Chapter 5

Dimension for Posets and Chromatic Number for Graphs

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5.1 Introduction

We survey three important research themes involving dimension for partially ordered sets (posets). In each case, there are analogous results involving chromatic number for graphs. These themes have been chosen to highlight recent research on the combinatorics of posets and to illustrate the broad range of connections with other areas of combinatorial mathematics. All of the major results are from papers published since 2015. We outline proofs for these results, and this approach yields a number of good exercises for students. Each exercise comes with a degree of difficulty scored by one chili pepper (easy) to three chili peppers (really challenging). We also include comments on open problems for future research.

We assume readers are familiar with basic concepts for graphs, such as would be covered in an undergraduate level class in discrete mathematics. These basics include paths, cycles, components, blocks, and cut-vertices. For a graph G, we will use the following standard notation: $|G|$, $\Delta(G)$, $\omega(G)$, and $\chi(G)$ denote, respectively, number of vertices, maximum degree, maximum clique size, and chromatic number.

Partially ordered sets (posets) have become standard topics in undergraduate courses, so for a poset P, we will also also assume readers are familiar with the following basic concepts: comparable and incomparable pairs of points, comparability and incomparability graphs, covers and cover graphs, order diagrams (also called Hasse diagrams), chains and antichains, maximal and maximum chains and antichains, height and width, maximal and minimal points, and the dual of a poset. We will also assume readers know Dilworth's theorem [8] and its dual, i.e., a poset of width w can be partitioned into w chains, and a poset of height h can be partitioned into h antichains. For a poset P, we let $|P|$, $\text{width}(P)$, $\text{Min}(P)$ and $\text{Max}(P)$ denote, respectively, the number of points, width, the set of minimal elements, and the set of maximal elements.

Readers who are completely new to the subject of combinatorics on posets may find additional information in the author's monograph [48] and survey article [49].

5.1.1 Basic Concepts and Results for Dimension

When x and y are distinct incomparable points in P, we will write $x \parallel y$ in P. Also, we let $\text{Inc}(P)$ denote the set of all ordered pairs (x,y) with $x \parallel y$ in P. In some situations, we will find it convenient to shorten the phrase $x < y$ in P to $x <_P y$. Notation such as $x \leq_P y$ and $x \parallel_P y$ can then be used.

For a poset P, a family $\{L_1,\dots,L_d\}$ of linear orders on the ground set of P is called a *realizer* of P if $x \leq_P y$ if and only if $x \leq y$ in L_i for each i with $1 \leq i \leq d$. Dushnik and Miller [10] defined the *dimension* of P, denoted $\dim(P)$, to be the least positive integer d for which there is a realizer $\{L_1,\dots,L_d\}$ of P. Here are three basic properties of dimension that follow immediately from the definition: Dimension is monotonic, i.e., if Q is a subposet of P, then $\dim(Q) \leq \dim(P)$; $\dim(P) = 1$ if and only if P is a chain; and $\dim(Q) = \dim(P)$ when Q is the dual of P.

Here are two statements that are easy exercises: If $x \in P$, then $\dim(P) \leq 1 + \dim(P - \{x\})$. If P is disconnected and has components $Q_1,\dots,Q_t$, then $\dim(P) = \max\{2, \max\{\dim(Q_i) : 1 \leq i \leq t\}\}$.

For an integer $d \geq 2$, a poset P is *d-irreducible* if $\dim(P) = d$ and $\dim(P - \{x\}) = d-1$ for every element $x \in P$. A poset P is *irreducible* if P is d-irreducible for some $d \geq 2$.

A poset P is called a *bipartite poset* when the ground set of P is the union of two disjoint antichains A and B with $A \subseteq \text{Min}(P)$ and $B \subseteq \text{Max}(P)$. For an integer $d \geq 2$, let S_d be the bipartite poset with $\text{Min}(S_d) = \{a_1,\dots,a_d\}$, $\text{Max}(S_d) = \{b_1,\dots,b_d\}$, and $a_i < b_j$ in S_d if and only if $i \neq j$. Posets in the family $\{S_d : d \geq 2\}$ are called *standard examples*. Two easy exercises: $\dim(S_d) = d$ for every $d \geq 2$. The standard example S_d is d-irreducible for all $d \geq 3$. Standard examples are the poset analogues of complete graphs in graph theory.

A graph G is 2-colorable if and only if it does not contain an odd cycle, i.e., the only 3-critical graphs are the odd cycles. Testing a graph G to determine whether

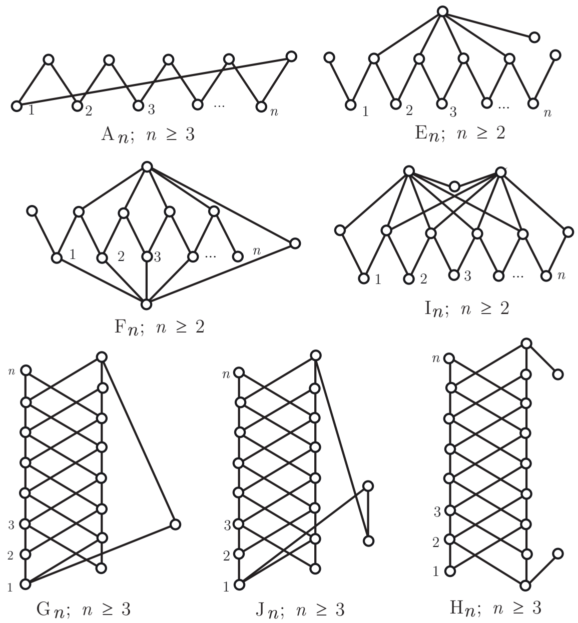

Figure 5.1
Seven Infinite Families of 3-Irreducible Posets

$\chi(G) \leq 2$ is in the class $\mathbb{P}$ of decision problems admitting polynomial time solutions. However, the decision problem $\chi(G) \leq 3$ is $\mathbb{NP}$-complete [17]. Testing a graph G to determine whether it is planar is also in the class $\mathbb{P}$ [23].

Kelly [30], and Trotter and Moore [50], working independently and using completely different methods, determined the list of all 3-irreducible posets. This list contains seven infinite families (see Figure 5.1) and 10 miscellaneous examples (see Figure 5.2).

Testing a poset P to determine whether $\dim(P) \leq 2$ is in $\mathbb{P}$ while the decision problem $\dim(P) \leq 3$ is $\mathbb{NP}$-complete [57]. By way of contrast, deciding whether a graph is a cover graph and deciding whether a poset has a planar order diagram are both $\mathbb{NP}$-complete problems [6, 18].

