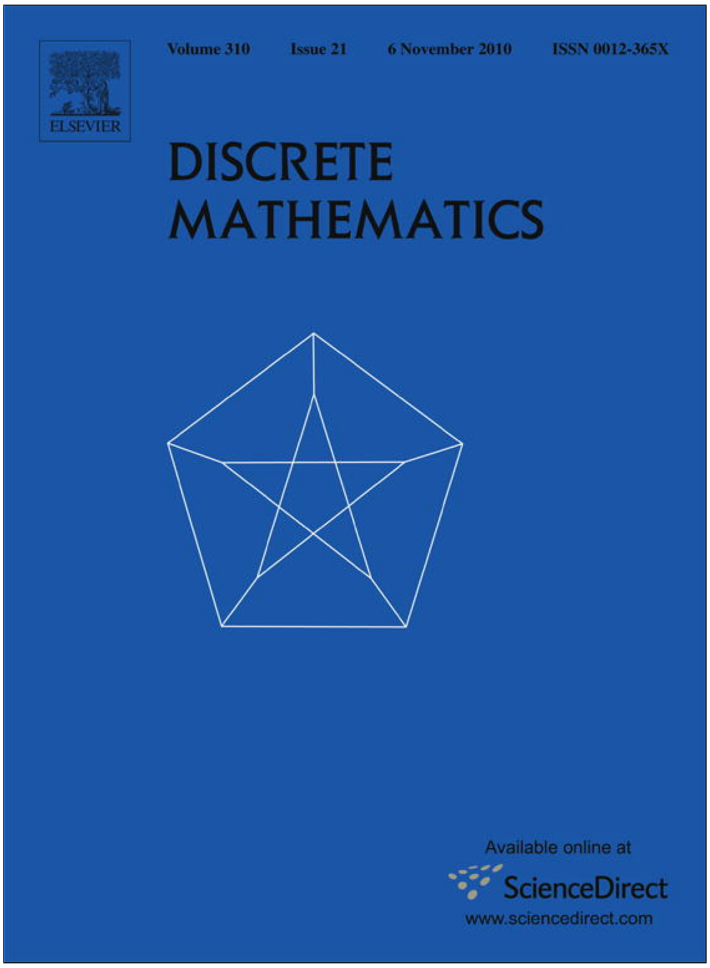
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On the size of maximal antichains and the number of pairwise disjoint maximal chains

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ABSTRACT

Fix integers n and k with $n \geq k \geq 3$. Duffus and Sands proved that if P is a finite poset and $n \leq |C| \leq n+(n-k)/(k-2)$ for every maximal chain in P, then P must contain k pairwise disjoint maximal antichains. They also constructed a family of examples to show that these inequalities are tight. These examples are two-dimensional which suggests that the dual statement may also hold. In this paper, we show that this is correct. Specifically, we show that if P is a finite poset and $n \leq |A| \leq n+(n-k)/(k-2)$ for every maximal antichain in P, then P has k pairwise disjoint maximal chains. Our argument actually proves a somewhat stronger result, and we are able to show that an analogous result holds for antichains.

1. Introduction

Let k be an integer with $k \geq 2$. In [1], Duffus and Sands investigated conditions that force a finite poset to have k-pairwise disjoint maximal antichains. They noted that the case $k = 2$ is easy to solve, since in order to have 2 pairwise disjoint maximal antichains, it is necessary and sufficient that P not contain a point which is incomparable with all other points, i.e., P cannot have a trivial maximal chain consisting of a single point.

But the situation when $k \geq 3$ is more complicated, as reflected in the following intriguing result [1].

Theorem 1.1 (*Duffus and Sands*). *Let n and k be integers with $n \geq k \geq 3$, and let P be a finite poset. If $n \leq |C| \leq n + (n-k)/(k-2)$, for every maximal chain C in P, then P has k pairwise disjoint maximal antichains.*

For each pair n and k with $n \geq k \geq 3$, Duffus and Sands also constructed a poset $P(n, k)$ satisfying the following properties:

(1) If C is a maximal chain in $P(n, k)$, then $n \leq |C| \leq n + 1 + \lfloor (n-k)/(k-2) \rfloor$.
(2) $P(n, k)$ does not have k pairwise disjoint maximal antichains.

These examples show that the inequality in Theorem 1.1 is best possible.

Duffus and Sands also initiated an investigation of the dual problem: conditions that force a poset to have k pairwise disjoint maximal chains. When $k = 2$, they proved that a poset P has 2 pairwise disjoint chains if and only if it does not contain a point which is comparable with all other points, i.e., P cannot have a trivial maximal antichain consisting of a single point.

