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The Dimension of Planar Posets

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A partially ordered set (poset) is planar if it has a planar Hasse diagram. The dimension of a bounded planar poset is at most two. We show that the dimension of a planar poset having a greatest lower bound is at most three. We also construct four-dimensional planar posets, but no planar poset with dimension larger than four is known. A poset is called a tree if its Hasse diagram is a tree in the graph–theoretic sense. We show that the dimension of a tree is at most three and give a forbidden subposet characterization of two-dimensional trees.

1. INTRODUCTION

A partially ordered set (poset) consists of a pair (X, P) where X is a set (always finite in this paper) and P is a reflexive, antisymmetric, and transitive relation on X. P is called a partial order on X. The notations $(x, y) \in P$, $x \leqslant y$ in P, and $y \geqslant x$ in P are used interchangeably. Distinct points x and y are said to be incomparable, denoted $x \mathrel{\mathrm{I}} y$, if neither (x, y) nor (y, x) is in P; we let $\mathscr{I}_P = \{(x, y)\colon x \mathrel{\mathrm{I}} y \text{ in } P\}$. If $\mathscr{I}_P = \varnothing$, then P is called a linear order on X and (X, P) is called a linearly ordered set or chain. If $x \neq y$ and $(x, y) \in P$ we write $x < y$ in P or $y > x$ in P. If $x > y$ in P and $x > z \geqslant y$ in P implies $z = y$, then we say x covers y in P and write $x \succ y$ in P.

If P and Q are partial orders on X and $P \subseteq Q$, then Q is called an extension of P. If Q is also a linear order, Q is called a linear extension of P. If P is a partial order on X and $Y \subseteq X$, the relation $P(Y)$ defined by $P(Y) = P \cap (Y \times Y)$ is a partial order on Y called the restriction of P to Y and $(Y, P(Y))$ is called a subposet of (X, P). A subset $C \subseteq X$ is called a chain if $(C, P(C))$ is a chain and a subset $A \subseteq X$ is called an antichain if $P(A) = \{(a, a)\colon a \in A\}$.

For an arbitrary relation $\mathscr{R}$ on X, the transitive closure of $\mathscr{R}$, denoted $\overline{\mathscr{R}}$, is defined by $\overline{\mathscr{R}} = \{(x, y)$: There exists an integer $n \geqslant 2$ and a sequence

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$x_1, x_2, \ldots, x_n$ such that $x = x_1$, $y = x_n$, and $(x_i, x_{i+1}) \in \mathscr{R}$ for every $i < n\}$. Note that if P is a partial order on X, then $\overline{P} = P$. The dual of a relation $\mathscr{R}$, denoted $\hat{\mathscr{R}}$, is defined by $\hat{\mathscr{R}} = \{(x, y): (y, x) \in \mathscr{R}\}$. If (X, P) is a poset, then $(X, \hat{P})$ is also a poset and is called the dual of (X, P).

It is sometimes convenient to use a single symbol, usually X, to denote a poset. We use the symbol R to denote the set of real numbers with the usual ordering.

2. A Theorem on Extensions of Partial Orders

If Q is an extension of P, let $S = Q - P$. Then $S \subseteq \mathscr{I}_P$ and $Q = \overline{P \cup S}$. Conversely we may ask when $S \subseteq \mathscr{I}_P$ implies $\overline{P \cup S}$ is a partial order. The answer is given in the following theorem.

Theorem 1. *Let (X, P) be a poset and $S \subseteq \mathscr{I}_P$. Then the following statements are equivalent.*

1. *$\overline{P \cup S}$ is not a partial order on X.*

2. *There exist an integer $n \geqslant 2$ and a set $\{(a_i, b_i): 1 \leqslant i \leqslant n\} \subseteq S$ so that $\{(b_i, a_{i+1}): 1 \leqslant i \leqslant n\} \subseteq P$.*

Proof. We comment that it is necessary to interpret the set $\{(b_i, a_{i+1}): 1 \leqslant i \leqslant n\}$ cyclically.

Now suppose that statement 2 holds. It follows that (a_1, b_1) and (b_1, a_1) are in $\overline{P \cup S}$ but since $(a_1, b_1) \in S \subseteq \mathscr{I}_P$, a_1 and b_1 are incomparable points. We conclude that $\overline{P \cup S}$ violates the antisymmetric requirement for partial orders.

