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Fractional Local Dimension

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Abstract
The original notion of *dimension* for posets was introduced by Dushnik and Miller in 1941 and has been studied extensively in the literature. In 1992, Brightwell and Scheinerman developed the notion of *fractional dimension* as the natural linear programming relaxation of the Dushnik-Miller concept. In 2016, Ueckerdt introduced the concept of local dimension, and in just three years, several research papers studying this new parameter have been published. In this paper, we introduce and study fractional local dimension. As suggested by the terminology, our parameter is a common generalization of fractional dimension and local dimension. For a pair (n, d) with $2 \leq d < n$, we consider the poset $P(1, d; n)$ consisting of all 1-element and d-element subsets of $\{1, \ldots, n\}$ partially ordered by inclusion. This poset has fractional dimension $d + 1$, but for fixed $d \geq 2$, its local dimension goes to infinity with n. On the other hand, we show that as n tends to infinity, the fractional local dimension of $P(1, d; n)$ tends to a value $\mathrm{FLD}(d)$ which we will be able to determine *exactly*. For all $d \geq 2$, $\mathrm{FLD}(d)$ is strictly less than $d + 1$, and for large d, $\mathrm{FLD}(d) \sim d/(\log d - \log\log d - o(1))$. As an immediate corollary, we show that if P is a poset, and d is the maximum degree of a vertex in the comparability graph of P, then the fractional local dimension of P, is at most $2 + \mathrm{FLD}(d)$. Our arguments use both discrete and continuous methods.

Keywords Dimension · Local dimension · Fractional dimension · Fractional local dimension

1 Introduction

A non-empty family $\mathcal{R}$ of linear extensions of a poset P is called a *realizer* of P when $x \leq y$ in P if and only if $x \leq y$ in L for each $L \in \mathcal{R}$. In their seminal paper, Dushnik and

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Miller [8] defined the *dimension* of P, denoted $\dim(P)$, as the least positive integer d for which P has a realizer $\mathcal{R}$ with $|\mathcal{R}| = d$.

For an integer $n \geq 2$, the *standard example* S_n is a height 2 poset with minimal elements $A = \{a_1, a_2, \ldots, a_n\}$ and maximal elements $B = \{b_1, b_2, \ldots, b_n\}$ such that $a_i < b_j$ in S_n if and only if $i \neq j$. As noted in [8], $\dim(S_n) = n$ for all $n \geq 2$. Dimension is clearly a monotone parameter, i.e., if Q is a subposet of P, then $\dim(Q) \leq \dim(P)$. Accordingly, a poset which contains a large standard example as a subposet has large dimension. On the other hand, posets which do not contain the standard example S_2 are called *interval orders*, and it is well known that interval orders can have arbitrarily large dimension.

In this paper, we introduce and study a new parameter for posets which we call *fractional local dimension*. To discuss our new parameter, we need some additional notation and terminology. A *partial linear extension*, abbreviated ple, of a poset P is a linear extension of a subposet of P. We do not require the subposet to be proper, so a linear extension is also a ple. We let $\text{ple}(P)$ denote the family of all ple's of P.

When P is a poset, a function w assigning to each $M \in \text{ple}(P)$ a real-valued weight $w(M)$, with $0 \leq w(M) \leq 1$, will be called a *local weight function for* P. For each $u \in P$, we then define the *local measure of* u *for* w, denoted $\mu(u, w)$ by setting:

$$\mu(u, w) = \sum \{w(M) : M \in \text{ple}(P), u \in M\}.$$

In turn, we define the *local measure of* P *for* w, denoted $\mu(P, w)$ by setting:

$$\mu(P, w) = \max\{\mu(u, w) : u \in P\}.$$

A local weight function w for P will be called a *fractional local realizer* of P provided for each ordered pair (u, v) of elements (not necessarily distinct) of P, if $u \leq v$ in P, we have

$$\sum \{w(M) : M \in \text{ple}(P), u \leq v \text{ in } M\} \geq 1,$$

and if u is incomparable to v in P, we have

$$\sum \{w(M) : M \in \text{ple}(P), u > v \text{ in } M\} \geq 1.$$

The *fractional local dimension of* P, denoted $\text{fldim}(P)$, is then defined by:

$$\text{fldim}(P) = \min\{\mu(P, w) : w \text{ is a fractional local realizer of } P\}.$$

Since $\text{fldim}(P)$ is the solution to a linear programming problem posed with coordinates which are integers, there is an optimal solution in which all weights are rational numbers.

Readers may note that our parameter becomes *fractional dimension* if we only allow positive weights on linear extensions. This concept was introduced in 1992 by Brightwell and Scheinerman [5] and has been studied by several groups of researchers in the intervening years (see [11, 22] and [2], for example).

On the other hand, if we only allow integer values of 0 or 1 for weights, then we obtain *local dimension*. This is a relatively new concept introduced by Ueckerdt [23] in 2016, but in the past two years, this concept is already the subject of several research papers (see [1, 21] and [4], for example). This research on local dimension has also sparked renewed interest in a related parameter, known as *Boolean dimension* introduced in 1989 by Nešetřil and Pudlák [17] (see also [1, 10, 13, 16] and [21], for example).

In the discussion to follow, we will denote the fractional dimension, local dimension and Boolean dimension of a poset P by $\text{fdim}(P)$, $\text{ldim}(P)$ and $\text{Bdim}(P)$, respectively.

We list below some basic properties of fractional local dimension. The first three are immediate consequences of the definition. The remaining items in the list are non-trivial.

However, in each case, a proof can be obtained via a relatively straightforward modification of an argument, or arguments, given elsewhere in the literature. Accordingly, for each of these statements, we give citations to the relevant papers. Our focus in this paper is on results which do not have analogues in the literature and seem to require proof techniques which are completely novel in the combinatorics of posets.

(1) For any poset P, $\mathrm{fldim}(P) \leq \mathrm{fdim}(P)$ and $\mathrm{fldim}(P) \leq \mathrm{ldim}(P)$.
(2) If P is a poset and Q is a subposet of P, then $\mathrm{fldim}(Q) \leq \mathrm{fldim}(P)$.
(3) If P is a poset and P^* is the dual of P, then $\mathrm{fldim}(P^*) = \mathrm{fldim}(P)$.
(4) If P is a poset, then the following statements are equivalent:
(a) $\dim(P) = 2$; (b) $\mathrm{fdim}(P) = 2$; (c) $\mathrm{ldim}(P) = 2$; (d) $\mathrm{fldim}(P) = 2$ and (e) $\mathrm{Bdim}(P) = 2$. The proof for fractional local dimension is an easy extension of ideas from the proof for fractional dimension given in [5] and the proof for local dimension is given in [1].
(5) If P is a poset with at least two points, then for every $x \in P$, $\mathrm{fldim}(P) \leq 1+\mathrm{fldim}(P-\{x\})$. Again, the proof is a blending of ideas from the proof of the analogous result for fractional dimension (see either [5] or [2]) and the proof for local dimension given in [1]. The question as to whether the analogous result holds for Boolean dimension is still open.
(6) For all $n \geq 3$, $\mathrm{ldim}(S_n) = 3$, as noted by Ueckerdt [23]. This observation was part of his motivation in developing the concept of local dimension. On the other hand, as noted in [5], $\mathrm{fdim}(S_n) = n$ for all $n \geq 2$. It is a nice exercise to show that $\mathrm{fldim}(S_n) < 3$ for all $n \geq 3$.
(7) If P is an interval order, then it is shown in [5] that $\mathrm{fdim}(P) < 4$. In [22], it is shown that this inequality is asymptotically tight. Both statements also hold for fractional local dimension, and the argument for the tightness of the inequality is a technical, but relatively straightforward, extension of the proof in [22]. On the other hand, it is shown in [1] that both local dimension and Boolean dimension are unbounded on the class of interval orders.
(8) The following conjecture, known as the *Removable Pair Conjecture*, has been open for nearly 70 years: If P is a poset with at least three points, then there is a distinct pair $\{x, y\}$ of points of P such that $\dim(P) \leq 1 + \dim(P - \{x, y\})$. In [2], Biró, Hamburger and Pór gave a very clever proof of the analogous result for fractional dimension, and it is straightforward to adapt their argument to obtain the parallel result for fractional local dimension. However, the Removable Pair Conjecture is open for both local dimension and Boolean dimension.

2 Forcing Large Fractional Local Dimension

We have already noted that the fractional local dimension of a standard example is less than 3 and the fractional local dimension of an interval order is less than 4. However, there does not appear to be an elementary argument to answer whether or not fractional local dimension is bounded on the class of *all* posets. Accordingly, our primary goal for the remainder of this paper is to analyze the fractional local dimension of posets in a well-studied family. As a by-product of this work, we will learn that there are indeed posets which have arbitrarily large fractional local dimension.

