ɔoxes $\cup_{j=1}^{m} B_j$, let $\mathbf{r}(j) = (r_1(j), \ldots, r_\beta(j)) \in B_j$ and define

$\wedge_{\{\gamma: r_\gamma(j)<s\}} [LDT(TR_\gamma, \mathbf{d}_\gamma, r_\gamma(j)) \wedge \neg LDT(TR_\gamma, \mathbf{d}_\gamma, r_\gamma(j)+1)]$

$\wedge_{\{\gamma: r_\gamma(j)\geq s\}} LDT(TR_\gamma, \mathbf{d}_\gamma, s)$,

(When $r_\gamma(j) = 0$ then the expression in the brackets above should be $LDT(TR_\gamma, \mathbf{d}_\gamma)$.) Then φ_j and, thus, $\varphi := \vee_{j=1}^{m} \varphi_j$ are themselves first es. Further, $\mathfrak{M}[\varphi] = \cup_{j=1}^{m} \mathfrak{M}[\varphi_j] = \cup_{j=1}^{m} B_j$, i.e. any disjoint union of $\varphi]$ for some first order sentence φ. (We note that φ may have higher ɔth than s.) By Remark 9 this completes the complete characterization robabilities.

) denote the limit probability of ψ as a function of the real number of $f_\psi(c)$ is simple to describe if we relax the condition of a complete ion. The basic sentence $LDT(TR_\gamma, \mathbf{d}_\gamma, i) \wedge \neg LDT(TR_\gamma, \mathbf{d}_\gamma, i+1)$ has obability f of the form $qe^{-\lambda e^{-c}} e^{-ci}$ where λ, q are positive rational e sentence $LDT(TR_\gamma, \mathbf{d}_\gamma, s)$ will have an f of the form one minus a such terms. To conclude this section, we summarize the results in the ɔrem.

n 22. *Let k, l be integers such that $l \geq k-1 \geq 0$. Let ψ be any sentence der language of graphs. Then, in general, the limiting probability $f_\psi(c)$ of a finite sum and difference of rational numbers times finite products he form above, i.e. a linear combination with rational coefficients of form $e^{-c\lambda_1} e^{-\lambda_2 e^{-c}}$ where the λ_1, λ_2 are themselves rational numbers. 1 as the null product here.] ular, $f_\psi(c)$ always exists and is always an infinitely differentiable func-*

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RAMSEY THEORY AND PARTIALLY ORDERED SETS

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ABSTRACT. Over the past 15 years, Ramsey theoretic techniques and concepts have been applied with great success to partially ordered sets. In the last year alone, four new applications of Ramsey theory to posets have produced solutions to some challenging combinatorial problems. First, Kierstead and Trotter showed that dimension for interval orders can be characterized by a single ramsey trail by proving that interval orders of sufficiently large dimension contain all small interval orders as subposets. Second, Winkler and Trotter introduced a notion of Ramsey theory for probability spaces and used the resulting theroy to show that interval orders can have fractional dimension arbitrarily close to 4. Third, Felsner, Fishburn and Trotter developed an extension of the product Ramsey theorem to show that there exists a finite 3-dimensional poset which is not a sphere order. Fourth, Agnarsson, Felsner and Trotter combined Ramsey theoretic techniques with other combinatorial tools to determine an asymtotic formula for the maximum number of edges in a graph whose incidence poset has dimension at most 4. In this paper, we outline how these applications were developed. Full details will appear in individual journal articles. This article also includes a brief sketch of how the applications of Ramsey theoretic techniques to posets have evolved.

1. Introduction

In recent years, there has been rapid growth in research activity centered on combinatorial problems for partially ordered sets, evidenced in part by the new AMS subject classification 06A07: Combinatorics of Partially Ordered Sets. In this article, we explore connections between Ramsey theory and partially ordered sets—especially with the poset parameter called *dimension.*

In this introductory section, we present only those concepts and notations essential to the results discussed in this paper. For additional background material, the reader is referred to the survey articles **[33]**, **[34]**, **[35]** **[36]**, the recent article by Brightwell **[4]** and the author's monograph on posets **[32]**.