When I is a non-empty set of incomparable pairs in a poset P, we let $\dim(I)$ denote the least nonnegative d such that there is a family $\mathcal{F}$ consisting of d linear extensions of P such that for every pair $(x,y) \in I$, there is some $L \in \mathcal{R}$ with $x > y$ in L. When A and B are subsets of P, we let $\mathrm{Inc}(A,B) = \mathrm{Inc}(P) \cap (A \times B)$. We then abbreviate $\dim(\mathrm{Inc}(A,B))$ as $\dim(A,B)$. We are particularly interested in the value of $\dim(\mathrm{Min}(P), \mathrm{Max}(P))$, especially in the case when P is bipartite, as we have the following elementary exercise: When P is a bipartite poset,

$$\dim(\mathrm{Min}(P), \mathrm{Max}(P)) \leq \dim(P) \leq 1 + \dim(\mathrm{Min}(P), \mathrm{Max}(P)).$$

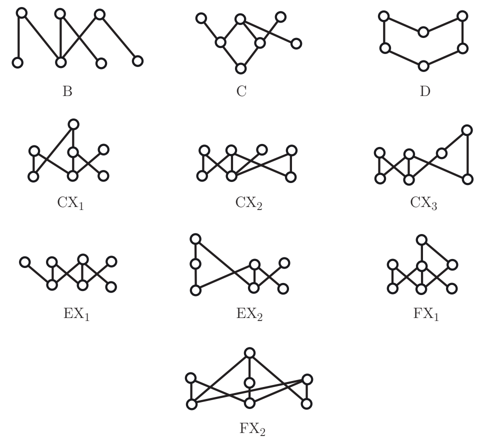

Figure 5.2
Ten Miscellaneous Examples of 3-Irreducible Posets

Given a poset P, we define * the *split of* P to be the bipartite poset Q with $\mathrm{Min}(Q) = \{x' : x \in P\}$, $\mathrm{Max}(Q) = \{x'' : x \in P\}$, and $x' <_Q y''$ if and only if $x \leq_P y$. Another exercise: If Q is the split of P, then

$$\dim(P) \leq \dim(\mathrm{Min}(Q), \mathrm{Max}(Q)) \leq \dim(Q) \leq 1 + \dim(P).$$

Throughout this paper, we will let $\mathbb{N}$ denote the set of positive integers and $\mathbb{R}$ the set of all real numbers. Also, for an integer $n \in \mathbb{N}$, we use $[n]$ to abbreviate $\{1, \ldots, n\}$.

5.2 Stability Analysis

The maximum chromatic number of a graph on n vertices is n, and this value is achieved only by the complete graph K_n. We may then ask whether this statement is "stable", i.e., if G is a graph on n vertices and $\chi(G)$ is close to n, must G be close to being a complete graph. An affirmative answer is provided by the following elementary exercise:

Proposition 5.1 *Let c and n be positive integers with $n \geq 2c+1$. If G is a graph on n vertices and $\chi(G) \geq n - c$, then G contains a clique of size $n - 2c$.*

Now for the analogous problems for posets. The following inequality of Hi-

*The idea of splitting a single point is due to R. Kimble and was shared in personal communications with the author in 1973–74. Other researchers expanded Kimble's idea to splitting all the points in a poset.

raguchi [21, 22] provides an upper bound on the dimension of a poset on n points, and the standard examples show that the inequality is best possible.

Theorem 5.2 *If $n \geq 4$ and P is a poset on n points, then* $\dim(P) \leq \lfloor n/2 \rfloor$.

The following theorem of Kimble [33] classifies the extremal posets in the general case—and this answer is *exactly* the analogue of the result for graphs.

Theorem 5.3 *If $n \geq 4$ and P is a poset with $|P| = 2n+1$, then $\dim(P) < n$ unless P contains the standard example S_n.*

Kimble's proof is an elegant inductive argument starting with $n = 4$ as the base case, but no complete proof of this base case has been written down. Note that the analogous statement is false when $n = 3$, since besides the standard example S_3, there is another 3-irreducible poset on six points. In addition, from Figures 5.1 and 5.2, we see that there are (up to duality) an additional 14 posets on seven points that are 3-irreducible. Accordingly, one would expect that there are *many* 4-irreducible posets on nine points, but in fact there are none.

In the summer of 2016, Derrick Stolee [43] used Sage [42] and its combinatorics package that generates (among other things) all unlabeled posets on a given number of points. Stolee devised a clever backtrack search algorithm to determine dimension and tested all $183,231$ unlabeled posets on nine points to verify that the standard example S_4 is the only 4-irreducible poset on at most nine points. With Stolee's contribution, we can now consider Theorem 5.3 as completely settled.

Although we do not include a detailed outline of Kimble's proof of Theorem 5.3 (with the base case $n = 4$ assumed to be valid), here are five lemmas that make nice exercises. Once they have been established, it is relatively straightforward to assemble an argument for Theorem 5.3. References are included at the beginning of each.

Lemma 5.4 ([8]) *If P is a poset, then* $\dim(P) \leq \mathrm{width}(P)$.

Lemma 5.5 ([21, 22]) *If* $(a,b) \in \mathrm{Inc}(P) \cap (\mathrm{Min}(P) \times \mathrm{Max}(P))$*, then* $\dim(P) \leq 1 + \dim(P - \{x,y\})$.

Lemma 5.6 ([33] and [46]) *Let A be a maximal antichain in a poset P. Then* $\dim(P) \leq \max\{2, |P - A|\}$.

Lemma 5.7 [33]) *If A is an antichain in a poset P, and* $\dim(P) = |P - A| \geq 4$*, then either* $P - A = \mathrm{Min}(P)$ *or* $P - A = \mathrm{Max}(P)$.

Lemma 5.8 ([54]) *If P is a poset that is not an antichain and* $w = \mathrm{width}(P - \mathrm{Min}(P))$*, then* $\dim(P) \leq w + 1$. *Furthermore, if $w \geq 2$ and $P - A$ can be covered by w chains, one of which is a singleton, then* $\dim(P) \leq w$.

Turning now to recent results, the following theorem of Trotter and Wang [54] was published in 2016. Ignoring the special (and trivial) case $d = 2$, each of the two statements of the theorem provides a modern proof of Hiraguchi's inequality. We will outline the proof of the statement for matchings in the comparability graph.

Theorem 5.9 *If P is a poset with $\dim(P) = d$ where $d \geq 3$, then there is a matching of size d in the comparability graph of P, and there is a matching of size d in the incomparability graph of P.*

The proof requires three claims/exercises. A subposet U of P is called an *up-set* if $y \in U$ whenever $x \in U$ and $x <_P y$. *Down-sets* are defined analogously. Note that U is an up-set in P if and only if $P - U$ is a down-set.

Claim 1. If U is an up-set in a poset P, then $\dim(P) \leq \dim(U) + \text{width}(P - U)$.

As an exercise , show that if s and t are positive integers, there is a poset P containing an up-set U such that $\dim(U) = s$, $\text{width}(P-U) = t$ and $\dim(P) = s+t$. For the next claim, students will need the second part of Lemma 5.8. For a maximum matching $\mathcal{M}$ in the comparability graph of a poset P, we let $A(\mathcal{M})$ denote the set of points *not* covered by $\mathcal{M}$. Note that $A(\mathcal{M})$ is an antichain in P.

Claim 2. Let P be a poset and let $\mathcal{M}$ be a maximum matching in the comparability graph of P. If $|\mathcal{M}| = m$ and either $A(\mathcal{M}) \subseteq \text{Min}(P)$ or $A(\mathcal{M}) \subseteq \text{Max}(P)$, then $\dim(P) \leq \max\{2, m\}$.