On the other hand suppose statement 1 holds. Since the relation $\overline{P \cup S}$ is reflexive and transitive, it must then violate the antisymmetric requirement. Choose distinct points x and y from X with $(x, y) \in \overline{P \cup S}$ and $(y, x) \in \overline{P \cup S}$. It follows that there exist an integer $m \geqslant 2$ and a sequence $x_1, x_2, \ldots, x_m$ containing at least two distinct points such that $(x_i, x_{i+1}) \in P \cup S$ for each $i \leqslant m$. Among such sequences we choose one with m minimum. It follows that $(x_i, x_j) \in P \cup S$ iff $j = i + 1$ and therefore $(x_i, x_{i+1}) \in P$ implies $(x_{i+1}, x_{i+2}) \in S$.

Let $x_{j_1}, x_{j_2}, \ldots, x_{j_n}$ be the subsequence consisting of those points for which $(x_{j_i}, x_{j_i+1}) \in S$. We note that $n \geqslant 2$; now define $a_i = x_{j_i}$ and $b_i = x_{j_i+1}$ for each $i \leqslant n$. Then $\{(a_i, b_i): 1 \leqslant i \leqslant n\} \subseteq S$; furthermore if $x_{j_{i+1}} = x_{j_i+1}$ then $a_{i+1} = b_i$ and if $x_{j_{i+1}} \neq x_{j_i+1}$, then $(x_{j_i+1}, x_{j_i+2}) \in P$ which implies that $a_{i+1} = x_{j_i+2}$. In either case we see that $(b_i, a_{i+1}) \in P$ and thus $\{(b_i, a_{i+1}): 1 \leqslant i \leqslant n\} \subseteq P$.

Suppose that $S \subseteq \mathscr{I}_P$ and $\overline{P \cup S}$ is not a partial order. Among the subsets $\{(a_i, b_i): 1 \leqslant i \leqslant n\} \subseteq S$ with $n \geqslant 2$ for which $\{(b_i, a_{i+1}): 1 \leqslant i \leqslant n\} \subseteq P$, choose one such subset with n minimum. It follows easily that $\{a_i: 1 \leqslant i \leqslant n\}$ and $\{b_i: 1 \leqslant i \leqslant n\}$ are antichains and that $(b_i, a_j) \in P$ iff $j = i + 1$ and $(a_i, b_j) \in P$ iff $i = j + 1$ and $a_i = b_j$. Hereafter we refer to a subset of S satisfying these properties as a TM cycle.

COROLLARY 1. *If P is a partial order on X and $(x, y) \in \mathscr{I}_P$, then $P \cup \{(x, y)\}$ is a partial order on X.*

COROLLARY 2 (Szpilrajn [7]). *If P is a partial order on X, then the collection $\mathscr{C}$ of all linear extensions of P is nonempty and $\cap\mathscr{C} = P$.*

If A and B are disjoint subsets of X and P is a partial order on X, then an extension Q of P is called an injection of B over A if $\{(a, b): a \in A, b \in B, a \mathrm{I} b \text{ in } P\} \subseteq Q$.

COROLLARY 3. *If A and B are disjoint subsets of X and P is a partial order on X, then there exists an injection of B over A iff there does not exist a* TM *cycle $\{(a_i, b_i): 1 \leqslant i \leqslant 2\}$ where $\{a_1, a_2\} \subseteq A$ and $\{b_1, b_2\} \subseteq B$.*

COROLLARY 4. *If C is a chain of a poset (X, P), then there exists an injection of C over $X - C$ and an injection of $X - C$ over C.*

A linear extension L of an injection Q of $X - C$ over C where C is a chain is called an upper extension [4] of C. Lower extensions are defined analogously.

We refer the reader to [10, 11] for additional uses of Theorem 1 and its corollaries.

3. The Dimension of a Poset

Dushnik and Miller [3] defined the dimension of a poset (X, P), denoted $\mathrm{Dim}(X, P)$, as the smallest positive integer t for which there exist linear extensions $L_1, L_2, \ldots, L_t$ of P such that $L_1 \cap L_2 \cap \cdots \cap L_t = P$. The dimension of a poset is one iff it is a chain; an antichain of two or more points has dimension two. If a maximum antichain of (X, P) has cardinality n, then $\mathrm{Dim}(X, P) \leqslant n$ since Dilworth's decomposition theorem [2] guarantees a partition $X = C_1 \cup C_2 \cup \cdots \cup C_n$ where each C_i is a chain and if L_i is an upper extension of C_i for each $i \leqslant n$, then $P = L_1 \cap L_2 \cap \cdots \cap L_n$.

A poset (X, P) is said to have a greatest lower bound if there exists a point $0 \in X$ such that $0 \leqslant x$ in P for every $x \in X$. A poset is said to have a least upper bound if there exists a point $1 \in X$ such that $x \leqslant 1$ in P for every $x \in X$. (X, P) is said to be bounded if it has both a greatest lower bound and a least upper bound. Clearly if 0 is a greatest lower bound, then $\mathrm{Dim}(X, P) = \mathrm{Dim}(X - \{0\}, P(X - \{0\}))$; a dual statement holds if X has a least upper bound.