For a pair (d, n) of integers with $2 \leq d < n$, let $P(1, d; n)$ denote the height 2 poset consisting of all 1-element and d-element subsets of $\{1, \ldots, n\}$ ordered by inclusion. It is customary to ignore braces in discussing the minimal elements of $P(1, d; n)$ so that the minimal elements are just the integers in $\{1, \ldots, n\}$. Accordingly, $u < S$ in $P(1, d; n)$ when $u \in S$. We will abbreviate $\dim(P(1, d; n))$ as $\dim(1, d; n)$, with analogous abbreviations for other parameters.

Here are some of the highlights of results for posets in this family:

(1) Dushnik [7] calculated $\dim(1, d; n)$ exactly when $d \geq 2\sqrt{n}$.
(2) Combining the upper bound of Spencer [19] with the lower bound of Kierstead [14], we have for fixed d:

$$\dim(1, d; n) = \Omega\left(2^d \log\log n\right) \qquad \text{and} \qquad \dim(1, d; n) = O(d2^d \log\log n).$$

(3) For $d = 2$, the value of $\dim(1, 2; n)$ can be computed *exactly* for almost all values of n (see the comments in [3]).
(4) It is shown in [1] that for each fixed $d \geq 2$, $\mathrm{ldim}(1, d; n) \to \infty$ with n.
(5) In [5], it was noted that for each pair (d, n) with $n > d \geq 2$, $\mathrm{fdim}(1, d; n) = d + 1$.

For fixed $d \geq 2$, $\mathrm{fldim}(1, d; n)$ is non-decreasing in n and bounded from above by $d+1$, so the following limit exists:

$$\mathrm{FLD}(d) = \lim_{n\to\infty} \mathrm{fldim}(1, d; n).$$

Although we know $2 < \mathrm{FLD}(d) \leq d + 1$ for all $d \geq 2$, it is not immediately clear that $\mathrm{FLD}(d) < d+1$. While this fact will emerge from our results, we know of no simple proof. Also, it is not clear whether $\mathrm{FLD}(d)$ grows with d. Again, we will show that it does, but there does not appear to be an elementary proof of this fact either.

For the remainder of this section, fix an arbitrary integer $d \geq 2$. To introduce our main results, we require some additional notation and terminology.

Define a real-valued function $f(x)$ on the closed interval $[0, 1]$ by setting $f(x) = (1 - x)^d$. Then there is a unique real number β with $0 < \beta < 1$ so that $f(\beta) = \beta$. For example, when $d = 2$,

$$\beta = (3 - \sqrt{5})/2 \sim 0.381966.$$

We then consider real-valued functions $r(x)$ and $g(x)$, both having the closed interval $[\beta, 1]$ as their domain, defined by:

$$r(x) = \frac{1}{d+1}\left[(d+1)x - dx^{\frac{d+1}{d}} - (1-x)^{d+1}\right] \quad \text{and} \tag{1}$$

$$g(x) = \frac{(d+1)r(x) - x}{(d+1)r(x) - x^2}. \tag{2}$$

Let x_{bst} and x_{bal} be the (uniquely determined) numbers in $[\beta, 1]$ such that (1) the value of $g(x)$ is maximized when $x = x_{\mathrm{bst}}$, and (2) $r(x) = x^2/2$ when $x = x_{\mathrm{bal}}$. Simple calculations show for all $d \geq 2$, $\beta < x_{\mathrm{bst}} < x_{\mathrm{bal}} < 1$. In turn, we define quantities c_{bal} and c_{bst} by:

$$c_{\mathrm{bal}} = \frac{x}{r(x)} \quad \text{evaluated when } x = x_{\mathrm{bal}} \text{ and}$$

$$c_{\mathrm{bst}} = 2 + \frac{(d-1)(x - x^2)}{(d+1)r(x) - x^2} \quad \text{evaluated when } x = x_{\mathrm{bst}}.$$

With these definitions in hand, we can now state our main theorem.

Theorem 2.1 *For every* $d \geq 2$, $\mathrm{FLD}(d) = \min\{c_{\mathrm{bal}}, c_{\mathrm{bst}}\}$.

Readers may note that our proof for Theorem 2.1 shows that $\mathrm{FLD}(d)$ is the minimum of the two quantities c_{bst} and c_{bal} but does not tell us which of the two is the correct answer. However, with some supplementary analysis, it is straightforward to determine which of the two values is actually the minimum:

Theorem 2.2 *When* $2 \le d \le 3$, $\mathrm{FLD}(d) = c_{\mathrm{bal}}$, *and when* $d \ge 4$, $\mathrm{FLD}(d) = c_{\mathrm{bst}}$.

For a representative sampling of choices for d, the following table gives the values of x_{bst}, x_{bal}, c_{bal} and c_{bst}, with all values truncated to six digits after the decimal point.

d	x_{bst}	x_{bal}	c_{bal}	c_{bst}
2	0.790715	0.800466	2.498543	2.499190
3	0.667279	0.675604	2.960314	2.960495
4	0.560972	0.589913	3.390330	3.389027
5	0.478918	0.526764	3.796767	3.790859
10	0.289481	0.356142	5.615731	5.560625
100	0.048626	0.071263	28.065087	26.812765
1000	0.007055	0.010988	182.020489	172.548703
10000	0.000932	0.001500	1333.454911	1269.860071
100000	0.000116	0.000191	10458.564830	10018.329933
1000000	0.000014	0.000023	85721.802797	82526.406188

Many extremal problems in combinatorics are relatively straightforward if there is a single extremal example. In our problem, there are several natural candidates, and as suggested by the statement of our main theorem, the optimum solution is always to be found among two of these examples.

The following elementary proposition, proved with standard techniques of calculus, shows the asymptotic behavior of $\mathrm{FLD}(d)$.

Proposition 2.3 *If* $d \ge 4$, *then* $\mathrm{FLD}(d) = \frac{d}{\log d - \log\log d - o(1)}$.

Arguments for the supporting lemmas for our main theorems require us to prove formally some highly intuitive statements. Some of these statements involve properties of posets and ple's while others involve properties of real-valued functions, their derivatives, and intervals where their concavity is upwards and downwards. In cases where elementary combinatorial arguments, algebraic manipulations and techniques from undergraduate analysis suffice, we will sometimes give only the statement of lemmas and propositions. In modestly more complex cases, we will simply sketch the details of the proof.

The remainder of the paper is organized as follows. In Section 3, we give the proof of the lower bound on $\mathrm{FLD}(d)$, i.e., we show that $\mathrm{FLD}(d)$ is at least the minimum of the quantities c_{bst} and c_{bal}. We consider this lower bound our main result. In Section 4, we give a construction showing that the lower bound is tight, a construction which capitalizes on the details of our proof for the lower bound. In particular, since the lower bound is the minimum of two quantities, we develop constructions which show that each of them is an upper bound on $\mathrm{FLD}(d)$.

In Section 5, we explain how our main theorem implies that the fractional local dimension of a poset P is at most $2 + \mathrm{FLD}(d)$ when P is a poset in which d is the maximum degree of its comparability graph. Finally, in Section 6, we close with some brief comments on open problems for fractional local dimension.

3 Proof of the Lower Bound

Fix an integer $d \geq 2$. In the first part of the proof, we also fix an integer n which is *extremely* large in comparison to d. With the values of d and n fixed, we abbreviate $P(1, d; n)$ to P.

Throughout the proof, we will denote members of $\mathrm{ple}(P)$ with the symbols M and L, sometimes with subscripts or primes. When we use M, the ple may or may not be a linear extension, but when we use L, the ple is *required* to be a linear extension.

To provide readers with an overview[1] of where the argument is headed, we will identify three ple's which will be called M_{bal}, M_{bst} and M_{sat}. Then we find that an optimal fractional local realizer will assign positive weight *only* to ple's which are isomorphic to one of these three. In fact, either (1) positive weight will be placed only on copies of M_{bal}; or (2) positive weight will be placed only on copies of M_{bst} and M_{sat}.

3.1 The Relaxed Linear Programming Problem LP

Recall that finding the value of $\mathrm{fldim}(1, d; n)$ amounts to solving a linear programming problem. Taking advantage of the natural symmetry of the poset P, we formulate a relaxed version of this problem using the following notation:

For each $M \in \mathrm{ple}(P)$,

(1) $a(M)$ counts the number of elements of $\mathrm{Min}(P)$ which are in M.
(2) $b(M)$ counts the number of elements of $\mathrm{Max}(P)$ which are in M.
(3) $r(M)$ counts the number of pairs $(u, S) \in \mathrm{Min}(P) \times \mathrm{Max}(P)$ with $u > S$ in M.
(4) $q(M)$ counts the number of pairs $(u, S) \in \mathrm{Min}(P) \times \mathrm{Max}(P)$ with $u < S$ in M.