With the notation of the last two claims, we let $\mathbb{U}(\mathcal{M})$ denote the set of all chains $\{x < y\}$ in $\mathcal{M}$ for which there is an element $a \in A(\mathcal{M})$ such that $a <_P y$. Analogously, $\mathbb{D}(\mathcal{M})$ consists of those chains $\{x < y\}$ in $\mathcal{M}$ for which there is some $a \in A$ such that $x <_P a$. The sets $\mathbb{U}(\mathcal{M})$ and $\mathbb{D}(\mathcal{M})$ need not be disjoint. However, we say the matching $\mathcal{M}$ is *pure* if these two sets are disjoint. Note that a perfect matching is pure since both $\mathbb{U}(\mathcal{M})$ and $\mathbb{D}(\mathcal{M})$ are empty. Here is the third exercise.

Claim 3. If P is a poset that is not an antichain, then there is a pure maximum matching in the comparability graph of P.

Assuming the three claims, we now prove the statement in Theorem 5.9 concerning matchings in the comparability graph, using an argument by contradiction. Let d be the least integer, with $d \geq 3$, for which the statement fails to hold. With d fixed, let P be a counterexample of minimum size. Note that P must be d-irreducible. Since every 3-irreducible poset has a matching of size 3 in its comparability graph, $d \geq 4$. Let m the maximum size of a matching in P. Our choice of d now implies $d - 1 = m \geq 3$.

Let $\mathcal{M}$ be a pure maximum matching in P, as guaranteed by Claim 3. By Claim 1, $A(\mathcal{M})$ must contain at least one point that does not belong to $\text{Min}(P)$, so $\mathbb{D}(\mathcal{M}) \neq \emptyset$. Let $m_1 = |\mathbb{D}(\mathcal{M})|$.

Dually, $A(\mathcal{M})$ must contain at least one point that does not belong to $\text{Max}(P)$. This implies $\mathbb{U}(\mathcal{M}) \neq \emptyset$. Let $m_2 = |\mathbb{U}(\mathcal{M})|$.

Now let D be the subposet of P consisting of the elements of P covered by the chains in $\mathbb{D}(\mathcal{M})$, and let $U = P - D$. Note that D is a down-set in P, and U is an up-set in P. From Claim 2, we know $\dim(U) \leq \max\{2, m - m_1\}$. Evidently, $\mathrm{width}(D) \leq m_1$. Using Claim 1, if $m - m_1 \geq 2$, then $\dim(P) \leq m$. Therefore, $m - m_1 = 1$.

Applying the same argument to the dual of P leads to the conclusion that $m - m_2 = 1$. In turn, this implies that $m = 2$, which is the final contradiction.

5.2.1 Stability Analysis for Dimension

Next we turn our attention to the stability analysis question for posets. For a poset P, define the *standard example size* of P, abbreviated to $\mathrm{se}(P)$, to be 1 if P does not contain the standard example S_2; otherwise, $\mathrm{se}(P)$ is the largest d for which P contains the standard example S_d. We now outline the proof of the following 2016 theorem of Biró, Hamburger, Pór, and Trotter [3], which is the poset analogue of Proposition 5.1.

Theorem 5.10 *There exists a function $f : \mathbb{N} \to \mathbb{N}$ with $f(c) = O(c^2)$ such that for each $c \geq 1$, if $n > 10f(c)$, P is a poset with $|P| \leq 2n+1$, and* $\dim(P) \geq n - c$*, then* $\mathrm{se}(P) > n - f(c)$.

Here is an outline of the proof. We assume [†] that c is a positive integer, and we set $s = 7c+6$, $t = 5s$ and $f(c) = c + t(7c+3)$, so that $f(c) = c + 5(7c+6)(7c+3) = O(c^2)$. We assume $n > 10f(c)$ and P is a poset on $2n+1$ points such that $\dim(P) \geq n - c$ and $\mathrm{se}(P) \leq n - f(c)$. Subsequently, we show that these assumptions lead to a contradiction.

When A is a maximal antichain in a poset P, it is natural to define $D(A)$ as consisting of those elements $x \in P$ for which there is some element $a \in A$ with $x <_P a$. The set $U(A)$ is defined analogously. Note that $\{A, D(A), U(A)\}$ is a partition of P. Note that we may have $D(A) = \emptyset$ or $U(A) = \emptyset$. Regardless, $D(A)$ and $A \cup D(A)$ are down-sets in P, while $U(A)$ and $A \cup U(A)$ are up-sets in P. These concepts are used in the following elementary lemma/exercise that is essentially the same as Claim 1 from the earlier proof concerning matchings.

Lemma 5.11 *If A is a maximum antichain in a poset P, then* $\dim(P) \leq 1 + \mathrm{width}(U(A)) + \mathrm{width}(D(A))$.

The next lemma/exercise will be a challenge for students, but a hint can be found in [52].

Lemma 5.12 *Let A be a maximal antichain in a poset P that is not an antichain. If $X = D(A)$ and $Y = U(A)$ are antichains in P with $|X| = s$ and $|Y| = s + t$ where $s, t \geq 0$, then* $\dim(P) \leq 1 + t + \lceil 4s/3 \rceil$.

We alert readers to the fact that as the argument proceeds, we will repeatedly delete points from P while continuing to refer to the remaining subposet as P. The

[†] Our outline will follow (essentially) the original proof, but we will get slightly better constants.

total number of points deleted will be a modest multiple of c, and we have assumed that the size of P is at least $10f(c)$.

We know from Lemma 5.4 that P must contain a maximum antichain A of size at least $n-c$. Since $\dim(P) \le |P-A|$, we also know $|A| \le n+1+c$. It follows from Lemma 5.11 that there are antichains $X \subseteq D(A)$ and $Y \subseteq U(A)$ such that $n-c-1 = |X|+|Y|$. We assume without loss of generality that $|X| \le |Y|$.

We update the meaning of P so that $P = A \cup X \cup Y$, so at most $2c+1$ points are discarded. Note that for the new P, we have $\dim(P) \ge n-c-(2c+1) = n-(3c+1)$.

Next, we apply Lemma 5.12. Let $|X| = \sigma$ and $|Y| = \sigma+\tau$. Since $n-c-1 = 2\sigma+\tau$, we have $\dim(P) \ge n-3c-1 = 2\sigma+\tau-2c$. Rounding up, we have $\dim(P) \le 2+\tau+4\sigma/3$. It follows that $\sigma \le 3c+3$.

We delete the points in X and update the meaning of P to be just $A \cup Y$. Now P is a bipartite poset, and we have $\dim(P) \ge n-(3c+1)-(3c+3) = n-(6c+4)$. To be consistent with other material, we relabel the antichain Y as B, and for the remainder of the argument, whenever we write $\dim(Q)$ for a bipartite subposet Q of P, we actually mean $\dim(\mathrm{Min}(Q), \mathrm{Max}(Q))$.

Let $d = \mathrm{se}(P)$. We assume $d \ge 2$. It will soon be clear that this assumption is quite safe. We have assumed that $d \le n - f(c)$, and we will proceed to show that this inequality does not hold. This contradiction will complete the proof.

After a relabeling, we may assume there are subsets $\{a_1, \dots, a_d\} \subset A$ and $\{b_1, \dots, b_d\} \subset B$ so that $a_i <_P b_j$ if and only if $i \ne j$. We then view the poset P as consisting of the standard example S_d with the remaining points forming a bipartite subposet Q.