We consider a *partially ordered set* (or *poset*) $\mathbf{P} = (X, P)$ as a structure consisting of a set X and a reflexive, antisymmetric and transitive binary relation P on X. We call X the *ground set* of the poset $\mathbf{P}$, and we refer to P as a *partial order* on

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ions $x \leq y$ in P, $y \geq x$ in P and $(x, y) \in P$ are used interchangeably, nce to the partial order P is often dropped when its definition is fixed ie discussion. We write $x < y$ in P and $y > x$ in P when $x \leq y$ in . When $x, y \in X$, $(x, y) \notin P$ and $(y, x) \notin P$, we say x and y are and write $x \| y$ in P.

we are concerned primarily with *finite* posets, i.e., those posets with sets, we find it convenient to use the familiar notation $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ pectively the reals, rationals, integers and positive integers equipped l orders. We also use $\mathbb{R}_0$ to denote the set of positive real numbers. se infinite posets are *total* orders; in each case, any two distinct points le. Total orders are also called *linear* orders, or *chains*. For a positive let $\mathbf{n}$ denote the n-element chain $0 < 1 < \cdots < n-1$, while $[n]$ denotes set $\{1, 2, \ldots, n\}$.

$= (X, P)$ is a poset, a linear order L on X is called a *linear extension* $< y$ in L for all $x, y \in X$ with $x < y$ in P. A set $\mathcal{R}$ of linear extensions a *realizer* of $\mathbf{P}$ when $P = \cap\mathcal{R}$, i.e., for all x, y in X, $x < y$ in P if and in L, for every $L \in \mathcal{R}$. The minimum cardinality of a realizer of $\mathbf{P}$ is *ension* of $\mathbf{P}$ and is denoted $\dim(\mathbf{P})$.

oset $\mathbf{P} = (X, P)$, let $\mathrm{inc}(\mathbf{P}) = \{(x, y) \in X \times X : x \| y \text{ in } P\}$. Then f linear extensions of P is a *realizer* of P if and only if for every), there exists $L \in \mathcal{R}$ so that $x > y$ in L. Call a pair $(x, y) \in \mathrm{inc}(\mathbf{P})$ if

in P implies $u < y$ in P, and
in P implies $v > x$ in P

X. Then let $\mathrm{crit}(\mathbf{P})$ denote the set of all critical pairs. It follows that $\mathcal{R}$ of linear extensions of P is a realizer of P if and only if for every), there exists some $L \in \mathcal{R}$ so that $x > y$ in L.

hat a linear extension L *reverses* the incomparable pair (x, y) when et $S \subset \mathrm{inc}(\mathbf{P})$. We say that L *reverses* S when $x > y$ in L, for every e say that a family $\mathcal{R}$ of linear extensions of P *reverses* S if for every ere is some $L \in \mathcal{R}$ so that $x > y$ in L. So the dimension of P is just ger t for which there exists a family $\mathcal{R}$ of linear extensions of P which R).

Three examples of Posets with Large Dimension

ction, we briefly discuss three well known examples of posets with on. These examples will help readers who are new to the subject of red sets with concepts discussed in subsequent sections.

ers $n \geq 3$, $k \geq 0$, define the *crown* $\mathbf{S}_n^k$ as the height 2 poset with l elements $a_1, a_2, \ldots, a_{n+k}$, $n+k$ maximal elements $b_1, b_2, \ldots, b_{n+k}$ for $j = i+k+1, i+k+2, \ldots, i-1$. In this definition, we interpret lically so that $n+k+1 = 1$, $n+k+2 = 2$, etc. The following formula sion of crowns is derived in [**30**].

1 2.1. *Let $n \geq 3$ and $k \geq 0$ be integers. Then*

$$\dim(\mathbf{S}_n^k) = \left\lceil \frac{2(n+k)}{k+2} \right\rceil.$$

□

The critical pairs in the crown $\mathbf{S}_n^k$ are just the incomparable pairs (a_i, b_j), where a_i is a minimal element and bj is a maximal element. In [**30**], Trotter proves that no linear extension of $\mathbf{S}_n^k$ can reverse more than $(k+2)(k+1)/2$ critical pairs. Since there are $(n+k)(k+1)$ critical pairs altogether, the lower bound in Theorem 2.1 follows immediately. It takes a little more work to show that this bound is tight.