We note that the count $q(M)$ is for all pairs (u, S) with $u < S$ in M, regardless of whether or not $u \in S$. Therefore, we have the following basic identity which holds for an arbitrary ple M: $r(M) + q(M) = a(M) \cdot b(M)$.

Here is the reduced problem which we will refer to as LP:

LP. *Minimize* the quantity c for which there is a local weight function $w : \mathrm{ple}(P) \to [0, 1]$ satisfying the following constraints:

$$A(w) = \sum_{M \in \mathrm{ple}(P)} a(M) \cdot w(M) \leq cn.$$

$$B(w) = \sum_{M \in \mathrm{ple}(P)} b(M) \cdot w(M) \leq c\binom{n}{d}.$$

$$R(w) = \sum_{M \in \mathrm{ple}(P)} r(M) \cdot w(M) \geq n\binom{n-1}{d}.$$

$$Q(w) = \sum_{M \in \mathrm{ple}(P)} q(M) \cdot w(M) \geq n\binom{n}{d}.$$

We refer to these four inequalities as constraints A, B, R and Q, respectively.[2] The following observation relates the optimal solution for LP and the value of $\mathrm{fldim}(1, d; n)$.

[1] As is to be expected, this overview is a slight distortion of the truth, but the errors disappear as $n \to \infty$.

[2] Writing the constraints in this in-line form serves to facilitate several arguments for properties of solutions. However, later in the proof, we will consider them in the "scaled" form obtained by dividing both sides of each constraint by the terms involving n and/or d.

Observation 3.1 *If w is a fractional local realizer of P and $c = \mu(P, w)$, then the pair (c, w) is a solution for* LP. *Therefore if c_0 is the minimum cost for* LP, *then* $\operatorname{fldim}(1, d; n) \geq c_0$.

Proof Observe that $A(w) = \sum_{a \in A} \mu(a, w)$. By the definition of c, and the fact that $|A(w)| = n$, we see $A(w) \leq cn$. Likewise, $B(w) = \sum_{b \in B} \mu(b, w) \leq \sum_{b \in B} c = c\binom{n}{d}$. In $P(1, d; n)$, each $a \in A$ is incomparable with $\binom{n-1}{d}$ elements of B. By the definition of a fractional local realizer, $\sum\{w(M) : M \in \operatorname{ple}(P), u > v \text{ in } M\} \geq 1$ for each incomparable pair (u, v). Therefore $R(w) \geq n\binom{n-1}{d}$. Similarly, the definition of a fractional local realizer requires that every ordered pair (u, v) of distinct elements with $u \leq v$ or $(v, u) \in \operatorname{Inc} P$ satisfies $\sum\{w(M) : M \in \operatorname{ple}(P), u \leq v \text{ in } M\} \geq 1$. Since there are $n\binom{n}{d}$ of these ordered pairs in $\operatorname{Min} P \times \operatorname{Max} P$ alone, $Q(w) \geq n\binom{n}{d}$. Therefore (c, w) is a solutions for LP. □

Accordingly, to obtain a lower bound for $\operatorname{fldim}(1, d; n)$, we focus on finding a good lower bound on c_0. Although, the value of c_0 depends both on d and n, we will show that with d fixed and $n \to \infty$, c_0 tends to the minimum of c_{bst} and c_{bal}.

3.2 Properties of Optimum Solutions

Let $\mathcal{W}_0$ denote the family of all local weight functions w such that (c_0, w) is an optimum solution for LP. In this part of the argument, we develop four tie-breaking rules: TB(1), TB(2), TB(3), and TB(4), which will be used to select a local weight function $w \in \mathcal{W}_0$ satisfying several useful properties. Since the rules will not be presented at the same time, we use the "computer science" tradition for notation so that $\mathcal{W}_0$ consists of those optimum local weight functions which remain tied after all conditions developed to this point in the argument have been applied.

The following elementary proposition is stated for emphasis.

Proposition 3.2 *If $w \in \mathcal{W}_0$, at least one of the two constraints A and B is tight, and at least one of the two constraints R and Q is tight.*

Here is our first tie-breaking rule for the selection of w:

TB(1). Maximize the quantity $R(w) + Q(w) - A(w) - B(w)$.

We note that TB(1) prefers optimum solutions where there is a "surplus" in the constraints R and Q and/or "slack" in the constraints A and B. The following elementary proposition follows from tie-breaker TB(1).

Proposition 3.3 *For $M \in \operatorname{ple}(P)$, if $w \in \mathcal{W}_0$ and $w(M) > 0$, then both $a(M)$ and $b(M)$ are positive, so that at least one of $q(M)$ and $r(M)$ are positive.*

Proof For contradiction, suppose $a(M) = 0$. (The case for $b(M) = 0$ is similar.) Therefore $b(M) > 0$ and $q(M) = r(M) = 0$. Create a new local weight function w' which agrees with w on all ple's except $w'(M) = 0$. Observe $A(w') = A(w)$, $B(w') = B(w) - b(M)w(M)$, $Q(w') = Q(w)$, and $R(w') = R(w)$. Therefore w' is a solution to LP with cost $c' \leq c_0$. Since $w(M)$ and $b(M)$ are positive, w' is preferred to w by TB(1), a contradiction. □

So for the remainder of the argument, we discuss only ple's $M \in \operatorname{ple}(P)$ for which both $a(M)$ and $b(M)$ are positive. This guarantees that at least one of $q(M)$ and $r(M)$ is positive. If M is a ple of this type, there is a uniquely determined integer $s = s(M)$, called the *block-length*, with $0 \leq s \leq n - d$ so that M has the following block-structure:

$$M = [A_0 < B_1 < A_1 < B_2 < A_2 < \cdots < B_s < A_s < B_{s+1}], \tag{3}$$

where (1) $\{B_1, B_2, \ldots, B_{s+1}\}$ is a family of pairwise disjoint subsets of $\mathrm{Max}(P)$ with $B_i \neq \emptyset$ when $1 \leq i \leq s$, and (2) $\{A_0, A_1, \ldots, A_s\}$ is a family of pairwise disjoint subsets of $\mathrm{Min}(P)$ with $A_i \neq \emptyset$ when $1 \leq i \leq s$. Note that we allow either or both of A_0 and B_{s+1} to be empty. Also note that $q(M)$ and $r(M)$ are determined entirely by the block-length s of M and the sizes of the blocks in the block-structure of M. Neither value is affected by the arrangement of elements in a block.

For the balance of the argument, whenever we refer to a ple M of P, we will use the letter $s = s(M)$ to denote the block-length of M and we will assume that the block-structure of M is given by Eq. 3. Also, when we refer to a pair (b, a), it will always be the case that a and b are positive integers with $1 \leq a \leq n$ and $1 \leq b \leq \binom{n}{d}$.

The statement of the following proposition, which again follows immediately from $\mathrm{TB}(1)$, uses a convention for subscripts that will be used for the balance of this section. When we are discussing a finite family $\{M_1, M_2, \ldots, M_t\}$ of ple's, for each $i \in \{1, \ldots, t\}$, we will use the terms in $(a_i, b_i, s_i, r_i, q_i, w_i)$ as abbreviations for the terms in $(a(M_i), b(M_i), s(M_i), r(M_i), q(M_i), w(M_i))$. If we make changes to a local weight function w to form a local weight function w', the value of $w'(M_i)$ is abbreviated as w'_i.

Proposition 3.4 *Let $w \in \mathcal{W}_0$. Then there do not exist* ple*'s M_1 and M_2 such that:*

(1) $w_1 > 0$,
(2) $r_1/a_1 \leq r_2/a_2$, $r_1/b_1 \leq r_2/b_2$, $q_1/a_1 \leq q_2/a_2$, *and* $q_1/b_1 \leq q_2/b_2$, *and*
(3) *at least one of the four inequalities in the preceding requirement is strict.*

Proof For contradiction, suppose there exist M_1 and M_2 with the listed properties. First assume both r_2 and q_2 are positive. By property (2), we can choose ε with $\max\{r_1/r_2, q_1/q_2\} \leq \varepsilon \leq \min\{a_1/a_2, b_1/b_2\}$. By (3), the inequality between ε and one of these four fractions is strict.

Define a new weight function w' which agrees with w on all ples except $w'_1 = 0$ and $w'_2 = w_2 + \varepsilon w_1$. Now the following inequalities verify that w' is a solution to LP.

$$\begin{aligned}
A(w') &= A(w) - w_1 a_1 + \varepsilon w_1 a_2 \leq A(w) - w_1 a_1 + \frac{a_1}{a_2} w_1 a_2 = A(w).\\
B(w') &= B(w) - w_1 b_1 + \varepsilon w_1 b_2 \leq B(w) - w_1 b_1 + \frac{b_1}{b_2} w_1 b_2 = B(w)\\
R(w') &= R(w) - w_1 r_1 + \varepsilon w_1 r_2 \geq R(w) - w_1 r_1 + \frac{r_1}{r_2} w_1 r_2 = R(w)\\
Q(w') &= Q(w) - w_1 q_1 + \varepsilon w_1 q_2 \geq Q(w) - w_1 q_1 + \frac{q_1}{q_2} w_1 q_2 = Q(w)
\end{aligned}$$

So w' is a solution to LP and, by (3), one of these inequalities is strict. Therefore

$$R(w') + Q(w') - A(w') - B(w') > R(w) + Q(w) - A(w) - B(w).$$

Thus w' is preferred to w by $\mathrm{TB}(1)$, a contradiction.