For a bipartite poset R, we let $m(R)$ denote the largest integer s for which there are s-element subsets $\{x_1, \dots, x_s\} \subseteq \mathrm{Min}(R)$ and $\{y_1, \dots, y_s\} \subseteq \mathrm{Max}(R)$ such that $x_i \parallel_R y_i$ for each $i \in [s]$.

At this point in the argument, we need the following two claims/exercises. The first is implicit in the proof of Lemma 5.5.

Claim 1. If $x \in \mathrm{Min}(P)$, $y \in \mathrm{Max}(P)$ and $x \parallel_P y$, then there is a linear extension $L = L(x,y)$ such (1) $x > v$ in L whenever $x \parallel_P v$ and (2) $u > y$ in L whenever $y \parallel_P u$. Therefore, $\dim(R) \le m(R)$ for every subposet R of P.

Claim 2. Let R be a subposet of Q and let $s = |\mathrm{Min}(R)|$. If $s \ge 2$, then $\dim(R) < s$ unless $\mathrm{se}(R) = s$.

From Claim 1, it follows that $\dim(P) \le d + \dim(Q)$. Since $|A| \le n+c+1$, it follows that $\dim(Q) \ge |\mathrm{Min}(Q)| - (7c+2)$. Let $W_1 \cup W_2 \cup \cdots \cup W_{7c+3}$ be a partition of $\mathrm{Min}(Q)$ into subsets whose sizes differ by at most one. For each $i \in [7c+2]$, we consider the subposet R_i consisting of W_i and $\mathrm{Max}(Q)$. Since $\dim(\mathrm{Min}(Q), \mathrm{Max}(Q)) \le \sum_{i=1}^{7c+3} \dim(W_i, \mathrm{Max}(Q))$, it follows that there is some $i \in [7c+3]$ such that $\dim(W_i, \mathrm{Max}(Q)) = |W_i|$. This implies that R_i contains a standard example of dimension $|W_i|$. We note that $|W_i| \ge (n-c) - d/(7c+3)$, and since $d \le n - f(c)$, we conclude that $|W_i| \ge [f(c)-c]/(7c+3) = t$.

After a relabeling, we take subsets $W = \{w_1, \dots, w_t\} \subset \mathrm{Min}(Q)$ and $Z = \{z_1, \dots, z_t\} \subset \mathrm{Max}(Q)$ such that $w_i <_P z_j$ if and only if $i \ne j$. Now we consider the poset P as consisting of a standard example S_d formed by $A \cup B$, a standard example S_t formed by $W \cup Z$, and the remaining points, which we take as a subposet R. From Claim 1, it follows that $\dim(P) \leq d + t + \dim(R)$. Furthermore, from the second statement in Claim 1, $\dim(R) \leq m(R)$.

We next define an auxiliary graph G that is a bipartite graph. The first part of G is $[d]$ and the second part is $[t]$. A pair ij is an edge in G if and only if the subposet of P determined by $\{a_i, b_i, w_j, z_j\}$ is *not* the standard example S_2. An application of Hall's theorem (another exercise) allows us to conclude that there is a complete matching in the bipartite graph G, i.e., a matching in which each integer in $[t]$ is paired with an integer in $[d]$. After a relabeling, we may assume that each $i \in [t]$ is paired with $i \in [d]$, which implies that for each $i \in [t]$, the subposet of P formed by $\{a_i, b_i, w_i, z_i\}$ is not the standard example S_2.

Let $A = \{a_1, a_2, \ldots, a_t\}$ and $B = \{b_1, b_2, \ldots, b_t\}$. We update the meaning of P to consist of two standard examples of dimension t, one formed by the points in $A \cup B$, and the other formed by the points in $W \cup Z$. The remaining points of the old standard example S_d are now added to the subposet R. Now we have $\dim(P) \leq 2t + m(R)$, but we ask readers to remember that the new R is larger than the old R. In fact, using Claim 2, we have $\dim(P) \leq m(R) + \dim(A \cup W, B \cup Z)$.

We note that $t = 5s$, so that $|A \cup W| = 2t = 10s$. Now for the killer claim/exercise. This one will be a challenge for students.

Claim 3. $\dim(A \cup W, B \cup Z) \leq 9s$.

Once Claim 3 has been established, we conclude that:

$$n - (6c + 4) \leq \dim(P) \leq m(R) + 9s \leq n + c + 1 - s.$$

This implies $s \leq 7c + 5$, when in fact $s = 7c + 6$. The contradiction completes the proof.

5.2.2 Open Problems for Stability Analysis

For the function $f(c)$ in Theorem 5.10, finite projective planes are used in [3] to prove that $f(c) = \Omega(c^{4/3})$. In a preliminary manuscript [4], Biró, Hamburger, Kierstead, Pór, Trotter, and Wang investigate the dimension of random bipartite posets, with an emphasis on the case where $p = p(n) \geq 1/2$. Readers may note that the dimension of random posets was first studied nearly 30 years ago by Erdős, Kierstead, and Trotter in [11], but their results concentrated on the case where $p = p(n) \leq 1/2$. One of the results extracted from [4] is an improvement in the exponent of c in $f(c)$ to $3/2$. It follows that the correct exponent on c in the function $f(c)$ is somewhere between $3/2$ and 2.

Here is another problem which has somewhat the same flavor. For integers n and d with $d \geq 2n + 1$, let $g(n,d)$ be the least integer t such that if $|P| = n$ and $\dim(P) = t$, then P contains the standard example S_d. We are primarily interested in estimating $g(n,d)$ when d is fixed and $n \to \infty$. Based on earlier comments, the behavior of $g(n,d)$ is essentially known when $d = 2$. For $d \geq 3$, our understanding is more limited. However, we do have the following lower bound, proved in [4].

Theorem 5.13 *For all $d \geq 3$,*

$$g(n,d) \geq \frac{n^{1-\frac{2d-1}{d(d-1)}}}{8\log n}.$$

Upper bounds are even more challenging, and all that we know is in the following theorem of Biró, Hamburger, and Pór [2].

Theorem 5.14 *For every $d \geq 3$ and every $\varepsilon > 0$, there is an integer n_0 such that if P is a poset with $|P| = n > n_0$, and P does not contain the standard example S_d, then $\dim(P) < \varepsilon n$.*

5.2.3 Open Problems on Size

The form of Hiraguchi's inequality makes the following conjecture quite natural.

Conjecture 5.15 (Removable Pair Conjecture) *If P is a poset with $|P| \geq 3$, then there is a pair $\{x,y\}$ of distinct elements of P such that $\dim(P) \leq 1 + \dim(P - \{x,y\})$.*

Apparently, the first time Conjecture 5.15 appears in print is 1975 in [46]. However, it is implicit in the papers [21, 22] by Hiraguchi. All substantive work to date on the conjecture involves showing that certain natural choices for the pair $\{x,y\}$ do not work. Here are two examples. In [52], a family of posets is constructed where the removal of any two elements from $\mathrm{Min}(P) \cup \mathrm{Max}(P)$ decreases dimension by 2. An ordered incomparable pair (x,y) in a poset P is called a *critical pair* when (1) $z <_P x$ implies $z <_P y$ and (2) $y <_P w$ implies $x <_P w$. Bogart and Trotter conjectured that the removal of a critical pair decreases dimension by at most 1. Reuter [37] gave a 14-element counterexample. Subsequently, Kierstead and Trotter [32] constructed an infinite family of counterexamples. Nevertheless, the author believes firmly that there is *some* critical pair whose removal decreases dimension by at most 1.