We refer the reader to [3, 8, 9] for additional results in the dimension theory of posets.

4. Poset Diagrams and Planarity

Lattice diagrams or Hasse diagrams (see [1, p. 4]) are a useful conceptual aid for posets of small order. In this paper we require our diagrams to satisfy the additional condition that each point in the plane lies in the interior of at most two arcs of the diagram. For a Hasse diagram D, we define the crossing number of D, denoted $\nu(D)$, as the number of

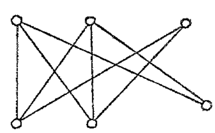

Figure 1

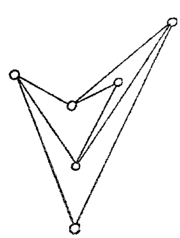

Figure 2

points in the plane which belong to the interior of exactly two arcs in D. A Hasse diagram D is planar if $\nu(D) = 0$. We define the crossing number of a poset (X, P), denoted $\nu(X, P)$, as $\min\{\nu(D)$: D is a Hasse diagram for $(X, P)\}$. A poset is said to be planar if it has a planar Hasse diagram. The diagram D in Fig. 1 has $\nu(D) = 9$ but D is a diagram of a planar poset as the diagram in Fig. 2 reveals.

5. The Dimension of a Planar Poset Having a Greatest Lower Bound

In this section, we consider a planar poset (X, P) which contains a greatest lower bound 0, and a Hasse diagram D of (X, P) with $\nu(D) = 0$. We then modify the development presented in [5] and employ Theorem 1 to obtain an upper bound on $\mathrm{Dim}(X, P)$.

For a chain C of covers $x_1 > x_2 > \cdots > x_n$ in (X, P), we define $J(C) = \{z \in R \times R$: There exists an integer $i < n$ such that z is a point on the arc from x_i to x_{i+1} in the diagram $D\}$. For each $x \in X$ with $x \neq 0$, we then define $J(x) = \cup \{J(C)$: C is a chain of covers from x to $0\}$. We now define for each $x \in X$ with $x \neq 0$, the trail of X, denoted $T(x)$, as the smallest subset of the plane satisfying the conditions: (i) $J(x) \subseteq T(x)$ and (ii) if s is a horizontal line segment in the plane whose end points are in $T(x)$, then all interior points of s are in $T(x)$.

We illustrate these definitions by shading the region $T(x)$ in Fig. 3.

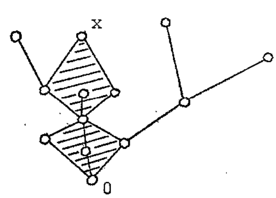

Figure 3

We note that $T(x)$ may include points of X which are incomparable with x in P. We also note that a horizontal line ℓ intersects $T(x)$ in either a closed line segment, a single point, or not at all. Furthermore for each $x \in X$, with $x \neq 0$ there exist unique chains of covers $C_L(x)$ and $C_R(x)$ from x to 0 so that for all horizontal lines ℓ which intersect $T(x)$, the left end point of $T(x) \cap \ell \in J(C_L(x))$ and the right end point of $T(x) \cap \ell \in J(C_R(x))$. The chains $C_L(x)$ and $C_R(x)$ are called the left and right boundary chains of x respectively.

Let π_1 and π_2 be the projection maps from $R \times R \to R$. We then define a relation $\mathscr{L}$ on X by $\mathscr{L} = \{(x, y)$: there exists a horizontal line ℓ for which $T(x) \cap \ell \neq \varnothing \neq T(y) \cap \ell$ and $\pi_1(z) < \pi_1(w)$ for every $z \in T(x) \cap \ell$ and every $w \in T(y) \cap \ell\}$. The line ℓ in this definition is called a test line for (x, y). Not all horizontal lines intersecting $T(x)$ and $T(y)$ need be test lines. In Fig. 4, $(x, y) \in \mathscr{L}$ and the line ℓ is a test line for (x, y). However $(y, z) \notin \mathscr{L}$.

Lemma 1. *If $(x, y) \in \mathscr{L}$ and ℓ is a horizontal line intersecting both*

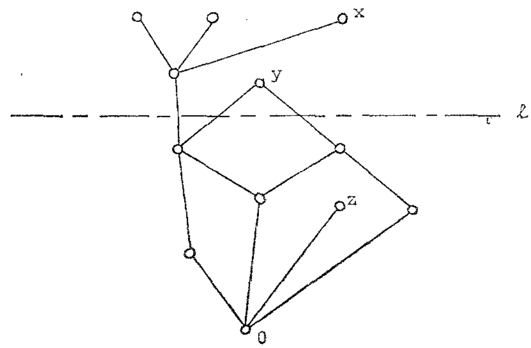

FIGURE 4

$T(x)$ and $T(y)$, then for every $z \in T(y) \cap \ell$, there exists $w \in T(x) \cap \ell$ such that $\pi_1(w) \leqslant \pi_1(z)$.