Now assume $r_2 = 0$. The argument for when $q_2 = 0$ is almost identical. By property (2), $r_1 = 0$ and therefore q_1 and q_2 are positive by Proposition 3.3. Since $r_1 = r_2 = 0$, property (3) states that $q_1/a_1 < q_2/a_2$ or $q_1/b_1 < q_2/b_2$. Select ε with $q_1/q_2 \leq \varepsilon \leq \min\{a_1/a_2, b_1/b_2\}$, noting that the inequality between ε and one of these three fractions is strict. When we define w' as above, we see $R(w') = R(w)$, $A(w') \leq A(w)$, $B(w') \leq B(w)$, and $Q(w') \geq Q(w)$ where one of the three inequalities is strict. As a result, w' is preferred to w by TB(1), a contradiction. $\square$

For a pair (b, a), we let ple(b, a) denote the set of all ple's M with $a(M) = a$ and $b(M) = b$. We say that a ple $M \in$ ple(b, a) is *maximal-preserving* if there is no $M' \in$ ple(b, a) with $q(M') > q(M)$. Determining the structure of maximal-preserving ple's is trivial, since a ple M is maximal-preserving if and only if it has block-length 0 and block-structure $M = [B_0 < A_1]$. Also, if $M \in$ ple(b, a) and M is maximal-preserving, then $r(M) = 0$ and $q(M) = ab$.

Lemma 3.5 *Let $M_1 \in$ ple(b, a). If M_1 is not maximal-preserving, there exists a* ple $M_2 \in$ ple(b, a) *with $q_2 = q_1 + 1$.*

Proof If $s_1 = 0$, then M_1 is maximal-preserving, so we know that $s_1 \geq 1$. Therefore, both B_1 and A_1 are non-empty and $r_1 > 0$. Let S be any set in A_1, and let u be any element of B_1. Form a ple M_2 by setting:

$$M_2 = [A_0 < B_1 - \{S\} < \{u\} < \{S\} < A_1 - \{u\} < B_2 < A_2 < \cdots < A_{s_1} < B_{s_1+1}].$$

Clearly, M_2 satisfies the requirements of the lemma. $\square$

Lemma 3.6 *If $w \in \mathcal{W}_0$, then constraint R is tight.*

Proof Let $w \in \mathcal{W}_0$. We assume that constraint R is not tight for w and argue to a contradiction. By Proposition 3.2, constraint Q must be tight. Thus there is a positive number δ so that $R(w) = n\binom{n-1}{d} + \delta$ while $Q(w) = n\binom{n}{d}$.

Let M_1 be any ple with r_1 and w_1 both positive. Then M_1 is not maximal-preserving. Let M_2 be the ple provided by Lemma 3.5.

Let $\varepsilon = \min\{w_1, \delta/2\}$. Form a local weight function w' by making the following two changes to w: set $w'_1 = w_1 - \varepsilon$ and $w'_2 = w_2 + \varepsilon$. Then $A(w') = A(w)$ and $B(w') = B(w)$. Furthermore, $R(w') = R(w) - \varepsilon$ and $Q(w') = Q(w) + \varepsilon$. Since $\varepsilon \leq \delta/2$, (c_0, w') is a feasible solution for LP with neither constraint R nor constraint Q being tight. The contradiction completes the proof of the lemma. $\square$

Analogous with our treatment for maximal-preserving ple's, we will say that a ple $M \in$ ple(b, a) is *maximal-reversing* if there is no ple $M' \in$ ple(b, a) with $r(M') > r(M)$. We are now ready for the analogue of Lemma 3.5.

Lemma 3.7 *Let $M \in$ ple(b, a) and let s be the block-length of M. Then M is maximal-reversing if and only if M satisfies the following five properties:*

Property 1 *If $0 \leq i \leq s$, $u \in A_i$ and $S \in B_{i+1}$, then $u \in S$.*

Property 2 *If $1 \leq i \leq s$ and $B_{i+1} \neq \emptyset$, then $|A_i| = 1$.*

Property 3 *If $1 \leq i \leq s$ and $B_{i+1} \neq \emptyset$, then B_i consists of all sets $S \in$ Max(P) such that (1) $A_{i-1} \subset S$ and (2) S contains no elements of $A_{i+1} \cup A_{i+2} \cup \cdots \cup A_s$.*

Property 4 $|B_{s+1}| = \max\{0, b - \binom{n-1}{d}\}$.

Property 5 $|A_0| = \max\{0, a - (n - d)\}$.

Furthermore, if $M_1 \in \text{ple}(b, a)$ *is not maximal-reversing, then there is* $M_2 \in \text{ple}(b, a)$ *with* $r_2 = r_1 + 1$.

Proof Before we present the proof, we pause to make two observations. First, it is easy to see that a ple satisfying Properties 1, 2 and 3, and one of Properties 4 and 5 satisfies all five properties. We have elected to state the hypothesis in this form so that the argument will be symmetric.

Second, given a pair (b, a), if M_1 and M_2 belong to $\text{ple}(b, a)$ and both M_1 and M_2 satisfy all five properties, then it is easy to see that M_1 and M_2 have the same block-length and the same block-structure. Therefore, $r_1 = r_2$ and $q_1 = q_2$. Accordingly, to complete the proof of the lemma, we need only show that if $M_1 \in \text{ple}(b, a)$ violates one of the five properties in the hypothesis, then there is a ple $M_2 \in \text{ple}(b, a)$ with $r_2 = r_1 + 1$.

Now suppose that M_1 does not satisfy Property 1. Then there is some i with $0 \le i \le s_1$ for which there is an element $u \in A_i$ and a set $S \in B_{i+1}$ such that $u \notin S$. In this case, we form a ple $M_2 \in \text{ple}(b, a)$ by setting:

$$M_2 = [A_0 < B_1 < A_1 < B_2 < \cdots < A_i - \{u\} < \{S\} < \{u\} < B_{i+1} - \{S\} < \\ < A_{i+1} < B_{i+2} < \cdots < A_{s_1} < B_{s_1+1}].$$

Then $r_2 = r_1 + 1$. Accordingly, we may assume that M_1 satisfies Property 1.

Now suppose that M_1 violates Property 2. Then there is some i with $1 \le i \le s_1$ such that $B_{i+1} \ne \emptyset$ and $|A_i| \ge 2$. Let u and u' be distinct elements of A_i. Then $u, u' \in S$ for every set $S \in B_{i+1}$ by Property 1. Fix a set $S \in B_{i+1}$. Since $i \ge 1$, we know $B_i \ne \emptyset$. Let S' be any set in B_i, and let v be any element of $S' - S$. Then let T be the set obtained from S by removing u' and replacing it with v. Clearly, the set T does not belong to M_1. Form $M_2 \in \text{ple}(b, a)$ by setting:

$$M_2 = [A_0 < B_1 < A_1 < B_2 < \cdots < A_i - \{u'\} < \{T\} < \{u'\} < B_{i+1} - \{S\} < \\ < A_{i+1} < B_{i+2} < \cdots < A_{s_1} < B_{s_1+1}].$$

Again, we note that $r_2 = r_1 + 1$. Accordingly, we may assume that M_1 satisfies Properties 1 and 2.

Now suppose that M_1 violates Property 3. Let i be the least integer with $1 \le i \le s_1$ for which $B_{i+1} \ne \emptyset$ and there is a set T such that (1) $A_{i-1} \subset T$; (2) T contains no element of $A_i \cup A_{i+1} \cup \cdots \cup A_{s_1}$; and (3) $T \notin B_i$. Clearly, the set T is not in M_1. Let S be any set in A_{i+1}. Form $M_2 \in \text{ple}(b, a)$ by setting:

$$M_2 = [A_0 < B_1 < A_1 < B_2 < \cdots < B_i \cup \{T\} < A_i < B_{i+1} - \{S\} < \\ < A_{i+1} < B_{i+2} < \cdots < A_{s_1} < B_{s_1+1}].$$

It follows that M_1 satisfies Properties 1, 2 and 3. In view of our observations at the start of the proof, we are left with the case that M_1 violates both Properties 4 and 5. So the inequalities $|B_{s_1+1}| \ge \max\{0, b - \binom{n-1}{d}\}$ and $|A_0| \ge \max\{0, a - (n - d)\}$ must both be strict. Now let $u \in A_0$ and $S \in B_1$. It follows that there is a set $T \in \text{Max}(P)$ such that T contains no element of $\{u\} \cup A_1 \cup \cdots \cup A_{s_1}$. Then T is not in M_1. Form a ple M_2 by setting:

$$M_2 = [A_0 - \{u\} < \{T\} < \{u\} < B_1 - \{S\} < A_1 < B_2 < A_2 < \cdots < B_{s_1} < A_{s_1} < B_{s_1+1}].$$

Since $r_2 = r_1 + 1$, the contradiction completes the proof of the lemma. □

For the balance of the proof, given a pair (b, a), we let $M(b, a)$ denote a maximal-reversing ple from $\text{ple}(b, a)$, and we abbreviate $r(M(b, a))$ and $q(M(b, a))$ as $r(b, a)$ and $q(b, a)$, respectively.