When $k = 0$, the crown $\mathbf{S}_n^0$ (also denoted $\mathbf{S}_n$), is called the *standard example* of an n-dimensional poset. To see that $\dim(\mathbf{S}_n) \leq n$, observe that there are n critical pairs, namely the pairs (a_i, b_i) for $i = 1, 2, \ldots, n$. Clearly, n linear extensions are enough to reverse them. Conversely, it is easy to see that no linear extension of $\mathbf{S}_n$ can reverse two or more linear extensions, so that the dimension of $\mathbf{S}_n^0$ is at least n.

Note that $\mathbf{S}_n$ is isomorphic to the set of 1-element and $(n-1)$-element subsets of $[n]$ ordered by inclusion. More generally, for integers k, r and n, with $1 \leq k < r \leq n-1$, let $\mathbf{P}(k, r; n)$ denote the poset consisting of all k-element and r-element subsets of $[n]$ ordered by inclusion. Also, let $\dim(k, r; n)$ denote the dimension of $\mathbf{P}(k, r; n)$. So $\mathbf{S}_n$ is isomorphic to $\mathbf{P}(1, n-1; n)$ and $\dim(1, n-1; n) = n$.

Our second example of a family of posets of large dimension is $\{\mathbf{P}(1, 2; n) : n \geq 3\}$. In this case, there are $n(n-1)(n-2)/2$ incomparable pairs; however, an easy exercise shows that a linear extension may reverse $n(n-1)(n-2)/6$ critical pairs. So the "pigeon hole" argument used for the first example shows only that $\dim(1, 2; n) \geq 3$. However, we claim that $\lim_{n \to \infty} \dim(1, 2; n) = \infty$, although the argument now requires some elementary Ramsey theory. Suppose to the contrary that there exists a positive integer $t \geq 3$ so that $\dim(1, 2; n) \leq t$, for every $n \geq 3$. We obtain a contradiction when n is sufficiently large. Let $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ be a realizer of $\mathbf{P}(1, 2; n)$. For each 3-element subset $\{i < j < k\} \subseteq [n]$, consider the critical pair $(\{j\}, \{i, k\})$, and choose an integer $\alpha \in [t]$ so that L_j reverses it, i.e., $\{j\} > \{i, k\}$ in L_α. Then we have a coloring of the 3-element subsets of $[n]$ with t colors. If n is sufficiently large, then (by Ramsey's theorem) there exists a 4-element subset $H = \{i < j < k < l\} \subseteq [n]$ and an integer $\alpha \in [t]$ so that all 3-element subsets of H are mapped to α. This means that $\{j\} > \{i, k\} > \{k\} > \{j, l\} > \{j\}$ in L_α, which is a contradiction.

Each of the first two examples is a height 2 poset, so posets of bounded height can have arbitrarily large dimension. Our third example is different. In this family, large height is required for large dimension. For each $n \geq 3$, let $\mathbf{I}(n) = (I_n, P_n)$ denote the poset defined by setting I_n to be the family of all 2-element subsets of $[n]$ with $\{i, j\} < \{k, l\}$ in P_n when $1 \leq i < j < k < l \leq n$. Again, we claim that $\lim_{n \to \infty} \dim(I_n, P_n) = \infty$. Suppose to the contrary that $\dim(I_n, P_n) \leq t$, for all $n \geq 3$. We obtain a contradiction when n is large.

For each 3-element subset $\{i < j < k\} \subseteq [n]$, $(\{i, j\}, \{j, k\})$ is a critical pair, so if $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ is a realizer of P_n, then we may choose $\alpha \in [t]$ so that $(\{i, j\}, \{j, k\})$ is reversed in L_α. This is a coloring of the 3-element subsets of $[n]$ with t colors, so that if n is sufficiently large, there exists a 4-element subset $H = \{i < j < k < l\}$ and an integer $\alpha \in [t]$ so that all 3-element subsets of H are mapped to α. This implies that $\{i, j\} > \{j, k\} > \{k, l\} > \{i, j\}$ in L_α, which is a contradiction.

This last example is drawn from the family of posets known as *interval orders* and we will have more to say about them in Sections 5, 6 and 7.

