that $x_2 > a_1$, $x_2 > a_2$, $x_2 > a_3$, and $\{x_2 > y_2\} \in \mathscr{M}$. We then repeat the argument given above to obtain a point $a_4 \in X - \{a_1, a_2, a_3\}$ such that a_1, a_2, a_3, and a_4 all have the same holdings.

We can now construct an embedding of X in $\underline{3}^{m+1}$ with the same argument for the case where $| A_{\mathscr{M}} | = 4$ and all points of $A_{\mathscr{M}}$ are minimal elements with the same holdings. The proof of our theorem is now complete.

5. Inequalities for $\mathrm{Dim}_4(X)$.

In this section, we show that $\mathrm{Dim}_4(X) \leqslant [| X |/2]$ when $| X | \geqslant 6$. We begin with:

Fact: If $| X | = 5$, then $\mathrm{Dim}_4(X) = 2$ unless X is $\bar{5}$ or $\underline{2} + \bar{3}$ in which case $\mathrm{Dim}_4(X) = 3$.

Fact 4: *If* $| X | = 7$, then $\mathrm{Dim}_3(X) \leqslant 3$.

It is possible to modify the development given in Section 4 to obtain the desired inequality for $\mathrm{Dim}_4(X)$. However we prefer to develop the result by removal theorems instead. We now state a number of such results without proof. In each case the reader may easily fashion an argument along the lines of the proofs given for Lemmas 1 and 2.

Lemma 3. *If x_1 is the greatest element of X, then* $\mathrm{Dim}_k(X) = \mathrm{Dim}_k(X - x_1)$ *for each $k \geqslant 2$ unless $X - x_1$ also has a greatest element. If x_2 is then the greatest element of $X - x_1$, then*

$$\mathrm{Dim}_k(X) \leqslant 1 + \mathrm{Dim}_k(X - \{x_1, x_2\}) \qquad \text{for each} \quad k \geqslant 3.$$

Lemma 4. *If x_1 and x_2 are distinct maximal elements of X and* $X = (X - \{x, x_2\} \oplus \{x_1, x_2\}$, *then* $\mathrm{Dim}_k(X) \leqslant 1 + \mathrm{Dim}_k(X - \{x_1, x_2\})$ *for every $k \geqslant 3$.*

Lemma 5. *If $X = Y + Z$ and $k \geqslant 3$, then there exists a pair $x, y \in X$ such that* $\mathrm{Dim}_k(X) \leqslant 1 + \mathrm{Dim}_k(X - \{x, y\})$.

If a is a maximal element, b is a minimal element and no element of X is incomparable with both a and b, we call the pair a, b a bounding pair.

LEMMA 6. *If a, b is a bounding pair and $X - \{a, b\}$ has at least two maximal elements and at least two minimal elements, then* $\mathrm{Dim}_k(X) \leqslant 1 + \mathrm{Dim}_k(X - \{a, b\})$.

LEMMA 7. *Suppose a and c are maximal elements, a covers b, and c covers d. If a and c are the only maximal elements or $X - \{a, b, c, d\}$ has at least two maximal elements, then* $\mathrm{Dim}_k(X) \leqslant 2 + \mathrm{Dim}_k(X - \{a, b, c, d\})$ *for all $k \geqslant 4$.*

THEOREM 4. *If $|X| \geqslant 6$, then* $\mathrm{Dim}_4(X) \leqslant [|X|/2]$.

Proof. It suffices to show that if $n \geqslant 3$ and $|X| = 2n + 1$, then $\mathrm{Dim}_4(X) \leqslant n$. We assume validity for $n \leqslant m$ where $m \geqslant 3$ and then let $n = m + 1$. Suppose now that $|X| = 2m + 3$ and $\mathrm{Dim}_4(X) \geqslant m + 2$. It follows from Lemma 5, that no maximal element is also a minimal element. We now proceed to show that X has at least four maximal elements and at least four minimal elements.

We conclude from Lemma 3, that X has at least two maximal elements and at least two minimal elements. Now suppose that a_1 and a_2 are the only maximal elements. If a_1 is not the greatest element of $X - a_2$ and a_2 is not the greatest element of $X - a_1$, we may choose $b_1, b_2 \in X$ such that a_1 covers b_1, a_2 covers b_2, $a_1 I b_2$, and $a_2 I b_1$. From Lemma 2, we conclude a_1 is the greatest element of $X - \{a_2, b_2\}$ and a_2 is the greatest element of $X - \{a_1, b_1\}$. From Lemma 7 we conclude that $|X| = 9$ and that $X - \{a_1, a_2, b_1, b_2\}$ is either $\underline{2} + \overline{3}$ or $\overline{5}$. We eliminate the first possibility by Lemma 2; in the second case we conclude that all maximal elements are greater than all minimal elements and therefore b_1 is not a minimal element. We choose a minimal element x such that $b_1 > x$. It follows that a_2, x is a bounding pair.