Now for the second tie-breaking rule:

TB(2). Minimize the number t_2 of ple's in $\text{ple}(P)$ which are assigned positive weight by w and are neither maximal-preserving nor maximal-reversing.

Lemma 3.8 *The value of t_2 is zero, i.e., there is a local weight function $w \in \mathcal{W}_0$ such that if $w(M) > 0$, then either M is maximal-preserving or maximal-reversing.*

Proof We argue by contradiction. Let $w \in \mathcal{W}_0$. Then there is a ple M_1 with $w_1 > 0$ such that M_1 is neither maximal-preserving nor maximal-reversing. Since M_1 is not maximal-preserving, $r_1 > 0$. If $q_1 = 0$, the block-length of M_1 is 1 and the block-structure of M_1 is $[B_1 < A_1]$ which implies M_1 is maximal-reversing. It follows that $q_1 > 0$.

Applying Lemma 3.5 repeatedly to M_1, let i_2 be the largest integer for which there is a ple M_2 such that $a_2 = a_1$, $b_2 = b_1$, $q_2 = q_1 + i_2$ (and $r_2 = r_1 - i_2$). Note that i_2 is positive and M_2 is maximal-preserving.

Applying Lemma 3.7 repeatedly to M_1, let i_3 be the largest integer for which there is a ple M_3 such that $a_3 = a_1$, $b_3 = b_1$, $r_3 = r_1 + i_3$ (and $q_3 = q_1 - i_3$). Again, we note that $i_3 > 0$ and M_3 is maximal-reversing.

We determine a local weight function w' by making the following three changes to w: Set $w'_1 = 0$, $w'_2 = w_2 + w_1 i_3/(i_2 + i_3)$ and $w'_3 = w_3 + w_1 i_2/(i_2 + i_3)$. With these values $A(w') = A(w)$, $B(w') = B(w)$, $Q(w') = Q(w)$ and $R(w') = R(w)$, so that $w' \in \mathcal{W}_0$. However, the value of t_2 has gone down by one. This observation completes the proof. □

Here is an another easy consequence of TB(1) and TB(2).

Lemma 3.9 *Let $w \in \mathcal{W}_0$ and let M be a* ple *with $w(M) > 0$. Then the following statements hold:*

(1) *If M is maximal-preserving, then $a(M) = n$ and $b(M) = \binom{n}{d}$ so that $M = L = [\text{Min}(P) < \text{Max}(P)]$ is a linear extension of P.*
(2) *If M is maximal-reversing, $s = s(M)$ and $k = |A_1| + |A_2| + \cdots + |A_s|$, then $|B_1| = \binom{n-k}{d}$.*

Proof To prove the first statement, we assume that M_1 satisfies $w_1 > 0$, $r_1 = 0$ and $a_1 + b_1 < n + \binom{n}{d}$. We note that $q_1 = a_1 b_1$ and $r_1 = 0$. Therefore (1) $q_1/a_1 = b_1 \le \binom{n}{d}$ and (2) $q_1/b_1 = a_1 \le n$. Then let M_2 be a maximal-preserving linear extension of P with block-structure $M_2 = [\text{Min}(P) < \text{Max}(P)]$. It follows that the pair (M_1, M_2) violates Proposition 3.4.

For the second statement, let M_1 be a maximal-reversing ple with $w_1 > 0$, and let $k = |A_1| + |A_2| + \cdots + |A_{s_1}|$. If $a_1 \ge n - d$, then $k = n - d$ and $|B_1| = 1$, as required. Now suppose that $a_1 < n - d$. Then $A_0 = \emptyset$. If $s_1 \ge 2$, then $|B_1| = \binom{n-k}{d}$ follows from Property 3.

Now suppose $s_1 = 1$. Then M_1 has block-structure $[B_1 < A_1]$. Therefore $r_1 = a_1 b_1$ and $q_1 = 0$. Suppose that $|B_1| < \binom{n-k}{d} = \binom{n-a_1}{d}$. Let M_2 be a ple obtained from M_1 simply by adding to B_1 a set S which does contain any of the minimal elements in A_1 and does not belong to B_1. Then $a_2 = a_1$, $b_2 = b_1 + 1$, $r_2 = r_1 + b_1$ and $q_2 = q_1 = 0$. It follows that the pair (M_1, M_2) violates Proposition 3.4. □

We will say that a maximal-reversing ple M of block-length s is *block-regular* if the following two conditions are met:

(1) Either $A_0 = \emptyset$ or $a = n$.
(2) Either $b = \binom{n}{d}$ or $b = \binom{n-|A_s|}{d}$.

Here is our third tie-breaking rule:

TB(3). Minimize the number t_3 of ple's in ple(P) which are assigned positive weight by w and are maximal-reversing but not block-regular.

Lemma 3.10 *The value of t_3 is 0, i.e., there is an optimum local weight function $w \in \mathcal{W}_0$ so that if $M \in$ ple(P) and $w(M) > 0$, then either M is a maximal-preserving linear extension of P, or M is block-regular.*

Proof We argue by contradiction and assume that $t_3 > 0$. Let $w \in \mathcal{W}_0$ and consider the non-empty family $\mathcal{F}$ of all ple's assigned positive weight which are maximal-reversing but not block-regular. We show that we can devise a sequence of weight shifts to decrease the value of t_3. Describing these shifts involves three cases, and we give full details for one of them. First, suppose that $M_1 \in \mathcal{F}$, $1 \le a_1 \le n-d$ and $1 \le b_1 < \binom{n-1}{d}$. In this case, we must have $0 < |B_{s_1}| < \binom{n-|A_{s_1}|}{d-1}$. Note that this requires $s_1 \ge 2$.

Let M_2 be a block-regular ple with $a_2 = a_1$ and $s_2 = s_1 - 1$. Also, let M_3 be a block-regular ple with $a_3 = a_1$ and $s_3 = s_1$. Note that $b_2 < b_1 < b_3$.

Then there are positive numbers λ_2 and λ_3 with $\lambda_2 + \lambda_3 = 1$ so that $b_1 = \lambda_2 b_2 + \lambda_3 b_3$. Simple calculations then show that $r_1 = \lambda_2 r_2 + \lambda_3 r_3$ and $q_1 = \lambda_2 q_2 + \lambda_3 q_3$. Form a local weight function w' from w by making the following three changes: $w_1' = 0$, $w_2' = w_2 + \lambda_2 w_1$ and $w_3' = w_3 + \lambda_3 w_1$. Then $A(w') = A(w)$, $B(w') = B(w)$, $R(w') = R(w)$ and $Q(w') = Q(w)$, so $w' \in \mathcal{W}_0$. However, the value of t_3 has gone down by 1. The contradiction shows that if $M_1 \in \mathcal{F}$, then either $n-d < a_1 < n$ or $\binom{n-1}{d} < b_1 < \binom{n}{d}$.

If $\binom{n-1}{d} < b_1 < \binom{n-1}{d}$, then we shift weight from M_1 to $M_2 = M(\binom{n-1}{d}, a_1)$ and $M_3 = M(\binom{n}{d}, a_1)$. If $n-d < a_1 < n$, we shift weight from M_1 to $M_2 = M(b_1, n-d)$ and $M_3 = M(b_1, n)$. After at most two applications of such shifts, the value of t_3 has been decreased. □

Considering the structure of LP, the following terminology is natural. Let $M \in$ ple(b, a) We will say that:

(1) M is *balanced* if $bn = a\binom{n}{d}$.
(2) M is *A-heavy* if $bn < a\binom{n}{d}$.
(3) M is *B-heavy* if $bn > a\binom{n}{d}$.

We note that linear extensions of P are balanced, and so is the block-regular ple $M(\binom{n-1}{d}, n-d)$. Due to the limitations of integer arithmetic, we may expect that many maximal-reversing ple's are not balanced. On the other hand, the following lemma asserts that collectively, the family of ple's assigned positive weight is balanced.

