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A combinatorial approach to height sequences in finite partially ordered sets

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ABSTRACT

Fix an element x of a finite partially ordered set P on n elements. Then let $h_i(x)$ be the number of linear extensions of P in which x is in position i, counting from the bottom. The sequence $\{h_i(x) : 1 \le i \le n\}$ is the *height sequence* of x in P. In 1982, Stanley used the Alexandrov–Fenchel inequalities for mixed volumes to prove that this sequence is log-concave, i.e., $h_i(x)h_{i+2}(x) \le h_{i+1}^2(x)$ for $1 \le i \le n-2$. However, Stanley's elegant proof does not seem to shed any light on the error term when the inequality is not tight; as a result, researchers have been unable to answer some challenging questions involving height sequences in posets. In this paper, we provide a purely combinatorial proof of two important special cases of Stanley's theorem by applying Daykin's inequality to an appropriately defined distributive lattice. As an end result, we prove a somewhat stronger result, one for which it may be possible to analyze the error terms when the log-concavity bound is not tight.

1. Introduction

Let P be a finite partially ordered set (poset) on n elements, and let $\mathcal{E}(P)$ denote the family of all linear extensions of P. For an element x of P and a linear extension $L \in \mathcal{E}(P)$, let $h_L(x) = |\{y : y \le x \text{ in } L\}|$. Note that $h_L(x)$ is just the position of x in the linear order L, starting the count from the bottom. When i is a positive integer with $1 \le i \le n$, we then set $h_i(x) = |\{L \in \mathcal{E}(P) : h_L(x) = i\}|$. The sequence $\{h_i(x) : 1 \le i \le n\}$ is called the *height sequence* of the element x in P.

In a paper that time has shown to be fundamentally important to the combinatorial theory of partially ordered sets, Stanley [10] used the Alexandrov–Fenchel inequalities for mixed volumes to prove the following result.

Theorem 1.1 (*Stanley*). *For each element x in a finite poset P, the height sequence of x in P is log-concave, i.e.,*

$$h_i(x)h_{i+2}(x) \le h_{i+1}^2(x)$$

for all i with $1 \le i \le n-2$.

For the remainder of the paper, we will fix a finite poset P and an element x of P. We then let $I(x)$ denote the set of elements in P that are incomparable with x in P. For each $1 \le i \le n$, let $H_i = \{L \in \mathcal{E}(P) : h_L(x) = i\}$. Since we have fixed the element x from P, we write h_i rather than $h_i(x)$.

The principal result of this paper will be a combinatorial proof of Stanley's theorem in the case where $I(x)$ is a chain, or a two-element antichain.

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Theorem 1.2 (*Special Case of Stanley's Theorem*)**.** *Let P be a finite poset, and let x be an element of P. If $I(x)$ is a chain or a two-element antichain, then the height sequence of x in P is log-concave, i.e.,*

$$h_i h_{i+2} \le h_{i+1}^2$$

for all i with $1 \le i \le n-2$.

In fact, our proof yields a somewhat stronger result than is stated in Theorem 1.2, and it is our hope that this stronger result (and the proof techniques we introduce to prove it) will have a broader range of applications.

2. Motivation for our research

Stanley's theorem is equivalent to the assertion that there is an injection from $H_i \times H_{i+2}$ into $H_{i+1} \times H_{i+1}$, when $1 \le i \le n-2$. So it is natural to ask whether one could construct such an injection in a purely combinatorial manner. While finding a combinatorial proof of Stanley's theorem seems a worthwhile goal just in terms of gaining a better understanding of the combinatorics of posets, this has not been the driving force for our research. Instead, we are attempting to answer questions for which current techniques do not seem sufficient. In order to put this issue into perspective, we pause to summarize briefly some closely related research.

For distinct elements x and y in a poset P, let $\Pr[x > y]$ denote the number of linear extensions of P in which x is greater than y divided by the total number of linear extensions. In [8,9], Shepp used a correlation inequality due to Fortuin et al. [5] (a result now known as the FKG Inequality) to prove the following theorem.