They also noted that for each pair n and k with $n \geq k \geq 3$, the poset $P(n, k)$ has dimension 2. As a consequence, there is a complementary poset $Q(n, k)$ such that:

(1) If A is a maximal antichain in $Q(n, k)$, then $n \leq |A| \leq n + 1 + \lfloor (n-k)/(k-2) \rfloor$.
(2) $Q(n, k)$ does not have k pairwise disjoint maximal chains.

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Accordingly, it is natural to ask whether the dual form of the Duffus/Sands theorem holds, and the principal goal of this paper will be to provide an affirmative answer by proving the following theorem.

Theorem 1.2. *Let n and k be integers with $n \geq k \geq 3$, and let P be a finite poset. If $n \leq |A| \leq n + (n-k)/(k-2)$ for every maximal antichain A in P, then P has k pairwise disjoint maximal chains.*

As the reader will note, the argument we present for Theorem 1.2 is completely different from the argument given by Duffus and Sands for Theorem 1.1. In fact, we will actually prove a more technical and somewhat stronger result and derive our principal theorem as a corollary. In Section 4, we will then show how a dual version of our technical result can be proved for families of pairwise disjoint maximal antichains. As a consequence, we provide here an alternative proof of their main theorem.

Although the results of this paper are very much in the spirit of Dilworth's theorem and its dual form, as well as the classic theorems of Greene [2] and Greene and Kleitman [3], we have not been able to establish any direct connection.

2. Cutsets and support structures

Let P be a finite poset. In discussions concerning families of pairwise disjoint maximal chains in P, we find it useful to apply well-known concepts and techniques from network flows. In particular, we will employ the following basic proposition.

Proposition 2.1. *The maximum number of pairwise disjoint maximal chains in P equals the minimum cardinality of a set intersecting all maximal chains in P.*

In view of Proposition 2.1, the following notation and terminology becomes natural. We will say that a chain C in a finite poset P is *saturated* if either $|C| = 1$ or if $|C| = r > 1$ and $C = \{x_1 < x_2 < \cdots < x_r\}$, then x_i is covered by x_{i+1} for each $i = 1, 2, \ldots, r-1$.

A saturated chain in P whose least element is a minimal element of P will be called an *initial* chain. Dually, a saturated chain whose greatest element is a maximal element of P will be called a *terminal* chain. A maximal chain is always saturated and is both an initial chain and a terminal chain. Trivially, for every point u in P, there is an initial chain whose greatest element is u, and there is a terminal chain whose least element is u. The union of these two chains is a maximal chain containing u.

Note also that whenever $u < v$ in P, there is always a saturated chain C with u the least element of C and v the greatest element of C. We say such a chain is a *linking* chain for u and v.

Let P be a finite poset and let W be a subset of P that intersects all maximal chains in P. We will refer to W as a *cutset* in P. Next, we will develop some additional structural information concerning cutsets.

First, recall that the *height* of an element x in a finite poset P, denoted $h_P(x)$, is the largest integer t for which there exists a chain of t elements in P with x the greatest element of this chain. Also, the *height* of the poset P is just the cardinality of a maximum chain in P.

Now let s be a positive integer, and let W be an s-element cutset in P. Then let r be the height of the subposet W, and let $W = W_1 \cup W_2 \cup \cdots \cup W_r$ be the partition of W determined by setting $W_i = \{w \in W : h_W(w) = i\}$, for each $i = 1, 2, \ldots, r$. Then for each $i = 1, 2, \ldots, r$, let A_i be the maximal elements of the set $\{x \in P : x \not> w, \text{for all } w \in W_i\}$. It is obvious that A_i is a maximal antichain in P and that $W_i \subseteq A_i$. Paralleling the discussion in [1], we refer to the maximal antichains in the family $\{A_i : 1 \leq i \leq r\}$ as *flat* antichains. Note that A_i and A_j need not be disjoint when $i \neq j$. However, the following important property does hold.

Claim 1. If $1 \leq i < j \leq r$, $u \in A_i$ and $v \in A_j$, then $u \not> v$ in P.

Proof. Suppose to the contrary that $u > v$ in P. Since v is a maximal element of the set $\{x \in P : x \not> w\}$, for all $w \in W_j$, then there exists some element $w \in W_j$ with $u > w$ in P. Since $i < j$, there is then some element $w' \in W_i$ with $w > w'$ in P. By transitivity, this implies that $u > w'$ in P with both u and w' belonging to the antichain A_i. The contradiction completes the proof of the claim. □

Let u be an element of P. We say u is *reachable* if there is an initial chain C having u as its greatest element so that $C \cap W = \emptyset$. Evidently, no point of W is reachable. Also, all minimal elements of P that do not belong to W are reachable. On the other hand, no maximal element of P is reachable, as this would imply that there is a maximal chain in P that does not intersect W.