Lemma 3.11 *If $w \in \mathcal{W}_0$, then both constraints A and B are tight.*

Proof We argue by contradiction and show that constraint B must be tight. The argument for constraint A is symmetric. Since constraint B is not tight, constraint A must be tight by Proposition 3.2. Accordingly, there is a positive real number δ so that $A(w) = nc_0$ and $B(w) = \binom{n}{d}c_0 - \delta$. In turn, this implies that w assigns positive weight to at least one ple which is block-regular and A-heavy.

Now suppose that w assigns positive weight to a block-regular ple $M_1 = M(b, a)$ with $a_1 < \binom{n-1}{d}$. Let M_2 be the maximal-reversing (but not block-regular) ple $M(b_1 + 1, a_1)$. We note that $r_2 > r_1$ and $q_2 > q_1$.

Let $\varepsilon_1 = \min\{w_1, \delta/2\}$. Then set $\varepsilon_2 = \max\{\varepsilon_1 r_1/r_2, \varepsilon_1 q_1/q_2\}$. Then make the following two changes to w: Set $w_1' = w_1 - \varepsilon_1$ and $w_2' = w_2 + \varepsilon_2$. Note first that $R(w') = R(w) - \varepsilon_1 r_1 + \varepsilon_2 r_2 \geq R(w)$. Similarly, $Q(w') \geq Q(w)$. On the other hand, since $\varepsilon_2 < \varepsilon_1$, we have:

$$A(w') = A(w) - \varepsilon_1 a_1 + \varepsilon_2 a_2 = A(w) - \varepsilon_1 a_1 + \varepsilon_2 a_1 < A(w) = nc_0.$$

Also, we have:

$$\begin{aligned} B(w') &= B(w) - \varepsilon_1 b_1 + \varepsilon_2 b_2 \\ &= B(w) - \varepsilon_1 b_1 + \varepsilon_2 (b_1 + 1) \\ &\leq B(w) - \varepsilon_1 b_1 + \varepsilon_1 (b_1 + 1) \\ &= B(w) + \varepsilon_1 \\ &\leq B(w) + \delta/2 \\ &< \binom{n}{d} c_0. \end{aligned}$$

It follows that $w' \in \mathcal{W}_0$ but neither constraint A nor constraint B is tight, contradicting Proposition 3.2. This forces $b(M) \geq \binom{n-1}{d}$ for every block-regular ple M assigned positive weight by w.

It follows that w assigns positive weight to the block-regular ple $M_1 = M(\binom{n-1}{d}, n)$. Let $M_2 = M(\binom{n-1}{d}, n - d)$ and $M_3 = M(\binom{n}{d}, n)$. We note that $r_1 = r_2 = r_3$ and $q_2 < q_1 < q_3$.

Let $\varepsilon_1 = \min\{w_1, \delta/(2\binom{n}{d})\}$. Then set $\varepsilon_2 = \varepsilon_1(q_3 - q_1)/(q_3 - q_2)$ and $\varepsilon_3 = \varepsilon_1(q_1 - q_2)/(q_3 - q_2)$. Form w' by making the following three changes. Set $w_1' = w_1 - \varepsilon_1$, $w_2' = w_2 + \varepsilon_2$ and $w_3' = w_3 + \varepsilon_3$.

Since $\varepsilon_1 = \varepsilon_2 + \varepsilon_3$, we have $R(w') = R(w)$. An easy calculation shows that we also have $Q(w') = Q(w)$. For constraint A, we have:

$$A(w') = A(w) - \varepsilon_1 n + \varepsilon_2 (n - d) + \varepsilon_3 n < A(w) - \varepsilon_1 n + \varepsilon_2 n + \varepsilon_3 n = A(w) = nc_0.$$

For constraint B, we have:

$$\begin{aligned} B(w') &= B(w) - \varepsilon_1 \binom{n-1}{d} + \varepsilon_2 \binom{n-1}{d} + \varepsilon_3 \binom{n}{d} \\ &= B(w) + \varepsilon_3 \left(\binom{n}{d} - \binom{n-1}{d} \right) \\ &< B(w) + \varepsilon_1 \binom{n}{d} \\ &\leq B(w) + \delta/2 \\ &< \binom{n}{d} c_0. \end{aligned}$$

Again, these calculations show that $w' \in \mathcal{W}_0$ but neither constraint A nor constraint B is tight. The contradiction completes the proof. $\square$

We close this subsection with a two useful observations about maximal preserving ple's. The elementary arguments are left as exercises.

Proposition 3.12 *Fix an integer a with* $1 \le a \le n$. *Then the ratio* $q(b,a)/r(b,a)$ *is strictly increasing in a. Furthermore, over all pairs* (b,a), *the maximum value of* $q(b,a)/r(b,a)$ *is d and this occurs if and only if* $a = n$ *and* $b = \binom{n}{d}$.

3.3 Transitioning to the Continuous Setting

In the next part of the discussion, we will keep the value of d fixed, but we will transition from the discrete to continuous settings by analyzing the behavior of solutions as $n \to \infty$. Since our emphasis is on combinatorics and not analysis, we proceed in an informal manner, but it should be clear to the reader how our arguments can be recast in a completely formal setting.

Fix a block-regular ple $M = M(b,a)$ with $1 \le a \le n-d$ and $b \le \binom{n-1}{d}$. With M fixed for the moment, we abbreviate the quantities in $(a(M), b(M), r(M), q(M), s(M))$ as (a, b, r, q, s), respectively. We also set $m = |A_s|$, so that $b = \binom{n-m}{d}$.

We then have the following *exact* formula for $r = r(M)$:

$$\begin{aligned} r &= \binom{n-a}{d} + \binom{n-a+1}{d} + \cdots + \binom{n-m-1}{d} + m\binom{n-m}{d} \\ &= m\binom{n-m}{d} + \binom{n-m}{d+1} - \binom{n-a}{d+1} \\ &= \frac{1}{d+1}\left[(n+md-d)\binom{n-m}{d} - (n-a-d)\binom{n-a}{d}\right]. \end{aligned} \tag{4}$$

We define quantities x and y by setting $x = b/\binom{n}{d}$ and $y = a/n$. Since $b = \binom{n-m}{d}$, it follows that (in the limit) $x = (1 - m/n)^d$. Since $a \ge m$, it follows that we have the following restrictions on the pair (x, y):

$$0 \le x, y \le 1 \qquad\qquad y \ge 1 - x^{1/d}. \tag{5}$$

Previously, we analyzed the quantity $r(b,a)$. Now we study the ratio $r(x,y)$ which we define by setting:

$$r(x,y) = \frac{r(b,a)}{n\binom{n}{d}}.$$

We then have the following formula for $r(x,y)$:

$$r(x,y) = \frac{1}{d+1}\left[(d+1)x - dx^{\frac{d+1}{d}} - (1-y)^{d+1}\right]. \tag{6}$$

It is not necessary to develop a parallel formula for $q(x,y)$ since $q(x,y) = xy - r(x,y)$. In the continuous setting, the linear programming problem LP becomes:

Minimize the quantity c subject to the following constraints:

$$A(w) = \sum_{M \in \mathrm{ple}(P)} y(M) \cdot w(M) \le c.$$

$$B(w) = \sum_{M \in \mathrm{ple}(P)} x(M) \cdot w(M) \le c.$$

$$R(w) = \sum_{M \in \text{ple}(P)} r(M) \cdot w(M) \geq 1.$$

$$Q(w) = \sum_{M \in \text{ple}(P)} q(M) \cdot w(M) \geq 1.$$

We continue to refer to these inequalities as constraints A, B, R and Q, respectively.

3.4 Ratio Curves

Let ρ be a non-negative number. We call the set of all pairs (x, y) such that $q(x, y)/r(x, y) = \rho$ the *ρ-ratio curve*. One special case is $\rho = 0$ and we note that the 0-ratio curve is just the set of pairs (x, y) with $y = 1 - x^{1/d}$.