Theorem 2.1 (*The XYZ Theorem*)**.** *If x, y, and z are distinct elements in a poset P, then*

$$\Pr[x > y | x > z] \ge \Pr[x > y].$$

The XYZ Theorem was first conjectured by Rival and Sands, who noted that the inequality holds trivially unless x, y, and z form a three-element antichain in P. They further conjectured that the inequality is strict in this case, a subtlety that does not follow from Shepp's approach.

Subsequently, Fishburn [4] used a generalization of the FKG Inequality due to Ahlswede and Daykin [1] to prove the strong form of the XYZ Theorem, i.e., the inequality in the XYZ Theorem is strict when x, y, and z form a three-element antichain. In fact, Fishburn's theorem also provides the error term in the inequality and characterizes those posets for which the strict inequality – with error term – is tight.

The FKG Inequality and the generalization due to Ahlswede and Daykin (also called the Four Functions Theorem) have been used by several authors (see [3,6,7], for example) to prove correlation inequalities for partially ordered sets.

By way of contrast, Stanley's convex geometry approach does not seem to provide techniques for analyzing the error term when the inequality is not tight. To explain why we are concerned with error terms, we discuss briefly a challenging problem posed by Kahn (personal communication). Let x be an element in a finite poset P with $|P| = n$. The *average* height of $x \in P$ is

$$h(x) = \frac{\sum_{i=1}^{n} i h_i(x)}{\sum_{i=1}^{n} h_i(x)}.$$

Also, for each element $x \in P$, let $D[x] = \{y \in P : y \le x \text{ in } P\}$.

Kahn made the following conjecture.

Conjecture 2.2. *Let x and y be elements of a finite poset P. If $m = |D[x] \cup D[y]|$, then*

$$\max\{h(x), h(y)\} \ge m - 1.$$

Kahn observed that his conjecture follows from Stanley's theorem when $|P| = m$, i.e. when all maximal elements of P are in $\{x, y\}$. To see this, let $n = |P|$ and let $e(P)$ be the number of linear extensions. It is clear that for one of x or y, $h_n \ge e(P)/2$. Using the fact that $\sum_{i=1}^{n} h_i = e(P)$ and that the sequence h_i is log-concave, one can argue that the average height is bounded below by the case when $h_i = e(P)/2^{n-i+1}$. So

$$\max\{h(x), h(y)\} \ge \frac{\sum_{i=1}^{n} i \frac{e(P)}{2^{n-i+1}}}{e(P)} = n - 1 + \frac{1}{2^n}.$$

When $n > m$ and additional elements are present in the poset, we still observe that one of x and y is in position m (or higher) in at least half the linear extensions of P. One might think that this would force the average height of such an element to be at least $m - 1$, but log-concavity alone only guarantees that it be at least $m \ln 2$, which is about $0.7m$.

It is reasonable to believe that if we had a better understanding of the behavior of the error terms in Stanley's inequality, we might have some chance of resolving Kahn's conjecture. Indeed, this over-arching goal motivated Brightwell and Trotter to investigate combinatorial approaches to correlation inequalities as presented in [2]. While their research managed to eliminate the role of the FKG Inequality and the Four Functions Theorem in the proof of the strong form of the XYZ Theorem (including the error term analysis), the FKG Inequality and the Four Functions Theorem are generally considered as part of a combinatorial mathematician's toolkit. With this perspective in mind, we derive a special case of Stanley's theorem as a consequence of the Four Functions Theorem.

The remainder of the paper is organized as follows. In Section 3, we develop some essential background material, including the concept of a Shepp lattice. Then, in Sections 4 and 5, we present the proof of our principal theorem.