If a_1 is the greatest element of $X - a_2$ and a_2 is the greatest of $X - a_1$, then $X = (X - \{a_1, a_2\}) \oplus \{a_1, a_2\}$. Therefore we assume that there exists a point b_1 such that a_1 covers b_1 but $a_2 I b_1$. Then it follows that a_2 is the greatest element of $X - \{a_1, b_1\}$ and a_1 is the greatest element of $X - a_2$. If there are three or more minimal elements, then we conclude by Lemma 1 that a_1 is greater than all minimal elements. In this case we choose any minimal element x with $x \neq b_1$ and see that a_1, x is a bounding pair.

We conclude that there are only two minimal elements, say d_1 and d_2. By duality we may also conclude that there exists a point e_1 such that e_1 covers d_1, $e_1 I d_2$, d_1 is the least element of $X - d_2$, and d_2 is the least element of $X - \{e_1, d_1\}$. Thus $b_1 \neq d_1$ and $a_1 \neq e_1$ and it follows that a_1, d_1 is a bounding pair.

We now conclude that X has at least 3 maximal elements and three minimal elements, every maximal element is greater than every minimal element, and every nonmaximal point is under at least two maximal elements. Suppose that X has exactly three maximal elements a_1, a_2, a_3. If anyone of these three elements, say a_i, is greater than all nonmaximal elements, then we may choose any minimal element b to obtain a bounding pair a_i, b.

Therefore for each $i \leqslant 3$ we may choose an element $b_i \in X$ with b_i covered by all maximal elements except a_i, and $a_i I b_i$. Now the poset $X - \{a_1, b_1, a_2, b_2\}$ has at least two maximal elements a_3 and b_3. We conclude that $|X| = 9$ and $X - \{a_1, b_1, a_2, b_2\}$ is $\bar{5}$ or $\underline{2} + \bar{3}$. However neither of these is possible because b_3 is not a minimal element in X and we may choose a minimal element $x \in X$ for which $x < b_3$ and $x < a_3$.

We now conclude that X has at least four maximal elements and at least four minimal elements, all maximal elements are greater than all minimal elements, and there does not exist a maximal element which is greater than all nonmaximal elements. For each maximal element a, let $L(a) = \{x \in X: x < a\}$. If for each distinct pair a_1, a_2 of maximal elements, we have $L(a_1) \subset L(a_2)$ or $L(a_2) \subset L(a_1)$, then that maximal element a for which $|L(a)|$ is maximum is greater than all nonmaximal elements. We then choose a pair a_1, a_2 of maximal elements for which $L(a_1) \not\subset L(a_2)$ and $L(a_2) \not\subset L(a_1)$. Then there exists points b_1, b_2 such that a_1 covers b_1, a_2 covers b_2, $a_1 I b_2$, and $a I_2 b_1$. By Lemma 7 we conclude that $|X| = 9$ and that $X - \{a_1, b_1, a_2, b_2\}$ is either $\bar{5}$ or $\underline{2} + \bar{3}$ but clearly this is not possible. The contradiction completes the proof.

6. Concluding Remarks and Open Problems

We have not been able to prove Theorem 3 using only removal theorems without the concept of matching. The primary obstacle is that Lemma 7 apparently holds only if $k \geqslant 4$. We have also been unable to prove Theorem 4 using only removal theorem which involve a pair of points. It remains an open question to answer whether for every $k \geqslant 3$, a poset X always contains a pair of points x, y such that

$$\mathrm{Dim}_k(X) \leqslant 1 + \mathrm{Dim}_k(X - \{x, y\}].$$

This same question is also unanswered for ordinary dimension, but the answer is no when $k = 2$ [9].

The collection of all posets for which $\mathrm{Dim}_2(X) = |X|$ has been determined [9] as has the collection of posets for which $\mathrm{Dim}(X) = [|X|/2]$[2];

[6]. Of the inequalities given here, the only manageable characterization problem is to determine those posets X for which $|X| = 2n + 1$ and $\mathrm{Dim}_3(X) = n + 1$.

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