Lemma 3.13 *For each $\rho \in \mathbb{R}_0$, the pairs (x, y) on the ρ-ratio curve determine a decreasing function of x.*

Proof The equation $q(x, y)/r(x, y) = \rho$ is equivalent to $xy/r(x, y) = 1 + \rho$. Using implicit differentiation and abbreviating $r(x, y)$ as r, we obtain the following formula for the derivative $\frac{dy}{dx}$ at the point (x, y):

$$\frac{dy}{dx} = -\frac{y[r - x(1 - x^{1/d})]}{x[r - y(1 - y)^d]}.$$

In this quotient, when $0 < x, y < 1$, the numerator is positive since $r > x(1 - x^{1/d})$. Note that, in the discrete setting, this is just saying that $r > bm$. Similarly, the denominator is positive since $r > y(1 - y)^d$. In the discrete setting, if b_1 is the size of the bottom block of maximal elements, this inequality is simply the statement that $r > ab_1$. Therefore $\frac{dy}{dx} < 0$ and the pairs on the ρ-ratio curve form a decreasing function in x. □

Lemma 3.14 *Let ρ be a fixed non-negative number and consider the set of all pairs (x, y) on the ρ-ratio curve with $x, y > 0$. Among these pairs, the quantity $r(x, y)$ is maximized and the quantity $(x + y)/r(x, y)$ is minimized when $x = y$.*

Proof We outline the proof for the first statement. The argument for the second is similar. On the ρ-ratio curve, y is a function of x, and we have a formula for $\frac{dy}{dx}$ from the proof of the preceding lemma. It follows that on this ratio curve, $r(x, y)$ is a function of x. Again, using implicit differentiation, we have:

$$\frac{dr}{dx} = 1 - x^{1/d} + (1 - y)^d \frac{dy}{dx}.$$

From this expression, it is straightforward to verify that $\frac{dr}{dx}$ is positive when $x < y$, 0 when $x = y$ and negative when $x > y$. Therefore, on the ρ-ratio curve, $r(x, y)$ is maximum when $x = y$. □

Lemma 3.15 *If $M = M(x, y)$, $w \in \mathcal{W}_0$ and $w(M) > 0$, then M is balanced.*

Proof We argue by contradiction and suppose that w assigns positive weight to a ple which is not balanced. Since constraints A and B are tight, it follows that there are ple's $M_1 =$

$N(x_1, y_1)$ and $M_2 = N(x_1, y_2)$, both with positive weight, so that M_1 is B-heavy and M_2 is A-heavy.

For $i \in \{1, 2\}$, let $\rho_i = q_i/r_i$. Let $M_3 = M(x_3, y_3)$ be the point on the ρ_1-ratio curve where $x_3 = y_3$. From the preceding lemma, we know that $r_3 > r_1$. Also, we know that $2x_3/r_3 < (x_1 + y_1)/r_1$. Also, let $M_4 = N(x_4, y_4)$ be the point on the ρ_2-ratio curve where $x_4 = y_4$. From the preceding lemma, we know that $r_4 > r_2$. Also, we know that $2x_4/r_4 < (x_2 + y_2)/r_2$.

Let w' be the local weight function obtained from w by making the following changes: $w'_1 = w'_2 = 0$, $w'_3 = w_3 + \varepsilon_1 r_3/r_1$ and $w'_4 = w_4 + \varepsilon_2 r_4/r_2$. Then $R(w') = R(w)$, $Q(w') = Q(w)$, $A(w') < A(w)$ and $B(w') < B(w)$. This is a contradiction since neither constraint A nor constraint B is tight. $\square$

For the balance of the argument, since we will be concerned exclusively with ple's on the balancing line (where $x = y$), we abbreviate $r(x, x)$ to $r(x)$. Similarly, $q(x, x)$ will be written as $q(x)$, and the ple $M(x, x)$ will now be just $M(x)$. Also, we will only be concerned with constraints R, Q and B.

Now for our fourth tie-breaker.

TB(4). Minimize the number t_4 of ple's in ple(P) which are assigned positive weight by w.

In view of the statement of our main theorem, and our outline for the proof, it is clear that ultimately we will show that $t_4 \leq 2$,

The following proposition consists of some relatively straightforward calculus exercises.

Proposition 3.16

(1) *The function* $r(x)/x$*, defined on the interval* $(0, 1]$*, is concave downwards on the interval* $(0, 1]$*, has a vertical asymptote at* $x = 0$ *and achieves a local maximum value at* $x = \delta$ *with* $\beta < \delta < x_{\text{bst}}$.

(2) *The function* $q(x)/x$ *is monotonically increasing on* $[\beta, 1]$.

Since $q(x)/x$ and $r(x)/x$ are increasing on the closed interval $[\beta, \delta]$, the following result is a straightforward application of Proposition 3.4 in the continuous setting.

Lemma 3.17 *There is no local weight function* $w \in \mathcal{W}_0$ *which assigns positive weight to a* ple $M = M(x)$ *with* $\beta \leq x < \delta$.

Next, we identify three balanced ple's, to be denoted M_{bst}, M_{bal} and M_{sat}. The subscripts in this notation are abbreviations, respectively, for best, balanced and saturated.

Recall, from Section 2, that x_{bst} is the unique point in the interval $[\beta, 1]$ where the function $g(x) = [(d + 1)r(x) - x]/[(d + 1)r(x) - x^2]$ achieves its maximum value. Also, x_{bal} is the unique point from $[\beta, 1]$ where $r(x) = q(x)$. Straightforward calculations show $\beta < \delta < x_{\text{bst}} < x_{\text{bal}} < 1$. Set $M_{\text{bst}} = M(x_{\text{bst}})$ and $M_{\text{bal}} = M(x_{\text{bal}})$.

With d fixed and $n \to \infty$, the block-regular ple $M(\binom{n}{d}, n)$ converges to the linear extension which we now denote as $M(1)$. We note that $r(1) = 1/(d + 1)$ and $q(1) = d/(d + 1)$. To be consistent with our notation for M_{bst} and M_{bal}, we set $x_{\text{sat}} = 1$ and $M_{\text{sat}} = M(1)$.

Lemma 3.18 *There is no* $w \in \mathcal{W}_0$ *which assigns positive weight to a* ple $M = M(x)$ *with* $x_{\text{bal}} < x < 1$.

Proof We argue by contradiction and assume some $w \in \mathcal{W}_0$ assigns positive weight w_0 to a ple $M_0 = M(x_0)$ with $x_{\text{bal}} < x_0 < 1$. Let $M_1 = M_{\text{bal}}$ and $M_2 = M_{\text{sat}}$. We will form a local weight function w' from w by removing all weight from M_0 and increasing the weight on M_1 and M_2 so that $R(w') = R(w)$, $Q(w') = Q(x)$ and $B(w') < B(w)$. To accomplish this task, we must find positive values $v_1 = w'_1 - w_1$ and $v_2 = w'_2 - w_2$ that satisfy the following three requirements:

$$\begin{aligned} w_0 r_0 &\le v_1 r_1 + v_2/(d+1) \\ w_0 q_0 &\le v_1 q_1 + v_2 d/(d+1) \\ w_0 x_0 &> v_1 x_1 + v_2. \end{aligned}$$

We consider the first two requirements as equations. Noting that $q_1 = r_1$, this results in the following solution:

$$\begin{aligned} v_1 &= w_0(dr_0 - q_0)/[(d-1)r_1] \\ v_2 &= (d+1)w_0(q_0 - r_0)/(d-1). \end{aligned}$$

From Proposition 3.12, we know $q_0/r_0 < d$ which shows that $v_1 > 0$. Also, since $x_{\text{bal}} < x_0 < 1$, we know $q_0 > r_0$ which implies $v_2 > 0$.

Using the fact that v_2 can be also be written as $(d+1)(w_0 r_0 - v_1 r_1)$, the third inequality is equivalent to:

$$\frac{(d+1)r_1 - x_1}{(d+1)r_1 - x_1^2} > \frac{(d+1)r_0 - x_0}{(d+1)r_0 - x_0^2}.$$

However, we know the function $g(x)$ is decreasing when $x \ge x_{\text{bst}}$, so this last inequality holds. The contradiction completes the proof of the lemma. □

Lemma 3.19 *Let $w \in \mathcal{W}_0$. Then there is no maximal-preserving linear extension which is assigned positive weight by w.*

Proof Suppose to the contrary that some $w \in \mathcal{W}_0$ assigns positive weight w_0 to the maximal-preserving linear extension $L_0 = [\text{Min}(P) < \text{Max}(P)]$. Since constraint R is tight, there must be some ple M_1 with $w_1 > 0$ and $r_1 > q_1$. Let $M_2 = M_{\text{sat}}$. We will then obtain w' from w by setting $w'_0 = w_0 - v_0$, $w'_1 = w_1 - v_1$ and $w'_2 = w_2 + v_2$. We require $0 < v_0 \le w_0$, $0 < v_1 \le w_1$ and $v_0/v_1 = dr_1 - q_1$. Clearly, appropriate values of v_0 and v_1 can be chosen. We then set $v_2 = (d+1)v_1 r_1$.

With these choices, it follows that $R(w') = R(w)$ and $Q(w') = Q(w)$. However,

$$\begin{aligned} B(w') &= B(w) - v_0 - v_1 x_1 + v_2 \\ &= B(w') - v_1(dr_1 - q_1) - v_1 x_1 + (d+1)v_1 r_1 \\ &= B(w') - v_1[dr_1 - (x_1^2 - r_1) - x_1 + (d+1)r_1] \\ &= B(w) - v_1(x_1 - x_1^2) \\ &< B(w). \end{aligned}$$

Recall that all ple's with positive weight are balanced, by Lemma 3.15, so we also have $A(w') < A(w)$, a contradiction which completes the proof of the lemma. □

Lemma 3.20 *Let $w \in \mathcal{W}_0$. If w assigns positive weight to a* ple $M = M(x)$ *with* $\delta \le x < x_{\text{bal}}$, *then* $x = x_{\text{bst}}$.