3. Background material

For completeness, we state the Four Functions Theorem of Ahlswede and Daykin [1]. In presenting this result, we use $\mathbb{R}_0$ to denote the set of all nonnegative real numbers. When L is a lattice and f is a function mapping L to $\mathbb{R}_0$, we let $f(X) = \sum_{x \in X} f(x)$. And when X and Y are subsets of L, we define:

$$X \wedge Y = \{x \wedge y : x \in X, y \in Y\}$$
$$X \vee Y = \{x \vee y : x \in X, y \in Y\}.$$

Theorem 3.1 (*Ahlswede and Daykin*)**.** *Let L be a distributive lattice, and let α, β, γ and δ be four functions mapping L to $\mathbb{R}_0$. If*

$$\alpha(x)\beta(y) \le \gamma(x \wedge y)\delta(x \vee y)$$

for all $x, y \in L$, then

$$\alpha(X)\beta(Y) \le \gamma(X \wedge Y)\delta(X \vee Y)$$

for all subsets X and Y of L.

Daykin's inequality is just the special case of Theorem 3.1 when all four functions are the constant function mapping all elements of L to 1. With this interpretation, the conclusion of the theorem becomes

$$|X||Y| \le |X \wedge Y||X \vee Y|$$

and this is the result we will need in this paper.

3.1. Constructing distributive lattices

For a positive integer k, let $\mathbf{k}$ denote the k-element chain $\{0 < 1 < \cdots < k-1\}$. Now let P be a finite poset and let L denote the set of all order-preserving maps from P to $\mathbf{k}$. It is natural to define a partial order on L by setting $f \le g$ if and only if $f(x) \le g(x)$ for all elements x in P. It is easy to see that under this partial order L is in fact a distributive lattice, with the meet $f \wedge g$ and join $f \vee g$ of elements f and g from L defined as follows:

$$(f \wedge g)(x) = \min\{f(x), g(x)\} \quad \text{and} \quad (f \vee g)(x) = \max\{f(x), g(x)\}$$

for all elements x in P.

In [8,9], Shepp applied the FKG Inequality to a distributive lattice defined on the same set of order-preserving functions from P to $\mathbf{k}$, but with an alternative partial order and consequently, alternative notions of meets and joins. First, he fixed an element x_0 from the poset P as a *root*, and then defined $f \le g$ when

1. $f(x_0) \ge g(x_0)$ and
2. $f(x) - f(x_0) \le g(x) - g(x_0)$, for all elements x in P with $x \ne x_0$.

It is straightforward to verify that, equipped with this definition, the set of all order preserving functions from P to $\mathbf{k}$ forms a distributive lattice with meets and joins determined as follows:

$$(f \wedge g)(x_0) = \max\{f(x_0), g(x_0)\}$$
$$(f \wedge g)(x) = \max\{f(x_0), g(x_0)\} + \min\{f(x) - f(x_0), g(x) - g(x_0)\} \quad \text{for all } x \ne x_0. \tag{1}$$

Similarly,

$$(f \vee g)(x_0) = \min\{f(x_0), g(x_0)\}$$
$$(f \vee g)(x) = \min\{f(x_0), g(x_0)\} + \max\{f(x) - f(x_0), g(x) - g(x_0)\} \quad \text{for all } x \ne x_0. \tag{2}$$

Intuitively, the meet operation pushes x_0 up, and everything else down, relative to x_0. Dually, the join operation pushes x_0 down, and everything else up, relative to x_0.

In what follows, we call this lattice the *Shepp lattice* with the root element x_0.

3.2. Algebraic properties of distributive lattices

In arguments to follow, we will take advantage of the following elementary result.

Proposition 3.2. *If L_1 and L_2 are distributive lattices, then their Cartesian product $L_1 \times L_2$ is also a distributive lattice.*

The plan is to apply Daykin's inequality to the appropriate subsets of the Cartesian product of two Shepp lattices.

4. When $I(x)$ is a chain

We fix a finite poset P and an element x from P. Suppose that $I(x) = \{y_1, \dots, y_l\}$. In this section we concentrate on the case when $I(x)$ is a chain, and we assume $y_1 < y_2 < \cdots < y_l$ in P.

We also fix a positive integer k with k much larger than $|P|$. Let Q be the poset obtained from P by removing the element x and adding a new element u with u incomparable to every element of P.