In another direction, the standard examples show that the inequalities in Lemmas 5.4 and 5.6 are tight, but there are other extremal examples for each of these lemmas. In [47], Trotter gave a forbidden subposet characterization of Lemma 5.6 by determining the minimum list $\mathcal{L}_n$ of posets such that if $n \geq 4$, A is an antichain in a poset P, $|P - A| = n$, and $\dim(P) = n$, then P is isomorphic to a poset in $\mathcal{L}_n$. For each $n \geq 4$, there are $2n+1$ posets in the list $\mathcal{L}_n$.

However, the situation with Lemma 5.4 is quite different. We show in Figure 5.3 two infinite families of posets for which dimension equals width. As an exercise , show that P_n and Q_n are n-irreducible when $n \geq 4$. However, P_3 and Q_3 are not 3-irreducible, as they contain the chevron D shown in Figure 5.2.

We do not know a forbidden subposet characterization of the inequality $\dim(P) \leq \mathrm{width}(P)$. In fact, we don't even know whether the list is finite for a fixed value of width. Also, the natural problem of determining whether the dimension of a poset is less than its width is clearly in $\mathbb{NP}$, but it is not known whether it is $\mathbb{NP}$-complete.

In yet another direction, it is conjectured that Theorem 5.9 also holds for critical

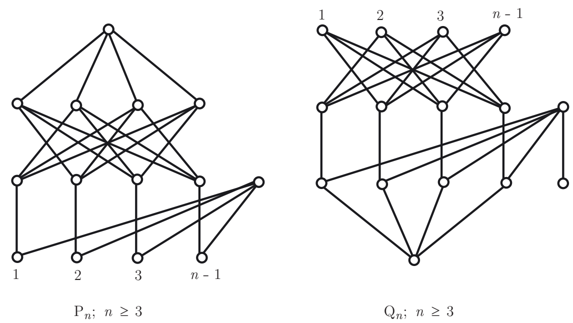

Figure 5.3
Posets with Dimension Equal to Width

pairs, i.e., if $d \geq 3$ and P is a poset with $\dim(P) = d$, then there is matching in the incomparability graph such that each matched pair is a critical pair. By inspection, this conjecture holds when $d = 3$.

5.3 Maximum Degree

Recall that for a graph G, we use $\Delta(G)$ to denote the maximum degree among the vertices of G. The inequality $\chi(G) \leq \Delta(G)+1$ is trivial. On the other hand, the following classic theorem of Brooks [7] characterizes the extremal graphs (see Lovász [35] for an elegant proof).

Theorem 5.16 *Let G be a connected graph with $\Delta(G) = k \geq 2$. If $\chi(G) = k+1$, then G is a complete graph on $k+1$ vertices, unless $k = 2$ and G is an odd cycle C_{2n+1} for some $n \geq 2$.*

The *maximum degree* of a poset P, denoted $\Delta(P)$, is defined to be the maximum degree of the comparability graph of P. For a positive integer k, we let $f(k)$ denote the maximum value of $\dim(P)$, where P is a poset with $\Delta(P) \leq k$. The standard examples show $f(k) \geq k+1$. Also, it is easy to verify $f(1) = 2$ and $f(2) = 3$. It is then natural to conjecture $f(k) = k+1$, for all $k \geq 1$, so that we have a poset analogue of Brooks' theorem. Our first challenge is that it is not immediately clear that the dimension of a poset is bounded in terms of its maximum degree.

In 1983, Rödl and Trotter showed that the function f is well defined and satisfies $f(k) \leq 2k^2+2$. A short proof of this inequality is given in [48]. In 1986, Füredi and Kahn [16] gave the following dramatic improvement: $f(k) < 50k\log^2 k$ for all $k \geq 1$.

These upper bounds left open the possibility that $f(k)$ might actually be $k+$

1. In 1991, this hope was destroyed by a superlinear lower bound given by Erdős, Kierstead, and Trotter in [11]: $f(k) = \Omega(k \log k)$.

After a gap of nearly 30 years, Scott and Wood [38] have just obtained the following significant improvement in the upper bound:

Theorem 5.17 $f(k) = k \log^{1+o(1)} k$.

Due to space limitations, we will not attempt to outline the proof of the lower bound on $f(k)$, but we will outline the proof of the new upper bound of Scott and Wood. Their proof uses a translation to boxicity provided by Adiga, Bhowmick, and Chandran [1], but our outline lives entirely in the poset world.

For a positive integer k, we define $g(k)$ to be the maximum value of $\dim(\mathrm{Min}(P), \mathrm{Max}(P))$ among all bipartite posets P with $\Delta(P) \leq k$. Using the observations concerning splits given in the introduction, $f(k) \leq g(k+1) \leq f(k+1)$, so in an asymptotic sense, determining $f(k)$ and $g(k)$ are equivalent problems.

We need three lemmas and some additional background material. For a pair (r,n) of integers with $2 \leq r < n$, let $P(1,r;n)$ denote the bipartite poset consisting of all 1-element and r-element subsets of $[n]$, ordered by inclusion. It is customary to consider a 1-element set $\{i\}$ as just the integer i so that $\mathrm{Min}(P(1,r;n)) = [n]$, and $i < S$ in $P(1,r;n)$ when $i \in S$. We abbreviate $\dim(P(1,r;n))$ as $\dim(1,r;n)$.

Let $\mathcal{F}$ be a family of linear orders on $[n]$. We say $\mathcal{F}$ is $(r+1)$-*suitable* if for each $r+1$-element subset $S \subset [n]$ and each integer $i \in S$, there is some $L \in \mathcal{F}$ such that i is the least element of S in L. Trivially, $\dim(1,r;n)$ is the minimum size of a $(r+1)$-suitable family of linear orders on $[n]$.

Dushnik [9] calculated $\dim(1,r;n)$ exactly for $r \geq 2\sqrt{n}$, and (see the comments in [3]) we can give the value of $\dim(1,2;n)$ *exactly* for almost all values of n. In all cases, we can give two consecutive integers and be certain that $\dim(1,2;n)$ is one of them. For intermediate values, asymptotic estimates are available (see the survey [31] by Kierstead), and two examples are given in the following lemma. The first inequality below is given in [16] and is a good exercise for students. The second is proved by Spencer in [41], but he credits the argument to Hajnal. A very modern and detailed proof is given by Scott and Wood in [38], and we state the inequality in the form used in their paper. Without a detailed hint, this one will be a challenge for students.

Lemma 5.18 *For all pairs* (r,n) *with* $2 \leq r < n$,

$$\dim(1,r;n) \leq r^2(1 + \log(n/r)).$$

$$\dim(1,r;n) \leq 1 + r2^r \log\Big(\frac{2e}{r} \log_2(2n)\Big).$$

Scott and Wood point out that the second inequality implies $\dim(1,r;n) \leq r2^r \log\log n$ when $n \geq 10^4$.