For each $i = 1, 2, \ldots, r$, let R_i denote the set of reachable points in the antichain A_i, and let $N_i = A_i - W_i - R_i$. Elements of N_i are not reachable.

Claim 2. $N_1 = R_r = \emptyset$.

Proof. Suppose first that $N_1 \neq \emptyset$, and let $u \in N_1$. We show that u is reachable. As noted previously, this statement holds trivially if u is a minimal element of P, so we may assume u is not a minimal element. Then let C an initial chain in P having u as its greatest element. We show that $C \cap W = \emptyset$ and thus that u is reachable. Suppose to the contrary that $w \in C \cap W$. Since $u \in A_1$ and $w < u$, we know that $w \notin A_1$. Thus, $w \in W_j$ for some j with $1 < j \leq r$, which then contradicts Claim 1. We conclude that $N_1 = \emptyset$.

Now suppose that $R_r \neq \emptyset$, and let $u \in R_r$. Choose an initial chain C whose greatest element is u so that $C \cap W = \emptyset$. Then let D be any terminal chain with u the least element of D. Just as before, we conclude that $D \cap W = \emptyset$, which would imply that $C \cup D$ is a maximal chain in P that does not intersect W. The contradiction shows $R_r = \emptyset$. □

Claim 3. For each $i = 1, 2, \ldots, r-1$, R_i and N_{i+1} are disjoint sets and $R_i \cup N_{i+1}$ is an antichain in P.

Proof. The two sets are evidently disjoint, since points in R_i are reachable, while points in N_{i+1} are not. We now show that $R_i \cup N_{i+1}$ is an antichain in P. Suppose to the contrary that it is not. Then it is clear that there must exist elements $u \in R_i$ and $v \in N_{i+1}$ that are comparable in P. By Claim 1, this requires $u < v$ in P. Let C be an initial chain having u as its greatest element with $C \cap W = \emptyset$, and let D be a saturated chain linking u and v. Then $C' = C \cup D$ is an initial chain in P with v as its greatest element. If $C' \cap W = \emptyset$, then v is reachable. The contradiction shows that there must exist some $w \in C' \cap W$. Clearly this implies that $u < w < v$ in P. However, the fact that $u \in A_i$ implies (using Claim 1) that w cannot belong to $W_1 \cup W_2 \cup \cdots \cup W_i$. On the other hand, the fact that $v \in A_{i+1}$ implies that w cannot belong to $W_{i+1} \cup W_{i+2} \cup \cdots \cup W_r$. The contradiction completes the proof that $R_i \cup N_{i+1}$ is an antichain in P. □

Again, paralleling the discussion in [1], we refer to the antichains in the family $\mathcal{S} = \{R_i \cup N_{i+1} : 1 \leq i \leq r-1\}$ as *slanted* antichains. Note that slanted antichains need not be maximal. Also, we refer to the family $\{A_i = W_i \cup R_i \cup N_i : 1 \leq i \leq r\}$ as the *support structure* for the cutset W in P. Strictly speaking, the support structure of a cutset W is determined entirely by W and P, but we find it useful to carry along the additional information given by the family of flat antichains, and the set of reachable elements.

3. Proof of the principal theorem

We now have the tools necessary to provide Theorem 1.2. As noted previously, we elect to prove a more technical and somewhat stronger result and derive Theorem 1.2 as a corollary. First, recall that the *width* of a poset P is the maximum cardinality of an antichain in P.

Theorem 3.1. *Let P be a poset, let s denote the maximum number of pairwise disjoint maximal chains in P, and let W be an s-element cutset in P. If the height of W is r, the width of P is t and $n = \min\{|A_i| : 1 \leq i \leq r\}$, then the following inequality holds:*

$$rn \leq s + t(r-1). \tag{1}$$

Proof. Let $\{A_i = W_i \cup R_i \cup N_i : 1 \leq i \leq r\}$ be the support structure of W. Since $|N_1| = |R_r| = 0$, it is immediate that

$$\sum_{i=1}^{r} |A_i| = s + \sum_{i=1}^{r} |R_i| + |N_i| = s + \sum_{i=1}^{r-1} |R_i \cup N_{i+1}|.$$

Since $|A_i| \geq n$ for each $i = 1, 2, \ldots, r$ and $|R_i \cup N_{i+1}| \leq t$, for each $i = 1, 2, \ldots, r-1$, inequality 1 follows. □