Proof. Let ℓ_1 be a test line for (x, y). We assume first that ℓ_1 is lower than ℓ. Let a, b, c, and d be the left end points of $T(y) \cap \ell$, $T(x) \cap \ell$, $T(x) \cap \ell_1$, and $T(y) \cap \ell_1$ respectively. We show that $\pi_1(b) \leqslant \pi_1(a)$.

If $\pi_1(b) > \pi_1(a)$, then the arcs $J(C_L(x))$ and $J(C_L(y))$ intersect at a point z between ℓ and ℓ_1. Since D is planar it follows that z corresponds to a point of X and that $z < x$ and $z < y$ in P. Therefore $c \in T(y)$. The contradiction shows that $\pi_1(b) \leqslant \pi_1(a)$ and thus $\pi_1(b) \leqslant \pi_1(z)$ for every $z \in T(y) \cap \ell$.

The argument when ℓ is lower than ℓ_1 is similar. Of course if $\ell = \ell_1$, then $\pi_1(w) \leqslant \pi_1(z)$ for every $w \in T(x) \cap \ell$ and every $z \in T(y) \cap \ell$.

The following statement follows immediately from Lemma 1.

LEMMA 2. *$\mathscr{L}$ is an antisymmetric relation.*

LEMMA 3. *$\mathscr{L}$ is a transitive relation.*

Proof. Let $(x, y) \in \mathscr{L}$ and $(y, z) \in \mathscr{L}$; then let ℓ_1 and ℓ_2 be test lines for (x, y) and (y, z) respectively. Then let ℓ be the lower of the two lines ℓ_1 and ℓ_2. From Lemma 1, we conclude that ℓ is test line showing $(x, z) \in \mathscr{L}$.

Let $\Delta = \{(x, x): x \in X\}$; then $\mathscr{L}^1 = \mathscr{L} \cup \Delta$ is a partial order on X. If (X, P) is a bounded poset, it is easy to see that for distinct points $x, y \in X$, $(x, y) \in \mathscr{I}_P$ iff $(x, y) \notin \mathscr{I}_{\mathscr{L}^1}$. In this case, P and $\mathscr{L}^1$ are "complementary" partial orderings. The existence of a complementary partial ordering is a well-known characterization of a poset with dimension at most two [3]. Zilber first observed that a bounded planar poset had a complementary partial ordering [1, p. 32].

In this paper, we will be concerned primarily with those planar posets (X, P) with greatest lower bounds for which the poset (X^1, P^1) obtained by adding a least upper bound to (X, P) is no longer planar. For such posets, $\mathscr{L}^1$ is not a complementary partial ordering. For example, consider the posets illustrated in Fig. 5. Each of these posets has dimension three.

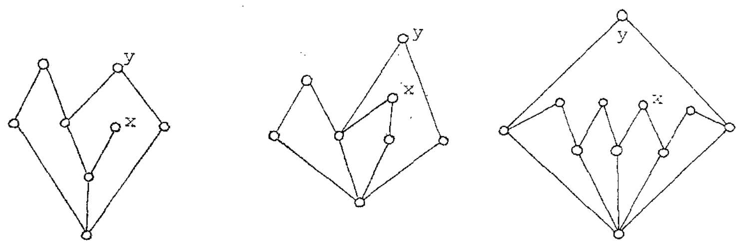

FIGURE 5

Now define a relation $\mathscr{M}$ on X by $(x, y) \in \mathscr{M}$ iff $(x, y) \in \mathscr{I}_P$ and $T(x) \subsetneq T(y)$. In Fig. 5, $(x, y) \in \mathscr{M}$ in each of the three diagrams. It follows easily that if $(x, y) \in \mathscr{I}_P$, then exactly one of the following statements is true $(x, y) \in \mathscr{L}$, $(y, x) \in \mathscr{L}$, $(x, y) \in \mathscr{M}$, $(y, x) \in \mathscr{M}$.

Since $x \leqslant y$ in P implies $T(x) \subseteq T(y)$, we conclude from Lemma 1 that the following statement holds.