Both the Füredi–Kahn and the Scott–Wood arguments use the the following well known result [12], now called the Lovász Local Lemma ‡.

‡In the literature, the condition $ep(D+1)$ is often replaced by $4pD < 1$. The proof of the lemma is the same in both cases.

Lemma 5.19 *Suppose that p is a real number with $0 < p < 1$, D is a positive integer, and $\mathcal{F}$ is a family of events in a probability space such that for all $E \in \mathcal{F}$, the probability of E is at most p, and E is mutually mutually independent of a set of all but D other events in $\mathcal{F}$. If $ep(D+1) < 1$, then the probability that none of the events in $\mathcal{F}$ holds is positive.*

When S is a set and m is a positive integer, we refer to a function $c : S \to [m]$ as a *coloring* of S, with the integers in $[m]$ viewed as colors. The next lemma, a key ingredient of the Scott–Wood proof, features repeated applications of the Lovász Local Lemma so that a desired property holds for at least one application. This is a particularly novel idea and is likely to have many other uses.

Lemma 5.20 *Let k and d be positive integers, and let P be a bipartite poset with $X = \mathrm{Min}(P)$ and $B = \mathrm{Max}(P)$. Suppose that (1) each element of X is comparable with at most d elements of B and (2) each element of B is comparable with at most k elements of X. If r, t, and m are positive integers such that*

$$m \geq e^{1/r}\Big(\frac{ed}{r+1}\Big)^{1+1/r} \quad \textit{and} \quad t \geq \log(edk),$$

then there is a list $(c_1, \ldots, c_t)$ of colorings of Y with colors from $[m]$ and a coloring ϕ of X using colors from $[t]$ such that for all elements $a \in X$, if $\phi(a) = i$, then for each $\alpha \in [m]$, at most r elements of Y are comparable with a in P and assigned color α by c_i.

To see that this lemma holds, we make the following observations. For each pair $(a,i) \in \mathrm{Min}(P) \times [t]$, let $E_{a,i}$ be the event that there is some $\alpha \in [m]$ such that at least $r+1$ elements of $\mathrm{Max}(P)$ that are comparable with a in P are assigned color α by c_i. Evidently,

$$\Pr(E_{a,i}) \leq \binom{d}{r+1} m^{-r} \leq \Big(\frac{ed}{r+1}\Big)^{r+1} m^{-r} \leq e^{-1}.$$

For each $a \in \mathrm{Min}(P)$, let $E_a = \bigcap_{i=1}^{t} E_{a,i}$. The probability p of E_a is at most e^{-t}. Clearly, the event E_a is dependent on at most $d(k-1)$ other events of the same form. Since $D+1 < dk$ and $t > \log(edk)$, it follows that $ep(D+1) < 1$. These remarks complete the proof of the lemma.

We are now ready to begin the outline of the proof of the new Scott–Wood upper bound, which we state below in the technical form needed for the argument. The simpler form given in Theorem 5.17 follows as an immediate corollary. Note that the form of the inequality allows us to assume k is large.

Theorem 5.21 *If $k \geq 10^4$ and $k \to \infty$, then*

$$g(k) \leq (2e^3 + o(1))(k \log k)\big(e^{2\sqrt{\log\log k}}\big).$$

For the balance of the outline, since we have assumed k is large, we will treat quantities like $\log k$ and $\sqrt{\log k}$ as if they were integers. Also, in working with the

Lovász Local Lemma, we will treat inequalities as equations. The small errors this approach produces can be readily repaired but in the interim, rounding up and rounding down only serves to obscure the line of reasoning.

Fix a bipartite poset P with $\Delta(P) \leq k$ and $\dim(\mathrm{Min}(P), \mathrm{Max}(P)) = g(k)$. Let $X = \mathrm{Min}(P)$ and $Y = \mathrm{Max}(P)$. The following lemma is a straightforward application of the Lovász Local Lemma 5.22.

Lemma 5.22 *If d and v are integers such that*

$$e\frac{\binom{k}{d+1}}{v^d}k^2 < 1,$$

then there is a coloring $c : Y \to [v]$ such that for each $a \in X$ and each color $i \in [v]$, there are at most d elements of P which are comparable to a in P and are assigned color i by c.

We leave it as an exercise to verify that the inequalities in the preceding lemma are satisfied when $d+1 = 3\log k$ and $v = e^2k/(3\log k)$, provided that k is large.

For each $j \in [v]$, let Y_j consist of all $y \in Y$ with $c(y) = j$. Note that $\dim(X,Y) \leq \sum_{j=1}^{v}\dim(X,Y_j)$. Choose $j \in [v]$ such that $\dim(X,Y_j) \geq \dim(X,Y)/v$, and set $B = Y_j$.

Set $r = \sqrt{\log\log k}$. We turn the inequalities of Lemma 5.20 into equations and set

$$m = e^{1/r}\left(\frac{ed}{r+1}\right)^{1+1/r} \quad \text{and} \quad t = \log(edk).$$

For each $i \in [t]$, let X_i consist of all elements $a \in X$ with $\phi(a) = i$. Choose a integer $i \in [t]$ such that $\dim(X_i,B) \geq \dim(X,B)/t$. Set $A = X_i$. For each $\alpha \in [m]$, let B_α consist of all $b \in B$ for which $c_i(b) = \alpha$. Choose $\alpha \in [m]$ such that $\dim(A,B_\alpha) \geq \dim(A,B)/m$. Set $Z = B_\alpha$. Note that we have $\dim(X,Y) \leq vmt\dim(A,Z)$.

Since the setup for and the proof of the next claim are fundamental to the Scott–Wood approach (as well as to the earlier arguments of Füredi–Kahn and Rödl–Trotter), we give a complete proof. Define an auxiliary graph G whose vertex set is Z. Distinct elements b and b' of Z form an edge in G when there is an element $a \in A$ with $a <_P b$ and $a <_P b'$. Clearly, the maximum degree in G is at most $k(r-1)$, so there is a partition $\{Z_1, \ldots, Z_{rk}\}$ of Z such that Z_β is an independent set in G for each $\beta \in [rk]$.

Claim 1. $\dim(A,Z) \leq 2\dim(1,r;rk)$.

Let $s = \dim(1,r;rk)$, as demonstrated by the $(r+1)$-suitable family $\{M_1, \ldots, M_s\}$ of linear orders on $[rk]$. We extend each M_j to two linear orders L_{2j-1} and L_{2j} of Z. There are two requirements. First, each of L_{2j-1} and L_{2j} respects blocks in the partition of Z, i.e., if β and γ are distinct integers in $[rk]$ with $b \in Z_\beta$ and $b' \in Z_\gamma$, then $b < b'$ in L_{2j-1} *and* in L_{2j} if and only if $\beta < \gamma$ in M_j. The second requirement is that for each $\beta \in [rk]$, the restriction of L_{2j-1} to the block Z_β is the dual of the restriction of L_{2j} to Z_β.

Each of these linear orders is extended to a linear order on Y by adding the elements of $Y - Z$ above Z. The order of elements of $Y - Z$ is arbitrary. In turn, these linear orders are expanded to linear extensions of P by inserting the elements of X as high as possible. The order of elements of X in a gap between consecutive elements of Y is arbitrary.