To see how our main theorem now follows easily as a corollary to Theorem 3.1, let n and k be integers with $n \geq k \geq 3$. Then let P be a finite poset in which every maximal antichain has at least n elements, and suppose that the width t of P is at most $n + (n-k)/(k-2)$. If P does not have k pairwise disjoint chains, then there is some positive integer s with $s < k$ for which there is an s-element cutset W in P. Let r denote the height of W and let t denote the width of P. From Theorem 3.1, we know that $rn \leq s + t(r-1)$, and this inequality may be rewritten as $t \geq n + (n-s)/(r-1)$. Since $r \leq s$ and $s \leq k-1$, this implies

$$t \geq n + \frac{n-s}{r-1} \geq n + \frac{n-s}{s-1} \geq n + \frac{n-k+1}{k-2}.$$

This is a contradiction, since $t \leq n + (n-k)/(k-2)$, and this remark completes the proof.

4. Some notes on the original problem

It is worth noting that the approach we have followed in proving Theorems 3.1 and 1.2 cannot be applied (at least not without modification) to the original problem studied by Duffus and Sands. The reason is that the dual version of Proposition 2.1 is not valid. Specifically, it is not true that the maximum number of pairwise disjoint antichains in a finite poset P equals the minimum cardinality of a set intersecting all maximal antichains in P.

Lemma 4.1. *For every $n \geq 2$, there exists a poset P_n in which the maximum number of pairwise disjoint antichains is 2, but the minimum cardinality of a set of points intersecting all maximal antichains is $2n$.*

Proof. Consider a finite projective plane F_n of order n. Let X denote the set of points in F_n and let Y denote the set of lines in F_n. Then $|X| = |Y| = n^2 + n + 1$; each line contains $n+1$ points; each point is on $n+1$ lines; each pair of distinct points determines a unique line; and each pair of distinct lines intersect in a unique point.

We construct a poset P_n of height 2 with X as the set of minimal elements and Y as the set of maximal elements. Furthermore, if $x \in X$ and $y \in Y$, we set $x < y$ in P_n if and only if point x is not on line y in F_n.

It follows immediately that in addition to the set of minimal elements and the set of maximal elements, P_n has $2(n^2+n+1)$ other maximal antichains. They come in two different types, with Type 1 corresponding to points in F_n and Type 2 corresponding to lines in F_n.

Type 1: For each $x \in X$, the set $A_x = \{x\} \cup \{y \in Y : x \not< y\}$ is a maximal antichain.
Type 2: For each $y \in Y$, the set $B_y = \{y\} \cup \{x \in X : x \not< y\}$ is a maximal antichain. □

Claim 1. The maximum number of pairwise disjoint maximal antichains in P_n is two.

Proof. It is easy to see that the poset P_n has 2 pairwise disjoint maximal antichains, for example, the set of minimal elements and the set of maximal elements. We now show that P_n does not have 3 pairwise disjoint maximal antichains.

Suppose to the contrary that $\mathcal{F} = \{I_1, I_2, I_3\}$ is a family of 3 pairwise disjoint maximal antichains in P_n. Then at most one antichain in $\mathcal{F}$ is a Type 1 antichain, since if x and x' are distinct points, then the line y which they determines belongs to A_x and to $A_{x'}$. Dually, at most one member of $\mathcal{F}$ is a Type 2 antichain.

So one of the members of $\mathcal{F}$ is either the set of minimal elements or the set of maximal elements. But in this case, no member of $\mathcal{F}$ can be either a Type 1 or a Type 2 antichain, since each of these contains both a minimal element and a maximal element. The contradiction completes the proof of the claim. □

Claim 2. The minimum cardinality of a set intersecting all maximal antichains in P_n is $2n$.

Proof. Let W be a set which intersects all maximal antichains in P_n. There are $2(n^2+n+1)$ antichains of Types 1 and 2, but any element of P_n belongs to exactly $n+2$ maximal antichains from these two types. This implies that

$$|W| \geq \left\lceil \frac{2(n^2+n+1)}{n+2} \right\rceil \geq 2n-1 \geq 3.$$

Furthermore, if x and x' are distinct points, then there is one maximal antichain of Type 2 to which both belong. Also, if y and y' are distinct lines, then there is one maximal antichain of Type 1 to which both belong. An easy calculation shows that if W consists entirely of minimal elements or entirely maximal elements, then $|W| \geq 2n+1$. But if W contains at least one maximal element and at least one minimal element, then $|W| \geq 2n$.