3. Early applications of Ramsey theory to posets

finite set S and an integer k with $0 \le k \le |S|$, we denote the set of all ıbsets of S by $\binom{S}{k}$. Given integers t and k and finite sets $S_1, S_2, \dots, S_t$, of $\binom{S_1}{k} \times \binom{S_2}{k} \times \cdots \times \binom{S_t}{k}$ is called a *grid* (also, a $\mathbf{k}^t$ grid), and the sets S_t are called *factor sets* of the grid. Using the natural order, a set of n ust an n-element chain, so considered as a poset, $S_1 \times S_2 \times \cdots \times S_t$ is to $\mathbf{n}_1 \times \mathbf{n}_2 \times \cdots \times \mathbf{n}_t$, where $n_i = |S_i|$ for $i = 1, 2, \dots, t$.
owing theorem, called the Product Ramsey Theorem and stated here n, has been applied in several different settings to posets. We refer the 4] for the proof.

ΞM 3.1. *Given positive integers m, k, r and t, there exists an integer f $n \ge n_0$ and f is any map which assigns to each $\mathbf{k}^t$ grid of $\mathbf{n}^t$ a color en there exists a subposet $\mathbf{P}$ isomorphic to $\mathbf{m}^t$ and a color $\alpha \in [r]$ so α for every $\mathbf{k}^t$ grid g from $\mathbf{P}$.* □

;h it was not originally stated in these terms, most likely the first ap- the product Ramsey theorem to posets can be traced to the proof of ʒ theorem [**31**].

ΞM 3.2. *Let $\mathbf{P} = (X, P)$ be a poset and let $A \subseteq X$ be an antichain with Then*

$$\dim \mathbf{P} \le 1 + 2\,\mathrm{width}(X - A, P(X - A)).$$

□

ıuality in Theorem 3.2 is quite straightforward, but it takes a non-trivial oretic argument to show that it is best possible. It is interesting to note he proof of Theorem 3.2 was first published, the use of Ramsey theory ıt the inequality is best possible was not the main point. Instead as / the title of the paper, it was the ensuing corollary: irreducible posets ;ht exist. Some years later, explicit constructions for irreducible posets eight would be given by Trotter and Ross [**37**], [**38**] and by Kelly [**16**]. ınother application of Ramsey theory, one which deals with the concept *ınctions* as developed by Walker in [**41**]. For integers n and r, let $L(n, r)$ ɜt of all r-term non-decreasing sequences from $[n]$. We then let $\mathbf{L}(n, r) =$ $n, r))$ be the subposet of $\mathbb{R}^r$ induced by $L(n, r)$. We view the elements s possible results of a series of races among n competitors in which ved. For example, the element $(5, 5, 3, 3, 2, 1)$ of $L(10, 6)$ represents an a single competitor of two fifth place finishes, two third place finishes, place finish, and one first place finish. Note that the notation does not ndividual races in which these respective placings were achieved.
ɔpose prize money is assigned to the finishing positions so that smaller sitions (corresponding to better achievement) receive higher monetary en total prize money determines a ranking function among the com- ain, ties are allowed). Walker [**41**] proposed to call a linear extension *consistent* if there was a way to assign monetary awards so that if x noney than y, then $x > y$ in L. He then proved that $L(n, r)$ is then ion of all its consistent linear extensions. Furthermore, when $r = 2$, -dimensional poset and Walker showed that it is also the intersection of linear extensions. Walker also showed that $P(4, 3)$ is the intersection of

3 consistent linear extensions and conjectured that $L(n, r)$ is always the intersection of r consistent linear extensions.

However, in [**25**], Rödl and Trotter used Ramsey theory to show that for every $t \ge 3$, there exists an integer n_0 so that if $n > n_0$, $P(n, 3)$ is not the intersection of t or fewer consistent linear extensions—even though it is 3-dimensional and thus the intersection of 3 linear extensions.

An *interval* in a poset $\mathbf{P} = (X, P)$ is just a subposet of the form $[x, y] = \{z \in X : x \le z \le y\}$, where $x < y$. In [**26**], Scheinerman defined the *poset boxicity* of a graph $\mathbf{G} = (V, E)$ as the least t for which $\mathbf{G}$ is the intersection graph of a family of intervals in a t-dimensional poset. In [**39**], Trotter and West show that a graph on n vertices has poset boxicity at most $O(\log \log n)$. They also use Ramsey theory to show that there exist posets of arbitrarily large poset boxicity—although the lower bound grows considerably more slowly than the upper bound.

4. Computational Aspects

Mirroring the general flavor of Ramsey theory, there are instances in which the major emphasis is on whether a Ramsey theoretic result is true—and in such cases, it is rarely possible to make precise estimates as to how large the parameters must be. However, in other instances, the existence question is relatively straightforward, so researchers try to make a precise determination for the parameters (or at least a relatively accurate asymptotic estimate). Here are a few examples involving posets.