LEMMA 4. *If ℓ is a horizontal line intersecting both $T(x)$ and $T(y)$ and there exists a point $z \in T(y) \cap \ell$ such that $\pi_1(z) > \pi_1(w)$ for every $w \in T(x) \cap \ell$, then either $x < y$ in P or $(x, y) \in \mathscr{L} \cup \mathscr{M}$.*

If $x < y$ in P, then $\pi_2(x) < \pi_2(y)$. More generally we have the following result, which may be proved by an argument very similar to the one used for Lemma 1.

LEMMA 5. *If $(x, y) \in \mathscr{M}$ and $x \leqslant z$ in P, then $\pi_2(z) < \pi_2(y)$.*

As a consequence of Lemma 5 we conclude that $\mathscr{M}$ is an antisymmetric and transitive relation on X.

We now proceed to establish an upper bound on the dimension of a planar poset with a greatest lower bound.

THEOREM 2. *Let (X, P) be a planar poset with a greatest lower bound. Then* $\operatorname{Dim}(X, P) \leqslant 3$.

Proof. Let D be a poset diagram for (X, P) with $\nu(D) = 0$. Then let the relations $\mathscr{L}$ and $\mathscr{M}$ be defined as before. Now define $S_1 = \hat{\mathscr{M}}$,

$S_2 = \mathscr{L} \cup \mathscr{M}$, and $S_3 = \hat{\mathscr{L}} \cup \mathscr{M}$. We show that $\overline{P \cup S_i}$ is a partial order on X for each $i \leqslant 3$ by proving that no S_i generates a TM cycle.

First suppose that $\{(a_i, b_i)\colon 1 \leqslant i \leqslant n\}$ is a TM cycle for S_1. Then $(b_i, a_i) \in \mathscr{M}$ for each $i \leqslant n$. Since $(b_i, a_{i+1}) \in P$ for every $i \leqslant n$, we conclude by Lemma 5 that $\pi_2(a_{i+1}) < \pi_2(a_i)$ for each $i \leqslant n$ and clearly this is not possible.

Now suppose that $\{(a_i, b_i)\colon 1 \leqslant i \leqslant n\}$ is a TM cycle for S_2. We note that $b_i \not> a_j$ for all $i \leqslant n$ and all $j \subseteq n$. We now prove the following statement by induction. For each $i \leqslant n$, $a_1 \mathrm{I}\, b_i$ and $(a_1, b_i) \in \mathscr{L} \cup \mathscr{M}$. This statement holds for $i = 1$ by definition of S_2. Suppose it is valid for $i \leqslant k$ where $1 \leqslant k < n$.

We assume first that $(a_1, b_k) \in \mathscr{M}$. If $(a_{k+1}, b_{k+1}) \in \mathscr{M}$, then $(a_1, b_{k+1}) \in \mathscr{M}$ since $\mathscr{M}$ is transitive. Now suppose that $(a_{k+1}, b_{k+1}) \in \mathscr{L}$. Since $T(b_{k+1}) \not\subseteq T(a_1)$ we cannot have $a_1 \geqslant b_{k+1}$ in P or $(b_{k+1}, a_1) \in \mathscr{M}$. Since $b_{k+1} \not> a_1$ in P, we have $a_1 \mathrm{I}\, b_{k+1}$ in P. If $(a_1, b_{k+1}) \notin \mathscr{L} \cup \mathscr{M}$, then $(b_{k+1}, a_1) \in \mathscr{L}$. However, $\mathscr{L}$ is transitive requiring $(a_{k+1}, a_1) \in \mathscr{L}$, but this is not possible because $T(a_1) \subseteq T(a_{k+1})$.

Now assume that $(a_1, b_k) \in \mathscr{L}$. If $(a_{k+1}, b_{k+1}) \in \mathscr{M}$, then $T(b_k) \subseteq T(a_{k+1}) \subseteq T(b_{k+1})$. Now let ℓ be a test line for (a_1, b_k); since $T(b_k) \cap \ell \subseteq T(b_{k+1}) \cap \ell$, it follows from Lemma 4 that $(a_1, b_{k+1}) \in \mathscr{L} \cup \mathscr{M}$. On the other hand, suppose $(a_{k+1}, b_{k+1}) \in \mathscr{L}$. If $(b_k, b_{k+1}) \in \mathscr{L}$ then $(a_1, b_{k+1}) \in \mathscr{L}$ because $\mathscr{L}$ is transitive. Therefore we may assume by Lemma 4 that $(b_k, b_{k+1}) \in \mathscr{M}$ and thus $T(b_k) \subseteq T(b_{k+1})$. As before we conclude that $(a_1, b_{k+1}) \in \mathscr{L} \cup \mathscr{M}$ and the inductive proof is complete. However this statement contradicts our assumption that $\{(a_i, b_i)\colon 1 \leqslant i \leqslant n\}$ is a TM cycle for S_2.