We now show that there is a set W with $|W| = 2n$ so that W intersects every maximal antichain in P_n. Choose a point x_0 and a line y_0 which passes through x_0. Then $W = \{x \in X : x \not< y_0, x \neq x_0\} \cup \{y \in Y : x_0 \not< y, y \neq y_0\}$ contains $2n$ elements. We now show that if I is a maximal antichain in P_n, then W intersects I. This is certainly true if I is either the set of maximal elements or the set of minimal elements. If $I = A_x$ is a Type 1 antichain, and x is not on the line y_0 in F_n, then $x \neq x_0$, and if y is the line determined by x and x_0, then $y \neq y_0$. It follows that y belongs to the antichain I as well as to W. On the other hand, if x is on the line determined by y_0 and $x \neq x_0$, then x belongs to I and to W. Furthermore, if $x = x_0$, then $y \in W \cap I$, for every line y passing through x_0, with $y \neq y_0$.

A dual argument shows that W intersects every Type 2 antichain, and with this observation, the proof of the claim is complete. □

In spite of the apparent difficulties presented by Lemma 4.1, there is a natural framework within which we can derive a dual version of Theorem 3.1 and then proceed to derive the Duffus/Sands result as a corollary.

Let P be a finite poset and let t denote the height of P. Then, for each $i = 1, 2, \ldots, t$, let $L_i = \max\{x : h_P(x) \leq i\}$. We refer to $\{L_i : 1 \leq i \leq t\}$ as the family of *level* antichains in P. It is straightforward to verify that each level antichain is a maximal antichain. Furthermore, we have the following important property:

Proposition 4.2. *If $1 \leq i < j \leq r$, $u \in L_i$ and $v \in L_j$, then $u \not> v$ in P.*

We then have the following basic result.

Theorem 4.3. *The maximum number of pairwise disjoint level antichains is equal to the minimum number of points in a set intersecting all of them.*

Proof. We show that there is a partition $\{1, 2, \ldots, t\} = B_1 \cup B_2 \cup \cdots \cup B_s$, so that for each $p = 1, 2, \ldots, s$:

(1) $B_p = [b_p, c_p]$ is a block of consecutive integers with $b_p = 1 + c_{p-1}$ when $p > 1$.
(2) There is a point x_p common to all antichains in $\{L_i : i \in B_p\}$.
(3) If $c_p < i \leq t$, then $L_i \cap L_{b_p} = \emptyset$.

Once this partition has been constructed, we will then have a family $\{L_{b_p} : 1 \leq p \leq s\}$ of s pairwise disjoint maximal antichains and an s-element set $W = \{x_p : 1 \leq p \leq s\}$ which intersects all level antichains.

The construction proceeds inductively Set $c_0 = 0$. Suppose for some $p \geq 1$, we have a value of c_{p-1} and if $p \geq 2$, the properties listed above hold for the blocks $B_1, B_2, \ldots, B_{p-1}$. If $c_{p-1} < t$, set $b_p = 1 + c_{p-1}$ and let c_p be the largest integer for which $c_p \leq t$ and $L_{b_p} \cap L_{c_p} \neq \emptyset$. Then choose x_p as an element from $L_{b_p} \cap L_{c_p}$. It follows from Proposition 4.2 that x_p belongs to every antichain in $\{L_i : i \in B_p\}$. Furthermore, if $c_p < i \leq t$, then $L_i \cap L_{b_p} = \emptyset$. □

Now we can state and prove a dual version for Theorem 3.1

Theorem 4.4. *Let P be a poset, let s denote the maximum number of pairwise disjoint antichains in the family of level antichains in P, and let W be an s-element set intersecting all level antichains in P. Let r be the width of W and let $C_1, C_2, \ldots, C_r$ be maximal chains in P that cover W. If $n = \min\{|C_i| : 1 \le i \le r\}$, then the following inequality holds:*

$$rn \le s + t(r-1). \tag{2}$$

Proof. Let $x \in W$ and let $B = [b, c]$ be the set of consecutive integers from $\{1, 2, \ldots, t\}$ so that $x \in L_j$ if and only if $j \in B$. It follows that $h_P(x) = b$. Furthermore, if $x \in C_i$, then there are no points in C_i that have height j where $b < j \le c$. Since $|C_i| \ge n$ for each $i = 1, 2, \ldots, r$ and we have eliminated points at all heights from $\{1, 2, \ldots, t\}$, except for the heights of elements of W, we conclude that $rn \le rt - t + s$, which is equivalent to inequality 2. □

Note that Theorem 1.1 again follows immediately from this more technical result.

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