We now construct a distributive lattice L. The elements of the ground set of L are pairs of functions (f_1, f_2) such that f_1 and f_2 are order preserving functions from Q to $\mathbf{k}$.

To complete the definition of the lattice L, we define lattice operations as follows. The meet and join operations on the right hand side are based on Shepp lattices with root element u.

$$(f_1, f_2) \wedge (g_1, g_2) = (f_1 \wedge g_1, f_2 \wedge g_2)$$
$$(f_1, f_2) \vee (g_1, g_2) = (f_1 \vee g_1, f_2 \vee g_2).$$

By appealing to Proposition 3.2, and the fact that the Shepp lattice is distributive, we may conclude the following basic fact.

Proposition 4.1. *L is a distributive lattice.*

Fix i with $1 \le i \le n-2$. Let $A = \{y \in P : x < y \text{ in } P\}$, $B = \{z \in P : z < x \text{ in } P\}$ and $I(x) = \{y_1, \dots, y_l\}$. Also, let $a = |A|$, $b = |B|$, and $m = i - b$. If x is in the ith position in a linear extension, then it will be above $y_1, \dots, y_{m-1}$ and below $y_m, \dots, y_l$. In the following definition, let f be an order-preserving function from Q to $\mathbf{k}$. When we write $f(S) = c$ with S a subset of the ground set of Q and c an integer, we mean that $f(s) = c$ for all $s \in S$.

We then define subsets X and Y of the ground set of L as follows (see Fig. 1):

$$\begin{aligned}
X = \{&(f_1, f_2) : \\
&f_1(\{y_1, \dots, y_m\}) = f_1(u), \\
&f_1(B) = 0, \\
&f_2(A) = k-1, \\
&f_2(\{y_m, \dots, y_l\}) = f_2(u), \\
&f_2(u) > f_2(p) \quad \text{for all } p \in B, \\
&f_j(p_1) \neq f_j(p_2) \quad \forall p_1, p_2 \in P, j \in \{1, 2\}, \\
&\quad \text{if it is not required otherwise by the conditions above}\} \\
Y = \{&(g_1, g_2) : \\
&g_2(\{y_{m+1}, \dots, y_l\}) = g_2(u), \\
&g_2(A) = k-1, \\
&g_1(B) = 0, \\
&g_1(\{y_1, \dots, y_{m+1}\}) = g_1(u), \\
&g_1(u) < g_1(p) \quad \text{for all } p \in A, \\
&g_j(p_1) \neq f_j(p_2) \quad \forall p_1, p_2 \in P, j \in \{1, 2\}, \\
&\quad \text{if it is not required otherwise by the conditions above}\}.
\end{aligned}$$

We will apply Daykin's inequality, using the sets X and Y. The theorem tells us that $|X||Y| \le |X \vee Y||X \wedge Y|$, and we will then interpret this result in terms of linear extensions of P. First, we focus on computing $|X|$ and $|Y|$, as this is the easy part. Let X' be the set of quadruples (E_1, E_2, S_1, S_2) where E_1 is a linear extension of $A \cup \{y_m, \dots, y_l\}$, E_2 is a linear extension of the $B \cup \{y_1, \dots, y_{m-1}\}$, S_1 is an $(a + l - m + 1)$-element subset of $\{1, \dots, k-1\}$, and S_2 is a $(b + m)$-element subset of $\{0, \dots, k-2\}$. We define a bijective mapping between X and X' in the following way. For $(f_1, f_2) \in X$, let $S_1 = \text{Range}(f_1) \setminus \{0\}$, $S_2 = \text{Range}(f_2) \setminus \{k-1\}$. Let E_1 be the linear extension defined by the pre-images of the elements of S_1 in increasing order,