From the definition of $\mathscr{L}$ it is clear that the argument to show that $S_3 = \hat{\mathscr{L}} \cup \mathscr{M}$ generates no TM cycles is dual to the argument for S_2 and the proof of our theorem is complete.

6. Four-Dimensional Planar Posets

In this section we show that the inequality given in Theorem 2 is best possible by exhibiting an infinite family of four-dimensional planar posets. We use the notation for crowns and dimension products introduced in [8].

Fact. For each $k \geqslant 0$, the poset $S_3^{\,k} \otimes S_1^{\,0}$ is a four-dimensional planar poset.

We illustrate with a planar diagram for $S_3^{\,2} \otimes S_1^{\,0}$.

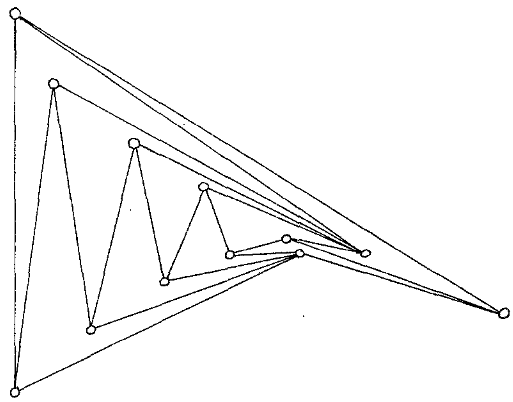

FIGURE 6

We have not been able to establish an upper bound on the dimension of a planar poset nor have we been able to construct a planar poset whose dimension is greater than four.

7. TREES

In this paper, we call a poset a tree[1] if its Hasse diagram is a tree in the graph theoretic sense. For example, the posets J_1 and J_2 whose Hasse diagrams are drawn in Fig. 1 are trees.

We now show that J_1 and J_2 each have dimension 3.

LEMMA 6. Dim $J_1 = 3$.

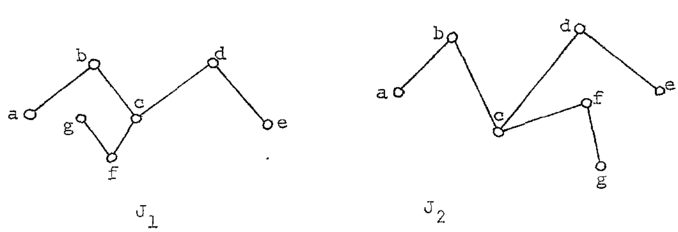

FIGURE 7

[1] In set theory, the word tree is used to describe a partially ordered set (X, P) for which $\{x \in X : x \leqslant y \text{ in } P\}$ is a well-ordered subposet of (X, P) for every $y \in X$. For an example, see Rudin [6]. In another setting it is only required that $\{x \in X : x \leqslant y \text{ in } P\}$ be a chain (see Wolk [12]).

Proof. Consider the linear extensions L_1, L_2, L_3, of J_1 defined by: $e < f < c < d < a < b < g$ in L_1, $a < f < g < c < b < e < d$ in L_2, and $f < g < a < c < e < b < d$ in L_3.

Since these extensions intersect to yield the partial order on J_1, we conclude $\operatorname{Dim} J_1 \leqslant 3$.

Now suppose $\operatorname{Dim} J_1 \leqslant 2$ and let M_1 and M_2 be linear orders whose intersection is the partial order on J_1. Since $b \mathrel{\mathrm{I}} e$ in J_1 we may assume $b < e$ in M_1 and $e < b$ in M_2. Now $x \mathrel{\mathrm{I}} y$ in J_1 implies $x < y$ in one of M_1 and M_2 and $y < x$ in the other; furthermore $x < y$ in J_1 implies $x < y$ in M_1 and M_2. We conclude that $a < f < c < b < e < d$ in M_1 and $e < f < c < d < a < b$ in M_2. Then g must be less than c in one of M_1 and M_2. If $g < b$ in M_1, then $b < g$ in M_2, which implies that $a < b$ in M_1 and M_2. On the other hand $g < c$ in M_2 implies $e < g$ in M_1 and M_2. The contradiction completes the proof that $\operatorname{Dim} J_1 = 3$.

A similar argument shows that $\operatorname{Dim} J_2 = 3$. For reasons of brevity, it is omitted.

We now show that the poset T_0 obtained from a tree T by attaching a zero is planar. As is often the case in combinatorics, the proof is accomplished by establishing an apparently stronger result. For a tree T with Hasse diagram $G(T)$, let $H(T)$ be the graph obtained from $G(T)$ by adding a new vertex 0 which is adjacent to every vertex in $G(T)$. We prove that there exists a plane drawing D of $H(T)$ without edge crossings so that the following two conditions are satisfied:

(i) Deleting the edges from 0 to nonminimal elements of T produces a Hasse diagram of T_0.