We claim that the resulting linear extensions demonstrate that $\dim(A,Z) \leq 2s$. To see this, let $(a,b) \in \mathrm{Inc}(A,Z)$, and let $b \in Z_\beta$. Also, let S be the set of all $\gamma \in [m]$ for which there is an element comparable with a that belongs to Z_γ. Note that $|S| \leq r$. If β does not belong to S, then there is some j such that β precedes all elements of S in M_j. It follows that $a > b$ in both L_{2j-1} and in L_{2j}. On the other hand, if $\beta \in S$, then there is some j such that β is the least element of S in M_j. It follows that $a > b$ in *exactly* one of L_{2j-1} and L_{2j}. These observations complete the proof of the claim.

To bound $\dim(1,r;rk)$, we use the second inequality in Lemma 5.18. Since $rk \geq 10^4$, we have

$$\dim(1,r;rk) \leq r2^r \log\log rk = (1+o(1)r2^k \log\log k.$$

We note that $t = \log(edk) = (1+o(1))\log k$. Also, since $r \to \infty$, it follows that

$$r\Big(\frac{e}{r+1}\Big)^{1+1/r} \to e.$$

Summarizing, we began by splitting the original problem into v subproblems. In turn, each of these subproblems was further split into mt subproblems. It follows that

$$\begin{aligned} g(k) &= \dim(X,Y) \\ &\leq vmt \dim(A,Z) \\ &\leq 2vmt \dim(1,r;rk) \\ &\leq (2e^3+o(1))(k\log k)d^{1/r}2^r \log\log k. \end{aligned}$$

With our choice of $r = \sqrt{\log\log k}$, simple calculations show that

$$d^{1/r}2^r \log\log k < e^{2\sqrt{\log\log k}}.$$

It follows that:

$$g(k) \leq (2e^3+o(1))(k\log k)\big(e^{2\sqrt{\log\log k}}\big).$$

Our outline for the proof of the Scott–Wood upper bound is now complete.

Readers who are familiar with the Füredi–Kahn proof will recognize that it stopped after the first coloring and passed immediately to the auxiliary graph step. Their bound reflected the inequality $g(k) \leq v\dim(1,d;kd)$ and the application of the first inequality in Lemma 5.18. Students may be interested ( ) to check that the values $d+1 = 3\log$ and $v = e^2k/(3\log k)$ are optimal for the Füredi–Kahn approach—but not for the Scott–Wood approach.

As for open problems on maximum degree, we can start with the fact that there is *still* a gap between the upper and lower bounds for $f(k)$. I believe that $k\log k = o(f(k))$ but suspect that this will be difficult to settle. Nevertheless, the improvement made by Scott and Wood was unexpected, so perhaps there is another surprise just around the corner.

Here are three problems that seem approachable. First, find the value $f(3)$, i.e., the maximum dimension among all posets P with $\Delta(P) = 3$. Analogously, find $g(3)$, the maximum value of $\dim(\mathrm{Min}(P), \mathrm{Max}(P))$ among all bipartite posets P with $\Delta(P) = 3$. Third, construct explicitly a poset P with $\dim(P) \geq 2 + \Delta(P)$.

5.4 Blocks in Posets and Graphs

The results of this section are a small part of a comprehensive series of papers exploring connections between dimension of posets and graph-theoretic properties of their cover graphs. Recent related papers include [44], [36], [55], [27], [28], [24], [5], and [40]. For reasons of space, we can only include here a single highlight from this series, and we have chosen one for which there is a clear analogue involving chromatic number.

Recall that when G is a connected graph, a *block* in G is a maximal subgraph that does not have a cut-vertex. Trivially, if G is disconnected and has components $C_1, \ldots, C_t$, then $\chi(G) = \max\{\chi(C_i) : i \in [t]\}$. Furthermore, if G is connected, then $\chi(G)$ is the maximum value of $\chi(B)$ taken over all blocks B of G.

Here are the analogous concepts for posets. A subposet B of a poset P is said to be *convex* if $y \in B$ whenever $x, z \in B$ and $x <_P y <_P z$. A convex subposet of P is called a *block* of P when the cover graph of B is a block in the cover graph of P. In our list of elementary properties of dimension, we gave the simple formula for the dimension of a disconnected poset. Now we consider the following problem. For a positive integer d, find the maximum dimension of a connected poset P such that $\dim(B) \leq d$ for every block B of P. It is not immediately clear that this problem is well defined, since there is no easy explanation that the answer is bounded in terms of d.

However, we will outline the following comprehensive solution given in 2016 by Trotter, Walczak, and Wang [53].

Theorem 5.23 *For every $d \geq 1$, if P is a poset and $\dim(B) \leq d$ for every block of P, then $\dim(P) \leq d+2$. Furthermore, this inequality is best possible.*

Before we begin the outline, we pause to comment that we have known this result for the case $d = 1$ since 1977. In [51], Trotter and Moore proved that $\dim(P) \leq 3$ if the cover graph of P is a tree. Note that the poset B in Figure 5.2 and the poset E_3 in Figure 5.1 are 3-irreducible and have cover graphs that are trees. Students are encouraged to tackle the following exercise before proceeding with the proof. For

$d \geq 1$, show that $\dim(P) \leq 3d$ if $\dim(B) \leq d$ for every block B of P. Also, in [14] an example is constructed of a poset P with $\dim(P) = 4$ and $\dim(B) = 2$ for every block B of P.

Now on with the outline. Fix a positive integer $d \geq 1$, and let P be a poset such that $\dim(B) \leq d$ for every block B of P. Let G be the cover graph of P. Since $d+2 \geq 3$, we may assume G is connected. We begin with an important proposition/exercise.

Proposition 5.24 *Let P be a poset, let w be a cut vertex in P, and let P' and P'' be subposets of P such that w is the unique point of P common to both P' and P''. Suppose further that M' and M'' are linear extensions of P' and P'' having block form $M' = [A < w < B]$ and $M'' = [C < w < D]$, respectively. If M is a linear order on the ground set of $P' \cup P''$ and M has block form $[A < C < w < D < B]$, then M is a linear extension of the subposet of P induced on $P' \cup P''$. Furthermore, the restriction of M to P' is M' and the restriction of M to P'' is M''.*

We refer to the block form $M = [A < C < w < D < B]$ as the *merge rule*. Let $\mathcal{B}$ be the family of blocks in P, and let $t = |\mathcal{B}|$. Also, let $\{B_1, \ldots, B_t\}$ be any labeling of the blocks of P such that for $2 \leq i \leq t$, one of the vertices of B_i belongs to at least one of the earlier blocks. Such a vertex of B_i is unique and is a cut vertex of P. We call this vertex the *root* of B_i and denote it by $\rho(B_i)$.

For every block $B_i \in \mathcal{B}$ and every element $u \in B_i$, we define the *tail* of u relative to B_i, denoted by $T(u, B_i)$, to be the subposet of P consisting of all elements $v \in \{u\} \cup B_{i+1} \cup \cdots \cup B_t$ for which every path in the cover graph of P from v to any vertex in B_i passes through u. Note that $T(u, B_i) = \{u\}$ if u is not a cut vertex. Also, if $u \in B_i$, $v \in B_{i'}$, and $(u,i) \neq (v,i')$, then either $T(u,B_i) \cap T(v,B_{i'}) = \emptyset$ or one of $T(u,B_i)$ and $T(v,B_{i'})$ is a proper subset of the other.