In [**23**], Nešetřil and Rödl show that for every positive integer h, if $\mathbf{P} = (X, P)$ is a poset of height h, then there exists a poset $\mathbf{Q} = (Y, Q)$ of height $2h - 1$ so that if the points of $\mathbf{Q}$ are assigned to two colors (say by a mapping to [2]), then there is a monochromatic subposet isomorphic to $\mathbf{P}$. The value $2h - 1$ is clearly best possible. This work is closely related to Nešetřil and Rödl's well known construction of graphs (and hypergraphs) with large chromatic number and large girth [**22**].

In [**18**], Kierstead and Trotter study the dual problem. Given an integer w, find the least integer $f(w)$ so that if $\mathbf{P} = (X, P)$ is a poset of width w, then there exists a poset $\mathbf{Q} = (Y, Q)$ of width $f(w)$ so that if the points of $\mathbf{Q}$ are two colored, then there exists a monochromatic subposet isomorphic to $\mathbf{P}$. It is elementary to show that $2w - 1 \le f(w) \le w^2$, but Kierstead and Trotter prove that $f(w) > 2w - 1$. Subsequently, Kierstead [**17**] showed that $f(w) > 5w/2$, but it is still not known whether $f(w) = O(w)$.

Given a poset $\mathbf{P} = (X, P)$, a function $f : X \to X$ is called a *regression* if $f(x) \le x$ for all $x \in X$. A regression is called a *choice function* if $f(x)$ is a minimal element for all $x \in X$. Given a regression f on a poset $\mathbf{P} = (X, P)$, a k-element chain $C = \{x_1 < x_2 < \cdots < x_k\}$ is *f-monotone* if $f(x_1) \le f(x_2) \le \cdots \le f(x_k)$. Note that if f is a choice function, then the statement that C is a f-monotone chain just means that f is constant on C.

For a positive integer n, let $\mathbf{B}_0(n)$ denote the poset consisting of all non-empty subsets of $[n]$. In [**24**], Perry proved that for each $k \ge 1$, if $n \ge 2^{k-1}$, then any choice function f on $\mathbf{B}_0(n)$ is constant on a chain of cardinality k. Furthermore, this result is best possible.

In [**42**], West, Trotter, Peck and Schor prove that if w and k are positive integers, then any regression on a poset of width at most w having at least $(w+1)^{k-1}$ points has a f-monotone chain of cardinality k. Furthermore, this result is best possible.

itive integer $n \geq 2$, let $\mathbf{P}_n$ denote the set of closed intervals of $\mathbb{R}$ with ›ints from $[n]$, partially ordered by inclusion. Evidently, $\mathbf{P}_n$ is a 2- ›oset. In [3], Alon, Trotter and West study the problem of determining teger $f(n)$ for which every regression on $\mathbf{P}_n$ has a monotone chain of $n)$. They show that

$$\log^*(n) - 2 \leq f(n) \leq \log^*(n).$$

5. Shift Graphs, Interval Orders and Layers

:egers n and k with $1 \leq k < n$, we call an ordered pair (A, B) of s a (k,n)-*shift pair* if there exists a $(k+1)$-element subset $C = \{i_1 <$ $_{+1}\} \subseteq [n]$ so that $A = \{i_1, i_2, \ldots, i_k\}$ and $B = \{i_2, i_3, \ldots, i_{k+1}\}$. We ıe (k,n)-*shift graph* $\mathbf{S}(k,n)$ as the graph whose vertex set consists of subsets of $[n]$ with a k-element set A adjacent to a k-element set B (A, B) is a (k,n)-shift pair. It is customary to call a $(2,n)$-shift graph ·aph; similarly, a $(3,n)$-shift graph is called a *double shift graph*. ; a $(1,n)$-shift graph is just a complete graph on n vertices, but for $k \geq$:iangle-free. Now it is an immediate consequence of Ramsey's theorem fixed $k \geq 1$, $\lim_{n\to\infty} \chi(\mathbf{S}(k,n) \to \infty$. However, as in the preceding vely accurate estimates are known for how fast the chromatic number growing. As mentioned previously, $\chi(\mathbf{S}(1,n) = n$, and for $k = 2$, it ıt $\chi(\mathbf{S}(2,n) = \lceil \lg n \rceil$. For $k = 3$, the chromatic number of $\mathbf{S}(3,n)$ is t for which there are n antichains in $\mathbf{2^t}$, the lattice of all subsets of $[t]$:lusion (see [32] for additional details and a discussion for larger values , the problem of estimating the number of antichains in $\mathbf{2^t}$ is a classical ın as *Dedekind's problem*. Although no closed form solution is known, tively accurate estimates, e.g., see Kleitman and Markowsky [20], and ugh to establish the following asymptotic formula:

$$\chi(\mathbf{S}(3,n)) = \lg\lg n + (1/2 + o(1)) \lg\lg\lg n.$$

› $= (X, P)$ is called an *interval order* if there exists a function F which :h element $x \in X$ a closed interval $[l_x, r_x]$ of the real line $\mathbb{R}$ so that and only if $r_x < l_y$ in $\mathbb{R}$. The poset $\mathbf{I}_n$ introduced in Section 2 is called *interval order*. Although posets of height 2 can have arbitrarily large ·owns, for example), this is not true for interval orders. For a positive $d(n)$ denote the maximum dimension of an interval order of height n. ‌li, Hajnal, Rödl and Trotter exploit the connection with double shift w that:

$$d(n) = \lg\lg n + (1/2 + o(1)) \lg\lg\lg n.$$

ıg techniques from Spencer [29] with the estimate for the chromatic ·uble shift graphs, Trotter [32] showed that the same estimate holds ı).

$$\dim(1, 2; n) = \lg\lg n + (1/2 + o(1)) \lg\lg\lg n.$$

6. Ramsey Trails in Interval Orders

·f the next four sections, we outline a recent application of Ramsey ets. Our first example involves the concept of Ramsey trails. Suppose finite structures with a well defined notion of substructure, which is

denoted $\subset$. Also suppose that f is a monotonic function mapping $\mathcal{C}$ to $\mathbb{R}_0$, i.e., if $G \subset H$, then $f(G) \leq f(H)$. A sequence $\mathcal{T} = \{G_n : n \geq 1\}$ of structures is called a *Ramsey trail* if

1. $G_n \subset G_{n+1}$ for all $n \geq 1$, and
2. $\lim_{n\to\infty} f(G_n) = \infty$.

Now suppose that r is a positive integer and that $\mathcal{T}_i = \{G_{i,n} : n \geq 1\}$ is a Ramsey trail for each $i = 1, 2, \ldots, r$. We say that this family *characterizes* f if for every integer t, there exists an integer s, so that if G is any structure in $\mathcal{C}$ with $f(G) > s$, then there is an integer $i \in [r]$ and an integer $n \geq 1$ so that $G_{i,n} \subset G$ and $f(G_{i,n}) > t$. The least r for which such a family exists is then called the *Ramsey complexity* of the function f. For example, consider the class of all graphs and the function f which assigns to a graph G the number of vertices in the graph. Then it follows that the Ramsey complexity of f is 2. This is evidenced by two ramsey trails, the set of all independent graphs and the set of all complete graphs. On the other hand, the existence of graphs with large girth and large chromatic number is enough to show that chromatic number cannot be characterized by any finite number of Ramsey trails. So the Ramsey complexity of the function χ is infinite.

Nevertheless, it is an important topic in graph theory to identify classes of discrete structures and monotonic functions defined on them for which the Ramsey complexity is finite. It is of special interest to recognize when it is 1. For example, a well studied problem in graph theory is to investigate classes of graphs for which chromatic number can be bounded as a function of maximum clique size. Such classes are said to be χ-*bounded*. As just a single example, Gyárfás [15] has shown that the set of circle graphs (intersection graphs of chords of a circle) is χ-bounded. To date the best result on this subject is due to Kostochka and Kratochvíl [21] who showed that a circle graph with maximum clique size ω has chromatic number $O(2^\omega)$.

For posets in general, dimension cannot be characterized by any finite number of interval orders, so dimension has infinite Ramsey complexity. But for many years, it was believed that the Ramsey complexity of dimension is 1 for the class of interval orders, and that dimension for interval orders could be characterized by a single Ramsey trail, namely the family of canonical interval orders. This conjecture has recently been settled in the affirmative by Kierstead and Trotter [19]. Since every interval order is a subposet of a sufficiently large canonical interval order, their theorem has the following attractive reformulation.