(ii) If $y > x$ in T, then the edge from y to x in D lies to the right of the edge from y to 0.

THEOREM 3. *For every tree T, there exists a drawing D of $H(T)$, without edge crossings, satisfying conditions* (i) *and* (ii) *as given above.*

Proof. If T is a tree on one or two vertices, such a drawing trivially exists. We then assume validity for all trees on k vertices and let T be a tree on $k + 1$ vertices.

Now let x be a vertex in T which is an end vertex of the diagram $G(T)$. Then the poset $T - x$ is also a tree. Choose a drawing D satisfying the required conditions for $T - x$ and let y be the unique vertex which is adjacent to x in $G(T)$.

Suppose first that x is a minimal element of T. Then let e be the edge from y to 0 in the drawing D and ℓ the perpendicular bisector of e. In a natural way, the edge e divides ℓ into a left and right half. Clearly, it is possible to choose a small positive number ϵ so that if we choose a point x on the right half of ℓ at a distance ϵ from e, the straight line segments from y to x and x to 0 do not cross any edges in D.

This construction produces the required drawing for T.

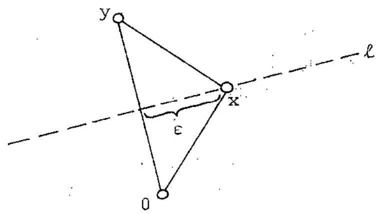

FIGURE 8

Now suppose that x is a maximal element of T. Let ℓ_e be the line in the plane containing the edge e and let ℓ_e' be that part of ℓ_e lying above y. Then among the collection of edges emanating upwards from y (including ℓ_e') there is one which is furthest to the left. Choose a line ℓ passing through y which is between the horizontal left ray through y and this left most line. Again it is easy to see that we can choose a point x on ℓ sufficiently close to y so that the line segments from x to 0 and x to y do not cross edges in D. And we have obtained the desired drawing of T and the proof of our theorem is complete.

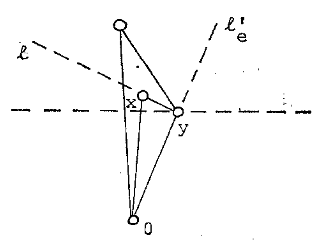

FIGURE 9

COROLLARY 5. *T_0 is planar for every tree T.*

COROLLARY 6. *The dimension of a tree is at most three.*

We invite the reader to compare Corollary 6 with [12, Theorem 5].

8. A CHARACTERIZATION OF 2-DIMENSIONAL TREES

In this section we prove that the 3-dimensional trees J_1 and J_2 constructed in Section 7 provide a forbidden subposet characterization of

2-dimensional trees by proving that every 3-dimensional tree contains at least one of J_1, J_2 or their duals as a subposet.

If x and y are distinct points in a poset X, we say that x and y have the same holdings if $z < x$ iff $z < y$ and $x < z$ iff $y < z$ for every $z \in X - \{x, y\}$. It is proved in [9] that if x and y have the same holdings in X, then $\mathrm{Dim}(X) = \mathrm{Dim}(X - x) = \mathrm{Dim}(X - y)$ unless $X - x$ is a chain and in this case $\mathrm{Dim}(X) = 2$.

If T is a tree and x a cut vertex of $G(T)$, let $N(x)$ denote the set of vertices of $G(T)$ which are adjacent to x in $G(T)$. If $y \in N(x)$ we also denote the set of vertices $\{z$: the unique path from x to z in $G(T)$ contains $y\}$ by $C_y(x)$ and call $C_y(x)$ a component. If $x < y$ in T we say $C_y(x)$ is an upper component of $T - x$; otherwise we say $C_y(x)$ is a lower component of $T - x$. If $C_y(x) = \{y\}$, we call $C_y(x)$ a degenerate component. If $C_y(x)$ is an upper component, we say $C_y(x)$ is a uniform component if $x < z$ in T for every $z \in C_y(x)$. Similarly a lower component $C_y(x)$ is said to be uniform if $z < x$ in T for every $z \in C_y(x)$.

Let $\mathscr{C}$ be the collection of all 3-dimensional trees which do not contain one or more of J_1 or J_2 or their duals as subposets. Then let n be the minimum number of vertices of any tree in $\mathscr{C}$.