By hypothesis, for every block $B_i \in \mathcal{B}$, there is a realizer $\{L_j(B_i) : j \in [d]\}$. Fix an integer j with $j \in [d]$ and set $M_j(1) = L_j(B_1)$. Next, repeat the following for $i \in [t]$. Suppose that we have a linear extension $M_j(i-1)$ of P_{i-1}. Let $w = \rho(B_i)$. Since $w \in P_{i-1}$, we can write $M_j(i-1) = [A < w < B]$. If $L_j(B_i) = [C < w < D]$, we then use the merge rule to set $M_j(i) = [A < C < w < D < B]$. When the procedure halts, take $L_j = M_j(t)$. This construction is performed for all $j \in [d]$ to determine a family $\mathcal{F} = \{L_j : j \in [d]\}$ of linear extensions of P.

The family $\mathcal{F}$ is a realizer for a poset P^* that is an extension of P. Set $R = \{(x,y) \in \mathrm{Inc}(P) : x < y \text{ in } L_j \text{ for every } j \in [d]\}$. To complete the proof, we show that $\dim(R) \leq 2$. First, note that for each $j \in [d]$ and each block $B_i \in \mathcal{B}$, the restriction of L_j to B_i is $L_j(B_i)$.

When L is a linear order on a set X and $S \subseteq X$, we say S is an *interval* in L if $y \in S$ whenever $x, z \in S$ and $x < y < z$ in L. An easy claim/exercise:

Claim 1. For every $j \in [d]$, and every pair (u,i) with $u \in B_i$, the tail $T(u,B_i)$ of u relative to B_i is an interval in L_j.

Let $(x,y) \in R$, and let i be the least positive integer for which every path from x to y in the cover graph of P contains at least two elements of the block B_i. We then define elements $u, v \in B_i$ by the following rules:

1. u is the unique first common element of B_i with every path from x to y;
2. v is the unique last common element of B_i with every path from x to y.

Note that $u \neq v$ and $u = x$ when $x \in B_i$. Also, $u \neq v$ and $v = y$ when $y \in B_i$. Here are two more claims/exercises.

Claim 2. The following two statements hold:

1. $x \in T(u, B_i)$, $y \notin T(u, B_i)$, $y \in T(v, B_i)$, and $x \notin T(v, B_i)$;
2. $u < v$ in P.

Claim 3. At least one of the following two statements holds:

1. for all y' with $y' \geq x$ in P, we have $y' \in T(u, B_i)$ and $y' < y$ in P^*;
2. for all x' with $x' \leq y$ in P, we have $x' \in T(v, B_i)$ and $x < x'$ in P^*.

Let R_{d+1} consist of all pairs $(x, y) \in R$ for which the first statement in Claim 3 applies. Analogously, let R_{d+2} consist of all pairs $(x, y) \in R$ for which the second statement in Claim 3 applies. Note that $R = R_{d+1} \cup R_{d+2}$. One more claim/exercise, and readers will note that with this claim in hand, we have completed the proof of the upper bound $\dim(P) \leq d + 2$.

Claim 4. For $j \in [2]$, there is a linear extension L_{d+j} of P such that $x > y$ in L_{d+j} when $(x, y) \in R_{d+j}$.

Our outline for the proof that the inequality in Theorem 5.23 is best possible will be quite brief. Fix an integer $d \geq 1$. Let $\mathbf{n}^d$ denote the Cartesian product of d copies of an n-element chain $\{0 < 1 < - \cdots < n-1\}$. Set $u \leq v$ in $\mathbf{n}^d$ if and only if $u_i \leq v_i$ in $\mathbb{N}$ for all $i \in [d]$. As is well known, $\dim(\mathbf{n}^d) = d$ for all $n \geq 2$.

For each $n \geq 2$, we construct a poset P as follows. We start with a base poset W that is a copy of $\mathbf{n}^d$. The base poset W will be a block in P, and W will also be the set of cut vertices in P. All other blocks in P will be "diamonds," i.e., copies of the 2-dimensional poset $\mathbf{2}^\mathbf{2}$. For each element $w \in W$, we attach a 3-element chain $x_w < y_w < z_w$ so that x_w is covered by w, w is covered by z_w, and w is incomparable to y_w.

The final claim/exercise requires an advanced topic in Ramsey theory, called the "Product Ramsey Theorem," and we refer students to Theorem 5 on page 113 of the text [20] by Graham, Rothschild, and Spencer. With this tool in hand, the claim should be accessible.

Claim. If n is sufficiently large relative to d, then $\dim(P) \geq d + 2$.

5.4.1 Open Problems involving Cover Graphs

In some sense, Theorem 5.23 is a complete solution. Nevertheless, it would be nice to find an explicit construction, say with a family of $(d+2)$-irreducible posets. This may be a challenging problem. Some of the difficulty is rooted in the application of the product Ramsey theorem. At the elementary level, this surfaced when we commented on the difficulty of showing that the inequality $\dim(P) \leq \dim(U) + \mathrm{width}(P-U)$ is best possible. For a more complex example, see the application made by Felsner, Fishburn, and Trotter in [13].

We remind readers that we have only scratched the surface of interesting and important problems linking the dimension of posets with graph theoretic properties of cover graphs. Here are two open problems in this area chosen from those that require no additional notation or terminology.

There is considerable interest in graph theory on classes of graphs where chromatic number is bounded in terms of maximum clique size, and we refer readers to the major survey paper by by Scott and Seymour [39], which lists more than 100 papers on this subject. For posets, the analogue would be classes of posets where dimension is bounded in terms of standard example size. Here is one of my absolute favorite conjectures: Dimension is bounded in terms of standard example size for posets that have planar cover graphs, i.e., for every $d \geq 2$, there is an integer t so that if P is a poset with a planar cover graph and $\dim(P) \geq t$, then $\mathrm{se}(P) \geq d$. If this conjecture holds, then it most likely holds for the class of posets whose cover graph does not have K_n as a minor, where n is a fixed positive integer.

In [44], Streib and Trotter proved that the dimension of a poset with a planar cover graph is bounded in terms of its height. Until recently the best upper bound was $2^{O(h^3)}$, a result extracted from a much more comprehensive paper by Joret, Micek, and Wiechert [28] establishing connections between dimension and weak coloring numbers. However, in 2019 Kozik, Micek and Trotter [34] have shown that dimension is polynomial in height for posets with planar cover graphs. Their upper bound is $O(h^6)$.

For the dimension of posets with planar order diagrams, much more can be said, as Joret, Micek, and Wiechert [27] have given the linear upper bound $192h+96$. Most likely, the real answer for posets with planar cover graphs is also linear in h, but the author hopes this is not the case. Since there are posets with planar cover graphs that do not have planar order diagrams, there is room for the true answer to be superlinear for posets with planar cover graphs.

5.5 Acknowledgment

The author is very grateful to F. Hoffman and D.B. West for detailed reviews of earlier versions of this chapter and for many helpful suggestions regarding exposition and presentation.

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