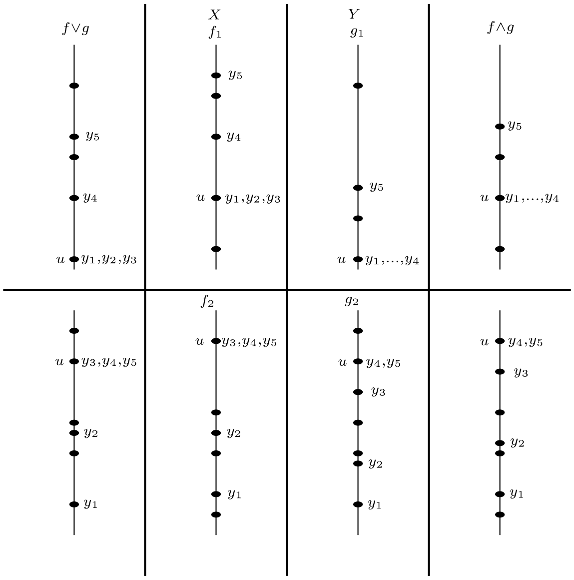

Fig. 1. Illustration of subsets $X, Y, X \vee Y$ and $X \wedge Y$. In this figure $l = 5, m = 3, a = 2$ and $b = 3$.

with the exception that instead of the pre-image of $f_1(u)$ (which would be the set $\{u, y_1, \ldots, y_m\}$), we use the single element y_m. Also, let E_2 be the linear extension defined by the pre-images of the elements of $S_2 \setminus f_2(u)$ in increasing order. Note the existence of another bijection from the set of pairs E_1, E_2 to H_i.

Define a set Y' and a similar bijection from Y to Y'. It is clear that:

$$|X| = |X'| = h_i \binom{k-1}{a+l-m+1}\binom{k-1}{b+m}$$
$$|Y| = |Y'| = h_{i+2} \binom{k-1}{a+l-m}\binom{k-1}{b+m+1}.$$

On the other hand, it is not quite so easy to write a simple formula for $|X \vee Y|$ and $|X \wedge Y|$, as there may be "collisions", i.e., cases where in taking the join or meet, two or more elements of Q wind up in the same position. However, when k is *very* large, such collisions are rare. With these remarks in mind, we note that we may write:

$$|X \vee Y| = h^*_{i+1} \binom{k-1}{a+l-m+1}\binom{k-1}{b+m} + R_\vee,$$

where h^*_{i+1} is the number of linear extensions of P such that x is in the ith lowest position *and* y_2 is in a position where y_1 could also go. This means $h^*_{i+1} \le h_{i+1}$.

Moreover, in this formula, the term $R_\vee$ is the number of elements of $X \vee Y$ in which at least one collision occurs.

Using a similar counting method and notation,

$$|X \wedge Y| = h^{**}_{i+1} \binom{k-1}{a+l-m}\binom{k-1}{b+m+1} + R_\wedge$$

with $h^{**}_{i+1} \le h_{i+1}$.

After applying the Daykin's Inequality and canceling the multiplicative terms from both sides, we have:

$$h_i h_{i+2} \le \left(h^*_{i+1} + r_\vee\right)\left(h^{**}_{i+1} + r_\wedge\right)$$

where $r_\vee = \frac{R_\vee}{\binom{k-1}{a+l-m+1}\binom{k-1}{b+m}}$ and $r_\wedge = \frac{R_\wedge}{\binom{k-1}{a+l-m}\binom{k-1}{b+m+1}}$.

It is easy to see that as $k \to \infty$, the probability of a collision tends to zero. More precisely, as $k \to \infty$, both $r_\vee \to 0$ and $r_\wedge \to 0$. Using the inequalities $h_{i+1} \le h^*_{i+1}$ and $h_{i+1} \le h^{**}_{i+1}$, the theorem follows. As promised, we actually proved a somewhat stronger result by stating the inequalities in terms of h^*_{i+1} and h^{**}_{i+1}.

5. When $I(x)$ is a two-element antichain

As in the previous section, we fix a finite poset P and an element x from P. This time we assume that $I(x)$ is a two-element antichain, i.e. $I(x) = \{y_1, y_2\}$ and $y_1 \parallel y_2$.