Lemma 7. *If x is a cut vertex of a tree $T \in \mathscr{C}$ on n vertices, $y \in N(x)$, and $C_y(x)$ is uniform, then $C_y(x)$ is degenerate.*

Proof. If $C_y(x)$ is nondegenerate, it is easy to see that there exists a distinct pair of end vertices $z, w \in C_y(x)$ which have the same holdings in T. Since $\mathrm{Dim}(T - z) = \mathrm{Dim}(T)$ and z is an end vertex of T, $T - z$ is a tree and the result follows.

Lemma 8. *If x is a cut vertex of a tree $T \in \mathscr{C}$ on n vertices and $|N(x)| \geq 4$, then $|N(x)| = 4$, x has two upper components exactly one of which is degenerate and two lower components exactly one of which is degenerate.*

Proof. No pair of end vertices in T can have the same holdings. Therefore x cannot have more than one degenerate upper component or more than one degenerate lower component. If x has only one upper component, $C_y(x)$, and it is degenerate, then x and y have the same holdings so $\mathrm{Dim}(T - y) = \mathrm{Dim}(T)$ but $T - y$ is a tree.

If x has three or more nondegenerate upper components $C_{y_1}(x)$, $C_{y_2}(x)$, $C_{y_3}(x)$, we may choose points $z_1, w_1, z_2, w_2, z_3, w_3$ such that $x < z_1$, $x < z_2$, $x < z_3$, $w_1 < z_1$, $w_2 < z_2$, and $w_3 < z_3$ in T. But this implies that T contains J_2 as a subposet. Similarly if x has three or more nondegenerate lower components, then T contains $\hat{J}_2$ as a subposet. If x has two or more nondegenerate upper components and one or more nondegenerate lower components, then T contains J_1. Finally if x has two or more nondegenerate lower components and one or more nondegenerate upper components then T contains $\hat{J}_1$.

THEOREM 4. *If T is a tree, then* $\mathrm{Dim}(T) \leqslant 2$ *unless T contains one or more of the trees J_1 and J_2 or their duals as subposets.*

Proof. Let T be a tree from $\mathscr{C}$ having n vertices. The argument used in the preceding lemma shows that no cut vertex has more than two nondegenerate components. Now let W be the tree formed from T by adding to each cut vertex in T a degenerate upper component if it does not have one and a degenerate lower component if it does not have one. The Hasse graph of W has the appearance of the tree shown in Fig. 10.

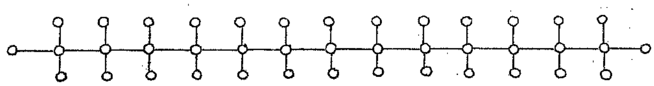

FIGURE 10

With this observation, it is easy to see that the poset formed from W by adding both a zero and a one is planar. By the remarks of Section 5, we conclude that $\mathrm{Dim}(W) \leqslant 2$. Since T is a subposet of W, we have $\mathrm{Dim}(T) \leqslant 2$. The contradiction completes the proof of our Theorem.

If X and Y are posets we say that X is a homeomorph of Y if $X = Y$ or if a Hasse diagram of X can be formed by inserting one or more vertices in an edge or edges of a Hasse diagram for Y. If D is a Hasse diagram of a poset X, then any diagram E formed by deleting edges in D or deleting vertices and incident edges is called a subdiagram of D. A subdiagram E of a Hasse diagram D of a poset X is always a Hasse diagram for some poset, but not necessarily a subposet of X. With the notions of homeomorph and subdiagram we can restate Theorem 2 in a form analogous to Kuratowski's Theorem.

THEOREM 5. *If D is a Hasse diagram for a tree T, then* $\mathrm{Dim}(T) \leqslant 2$ *unless D contains a subdiagram which a homeomorph of the Hasse diagrams of one of the posets J_1, J_2, $\hat{J}_1$, or $\hat{J}_2$.*

9. SOME COMMENTS ON THE ORIGIN OF THE PROBLEM

In 1972 R. Kimble discovered a natural device for transforming the computation of the dimension of an arbitrary poset X into the computation of the dimension of a poset of height one (the height of a poset is one less than the maximum number of vertices in a chain). For a poset X, he defined the split of X, denoted $S(X)$, as the poset whose point set is $\{x' : x \in X\} \cup \{x'' : x \in X\}$ with partial order defined by $y'' < z'$ in $S(X)$ iff $y \leqslant z$ in X. Kimble showed that $\operatorname{Dim}(X) \leqslant \operatorname{Dim} S(X) \leqslant 1 + \operatorname{Dim}(X)$ and asked if a poset is split repeatedly, can the dimension increase arbitrarily. It follows from Corollary 6 that repeated splitting of a poset can increase the dimension of a poset by at most two. Furthermore, this result is best possible.

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