THEOREM 6.1. *For every interval order* $\mathbf{P}$, *there exists an integer* t, *so that if* $\mathbf{Q}$ *is any interval order with dimension at least* t, *then* $\mathbf{P}$ *is isomorphic to a subposet of* $\mathbf{Q}$. □

7. Fractional Dimension for Interval Orders

Let $\mathbf{P} = (X, P)$ be a poset and let $\mathcal{F} = \{M_1, \ldots, M_t\}$ be a multiset of linear extensions of P. Brightwell and Scheinerman [5] call $\mathcal{F}$ a k-*fold realizer* of P if for each incomparable pair (x, y), there are at least k linear extensions in $\mathcal{F}$ which reverse the pair (x, y), i.e., $|\{i : 1 \leq i \leq t, x > y \text{ in } M_i\}| \geq k$. The *fractional dimension* of $\mathbf{P}$, denoted by $\text{fdim}(\mathbf{P})$, is then defined as the least real number $q \geq 1$ for which there exists a k-fold realizer $\mathcal{F} = \{M_1, \ldots, M_t\}$ of P so that $k/t \geq 1/q$ (it is easily verified that the least upper bound of such real numbers q is indeed attained). Using this terminology, the *dimension* of $\mathbf{P}$ is just the least t for which a 1–fold realizer of P. It follows immediately that $\text{fdim}(\mathbf{P}) \le \dim(\mathbf{P})$,)set **P**.

ıximum degree, denoted $\Delta(\mathbf{P})$, of a poset $\mathbf{P} = (X, P)$ is just the maxi- e in the associated comparability graph, i.e., the maximum number of ›arable to any one point. The dimension of a poset is bounded in terms ıum degree. The following upper bound is due to Füredi and Kahn [**13**].

EM 7.1. *If* $\mathbf{P} = (X, P)$ *is a poset and* $\Delta(\mathbf{P}) \le k$, *then* $\dim(\mathbf{P}) \le$ □

st lower bound to date is due to Erdös, Kierstead and Trotter [**7**].

EM 7.2. *There exists an absolute constant* $\epsilon > 0$ *so that for each* $k \ge 1$, *a poset* $\mathbf{P}$ *with* $\Delta(\mathbf{P}) = k$ *and* $\dim(\mathbf{P}) > \epsilon k \log k$. □

:tional dimension, the corresponding problem is much cleaner. Brightwell ›rman [**5**] proved that if $\mathbf{P}$ is a poset with $\Delta(\mathbf{P}) = k$, then $\text{fdim}(\mathbf{P}) \le$ · conjectured that this inequality could be improved to $\text{fdim}(\mathbf{P}) \le k+1$. ›roved by Felsner and Trotter [**9**], and the argument yielded a much nclusion, a result with much the same flavor as Brooks' theorem for

EM 7.3. *Let k be a positive integer, and let* $\mathbf{P}$ *be any poset with* $\Delta(\mathbf{P}) =$ m$(\mathbf{P}) \le k+1$. *Furthermore, if* $k \ge 2$, *then* $\text{fdim}(\mathbf{P}) < k+1$ *unless one*)onents of $\mathbf{P}$ *is isomorphic to* $\mathbf{S}_{k+1}$, *the standard example of a poset of* ꞔ + 1. □

and Trotter [**9**] derive several other inequalities for fractional dimen- hese lead to some challenging problems as to the relative tightness of similar to the one given in the preceding theorem.

nal dimension is also relatively well behaved on the class of interval 5], Brightwell and Scheinerman proved that the fractional dimension of order was less than 4, and they conjectured that this result was best : [**40**], Trotter and Winkler settled this conjecture in the positive.

EM 7.4. *For every* $\epsilon > 0$, *there exists an interval order* $\mathbf{P}$ *with* $\text{fdim}(\mathbf{P}) >$ □

:hniques and concepts introduced by Trotter and Winkler in [**40**] are › more important than the theorem which motivated the work in the Specifically, they ask what common patterns must appear in arbitrary spaces, provided that the space contain events corresponding to subsets ntly large finite set. This question must first be discretized, and this is ›d by considering approximations.