Among the linear extensions having x in the ith position, let $h_i(y_1 < y_2)$ be the number with $y_1 < y_2$ and $h_i(y_2 < y_1)$ be the number with $y_2 < y_1$. The inequality we need to show is

$$[h_i(y_1 < y_2) + h_i(y_2 < y_1)][h_{i+2}(y_1 < y_2) + h_{i+2}(y_2 < y_1)] \leq [h_{i+1}(y_1 < y_2) + h_{i+1}(y_2 < y_1)]^2.$$

Since

$$h_i(y_1 < y_2)h_{i+2}(y_1 < y_2) \leq h_{i+1}(y_1 < y_2)^2 \quad \text{and}$$
$$h_i(y_2 < y_1)h_{i+2}(y_2 < y_1) \leq h_{i+1}(y_2 < y_1)^2$$

by the previous section, it is sufficient to show

$$h_i(y_2 < y_1)h_{i+2}(y_1 < y_2) + h_i(y_1 < y_2)h_{i+2}(y_2 < y_1) \leq 2h_{i+1}(y_1 < y_2)h_{i+1}(y_2 < y_1).$$

We will do this by showing

$$h_i(y_2 < y_1)h_{i+2}(y_1 < y_2) \leq h_{i+1}(y_2 < y_1)h_{i+1}(y_1 < y_2) \text{ and} \tag{3}$$
$$h_i(y_1 < y_2)h_{i+2}(y_2 < y_1) \leq h_{i+1}(y_1 < y_2)h_{i+1}(y_2 < y_1). \tag{4}$$

We remark that (3) and (4) are not true for just any pair y_1 and y_2 of incomparable elements in a poset, but it is true in this particular case. The proof is very similar to the one presented in the previous section with $l = 2$.

To prove (3), we define A, B, a, b the same way as before. This time $l = 2$ and $I(x) = \{y_1, y_2\}$. The height sequence has only three nonzero elements: h_{b+1}, h_{b+2} and h_{b+3}, so we make $i = b + 1$. We modify the definition of X and Y appropriately to count the right linear extensions:

$$\begin{aligned}
X = \{(f_1, f_2) :\ & \\
& f_1(y_2) = f_1(u), \\
& f_1(y_2) < f_1(y_1), \\
& f_1(B) = 0, \\
& f_2(A) = k - 1, \\
& f_2(\{y_1, y_2\}) = f_2(u), \\
& f_2(u) > f_2(p) \quad \text{for all } p \in B, \\
& f_j(p_1) \neq f_j(p_2) \quad \forall p_1, p_2 \in P, j \in \{1, 2\}, \\
& \quad \text{if it is not required otherwise by the conditions above}\} \\
Y = \{(g_1, g_2) :\ & \\
& g_2(y_2) = g_2(u), \\
& g_2(y_1) < g_2(y_2), \\
& g_2(A) = k - 1, \\
& g_1(B) = 0, \\
& g_1(\{y_1, y_2\}) = g_1(u), \\
& g_1(u) < g_1(p) \quad \text{for all } p \in A, \\
& g_j(p_1) \neq f_j(p_2) \quad \forall p_1, p_2 \in P, j \in \{1, 2\}, \\
& \quad \text{if it is not required otherwise by the conditions above}\}.
\end{aligned}$$

Now

$$\begin{aligned}
|X| &= h_i(y_2 < y_1) \binom{k-1}{a+2} \binom{k-1}{b+1} \\
|Y| &= h_{i+2}(y_1 < y_2) \binom{k-1}{a+1} \binom{k-1}{b+2} \\
|X \vee Y| &= h_{i+1}(y_2 < y_1)^* \binom{k-1}{a+2} \binom{k-1}{b+1} + R_\vee \\
|X \wedge Y| &= h_{i+1}(y_1 < y_2)^{**} \binom{k-1}{a+1} \binom{k-1}{b+2} + R_\wedge,
\end{aligned}$$

and we finish the proof the same way as in the previous section. The inequality (4) can be shown similarly.

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