8. Circle Orders and Sphere Orders

ı partially ordered set (poset) $\mathbf{P} = (X, P)$, a function F which assigns $\in X$ a set $F(x)$ is called an *inclusion representation* of $\mathbf{P}$ if $x \le y$ l only if $F(x) \subseteq F(y)$. Every poset has such a representation. For ıst take $F(x) = \{y \in X : y \le x \text{ in } P\}$. We refer the reader to the ›r by Fishburn and Trotter [**11**] for additional background material on nclusion representations and an extensive bibliographic listing.

As is well known, the finite posets of dimension at most two are just those which have inclusion representations using closed intervals of the real line $\mathcal{R}$. Because a closed interval of $\mathbb{R}$ can also be considered as a sphere in $\mathbb{R}^1$, it is natural to ask which posets have inclusion representations using disks (circles) in $\mathbb{R}^2$. For historical reasons, these posets are called *circle orders.* Fishburn [**10**] showed that all interval orders are circle orders. Also, the so called *standard examples* of n-dimensional posets, the 1-element and $(n-1)$-element subsets of $\{1, 2, \dots, n\}$, ordered by inclusion, are circle orders. So among the circle orders are *some* posets of arbitrarily large dimension.

Call a poset $\mathbf{P}$ a *sphere order* if there is some $d \ge 1$ for which it has an inclusion representation using spheres in $\mathbb{R}^d$. Using the "degrees of freedom" theorem of Alon and Scheinerman [**2**], it follows that not all posets of dimension $d+2$ have inclusion representations using spheres in $\mathbb{R}^d$. In particular, when $d = 2$, we conclude that there are 4-dimensional posets which are not circle orders.

In [**27**], Scheinerman and Wierman used Ramsey theory to show that the countably infinite 3-dimensional poset $\mathbb{Z}^3$ is not a circle order.

These results leave open the following question:

QUESTION 8.1. *Is every finite 3-dimensional poset a circle order?* □

A somewhat more general question was posed by Brightwell and Winkler in [**6**].

QUESTION 8.2. *Is every finite poset a sphere order?* □

Using Ramsey theoretic techniqes which extend the product Ramsey theorem, both Question 1 and Question 2 are settled by the following theorem of Felsner, Fishburn and Trotter [**8**].

THEOREM 8.3. *There exists an integer* n_0 *so that if* $n > n_0$, *the finite 3-dimensional poset* $\mathbf{n}^3$ *is not a sphere order.* □

The techniques developed in [**8**] are likely to have applications to other combinatorial problems, especially the use of ramsey theory to control error in approximations and the concept of *uniform* induced functions.

9. Extremal Problems for Posets

In [**1**], Agnarsson, Felsner and Trotter study a natural extremal problem for posets, a problem which in fact was motivated by questions in ring theory. With a finite graph $\mathbf{G} = (V, E)$, associate a partially ordered set $\mathbf{P} = (X, P)$ defined by setting $X = V \cup E$ and $x < e$ in P if and only if x is an endpoint of e in $\mathbf{G}$. This poset is called the *incidence poset* of $\mathbf{G}$, and the extremal problem investigated in [**1**] is then to determine the maximum number $\text{M}(p, d)$ of edges in a graph on p nodes if its incidence poset has dimension at most d.

The starting point for this research is the following well known theorem of W. Schnyder [**28**].

THEOREM 9.1. *A graph* $\mathbf{G}$ *is planar if and only if the dimension of its incidence poset is at most* 3. □

As an immediate consequence of Schnyder's theorem, $\text{M}(p, 3)$ is just the maximum number of edges in a planar graph on p vertices, so $\text{M}(p, 3) = 3p - 6$ for all $p \ge 3$.

ı also determine the exact value of $M(p, 2)$, since the incidence poset of s dimension at most 2 if and only if it is either a path or a subgraph of follows that $M(p, 2) = p - 1$, for all $p \geq 2$.

≥ 4, it is likely to be very difficult to determine $M(p, d)$ precisely, except relatively small in comparison to d. For this reason, it seems more to concentrate on asymptotic results for fixed d with $p \to \infty$. For $d = 4$, Felsner and Trotter provide the following formula.

ɛEM 9.2.

$$\lim_{p \to \infty} \frac{M(p, 4)}{p^2} = \frac{3}{8}.$$

oof of this theorem requires several powerful combinatorial tools, includ- ›duct Ramsey theorem, Turán's theorem and the Erdős/Stone theorem. ıe reader to [**1**] for the proof and additional details on the connections heory.

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