Proof Suppose to the contrary that w assigns positive weight to a ple $M_0 = M(x)$ with $\delta \leq x < x_{\text{bal}}$ and $x \neq x_{\text{bst}}$. Since $r_0 > q_0$, w must also assign positive weight to $M_2 = M_{\text{sat}}$. We form w' by making three changes. First, we set $w'_0 = w_0 - v_0$, and we will require that $0 < v_0 \leq w_0$. Also, there may be an additional restriction on the size of v_0, a detail that will soon be clear. Second, we will take $M_1 = M_{\text{bst}}$ and set $w'_1 = w_1 + v_1$ with $v_1 > 0$. Third, we will take $M_2 = M_{\text{sat}}$ and set $w'_2 = w_2 + v_2$, however we place no restrictions on the sign of v_2, except that we need $w'_2 \geq 0$, so if v_2 is negative, we need $|v_2| \leq w_2$. This will be handled if necessary by restricting the size of v_0.

Our choices will be made so that $R(w') = R(w)$, $Q(w') = Q(w)$ and $B(w') < B(w)$. This requires:

$$\begin{aligned} v_0 r_0 &= v_1 r_1 + v_2/(d+1), && (7)\\ v_0 q_0 &= v_1 q_1 + v_2 d/(d+1), && (8)\\ v_0 x_0 &> v_1 x_1 + v_2. && (9) \end{aligned}$$

Solving the first two equations, we obtain $v_1 = (dr_0 - q_0)/(dr_1 - q_1)$ so that $v_1 > 0$. Note that this equation can be rewritten as $v_1 = [(d+1)r_0 - x_0^2]/[(d+1)r_1 - x_1^2]$. We also obtain:

$$v_2 = (d+1)(v_0 r_0 - v_1 r_1).$$

Accordingly, when $v_2 < 0$, we can maintain $|v_2| \leq w_2$ by scaling the system (7) to keep v_0 sufficiently small.

After substituting the specified values, the third constraint $v_0 x_0 > v_1 x_1 + v_2$ becomes $g(x_1) > g(x_0)$, which is true since $x_0 \neq x_1$. Since w' can be obtained with the desired properties, we have contradicted $w \in \mathcal{W}_0$, completing the proof of the lemma. □

With these observations in hand, our linear programming problem reduces to:
Minimize the quantity c subject to the following constraints:

$$\begin{aligned} B(w) &= w(M_{\text{bst}})x_{\text{bst}} + w(M_{\text{bal}})x_{\text{bal}} + w(M_{\text{sat}}) \leq c\\ R(w) &= w(M_{\text{bst}})r(x_{\text{bst}}) + w(M_{\text{bal}})r(x_{\text{bal}}) + w(M_{\text{sat}})/(d+1) \geq 1\\ Q(w) &= w(M_{\text{bst}})q(x_{\text{bst}}) + w(M_{\text{bal}})q(x_{\text{bal}}) + w(M_{\text{sat}})d/(d+1) \geq 1 \end{aligned}$$

By increasing $w(M_{\text{bal}})$, we can decrease $w(M_{\text{bst}})$ and $w(M_{\text{sat}})$ or vice versa to see that there are optimal solutions with at most two non-zero values among the variables $w(M_{\text{bst}})$, $w(M_{\text{bal}})$ and $w(M_{\text{sat}})$. Furthermore, the form of the constraints tell us that the only possible solutions are (1) to put weight only on M_{bal}, or (2) to put positive weight only on M_{bst} and M_{sat}. The values of c_0 given in the statement of our main theorem reflect the calculations of c_0 which these two options would yield. With these remarks, the proof of our lower bound is complete.

4 Proof of the Upper Bound

In this section, we show that the lower bound obtained in the preceding section is also an upper bound. Our lower bound is the minimum of two quantities, so in this section we show that each of the two options provides an upper bound. As before, our treatment is informal.

Recall that $M(x)$ is a block-regular ple with $x = b/\binom{n}{d} = a/n$. In particular, $M_{\text{bal}} = M(x_{\text{bal}})$, $M_{\text{bst}} = M(x_{\text{bst}})$, and $M_{\text{sat}} = M(1)$ where x_{bal} and x_{bst} were defined in Section 2.

We start with the first option in which all weight is put on the ple $M_0 = M_{\mathrm{bal}}$ so that $x_0 = x_{\mathrm{bal}}$. We have $r_0 = r(x_0) = x_0^2/2$. Therefore $w_0 = w(M_0) = 1/r_0 = 2/x_0^2$. Also, we have $c_0 = w_0 x_0 = x_0/r_0 = 2/x_0$. To construct a fractional local realizer for P, we then distribute the weight w_0 evenly among all ple's which are isomorphic to M_0. With this rule, we will have:

(1) For a given pair (S, S') of distinct sets in $\mathrm{Max}(P)$, the total weight assigned to ple's M with $S > S'$ in M is $w_0 x_0^2/2 = 1$.
(2) For a given pair (u, u') of distinct sets in $\mathrm{Min}(P)$, the total weight assigned to ple's M with $u > u'$ in M is $w_0 x_0^2/2 = 1$.
(3) For a pair $(u, S) \in \mathrm{Min}(P) \times \mathrm{Max}(P)$ with $u \notin S$, the total weight assigned to ple's M with $u > S$ in M is $w_0 r_0 = 1$.
(4) For a pair $(u, S) \in \mathrm{Min}(P) \times \mathrm{Max}(P)$ with $u \notin S$, the total weight assigned to ple's M with $u < S$ in M is $w_0(x_0^2 - r_0) = w_0 r_0 = 1$.

We are left to consider pairs $(u, S) \in \mathrm{Min}(P) \times \mathrm{Max}(P)$ with $u \in S$. For the ple M_0, the fraction of $\mathrm{Min}(P)$ that is in the top box is $1 - x_0^{1/d}$, so the total weight assigned to ple's M with $u < S$ in M is $w_0 x_0[x_0 - (1 - x_0^{1/d})]$. We need this quantity to be at least 1, but this is equivalent to requiring that $r_0 = x_0^2/2 \geq x_0(1 - x_0^{1/d})$, and a straightforward computation shows that this inequality is valid.

For the second option, we take $M_1 = M_{\mathrm{bst}}$, $x_1 = x_{\mathrm{bst}}$, $M_2 = M_{\mathrm{sat}}$ and $x_2 = 1$. Now we have:

$$\begin{aligned} w_1 r_1 + w_2/(d+1) &= 1, \\ w_1 q_1 + w_2 d/(d+1) &= 1, \\ w_1 x_1 + w_2 &= c_0. \end{aligned} \tag{10}$$

We take all ple's of P which are isomorphic to M_1 and distribute the weight w_1 evenly among them. Similarly, we take all ple's that are isomorphic to M_2 and distribute the weight w_2 evenly among them. We then have:

(1) For a given pair (S, S') of distinct sets in $\mathrm{Max}(P)$, the total weight assigned to ple's M with $S > S'$ in M is $w_1 x_1^2/2 + w_2/2$. However, if we add the first two of the equations in Eq. 10 and divide by 2, we obtain:

$$w_1(r_1 + q_1)/2 + w_2/2 = w_1 x_1^2/2 + w_2/2 = 1.$$

(2) For a given pair (u, u) of distinct sets in $\mathrm{Min}(P)$, the total weight assigned to ple's M with $u > u'$ in M is $w_1 x_1^2/2 + w_2/2$. Again, this sum is 1.
(3) For a pair $(u, S) \in \mathrm{Min}(P) \times \mathrm{Max}(P)$ with $u \notin S$, the total weight assigned to ple's M with $u > S$ in M is $w_1 r_1 + w_2/(d+1) = 1$.
(4) For a pair $(u, S) \in \mathrm{Min}(P) \times \mathrm{Max}(P)$ with $u \notin S$, the total weight assigned to ple's M with $u < S$ in M is $w_1(x_1^2 - r_1) + w_2 d/(d+1)$. This reduces to the second equation of Eq. 10.

Again, we are left to consider pairs $(u, S) \in \mathrm{Min}(P) \times \mathrm{Max}(P)$ with $u \in S$. The desired inequality is:

$$w_1 x_1[x_1 - (1 - x_1^{1/d})] + w_2 \geq 1.$$

To obtain this inequality, we first observe that the inequality $d+1-dw_1r_1 \geq 1$ is equivalent to $1 \geq w_1r_1$, but this inequality holds strictly in view of the first equation of Eq. 10. The first two equations in Eq. 10 imply $w_2 = 2 - w_1x_1^2$. It follows that:

$$\begin{aligned} w_1x_1[x_1 - (1 - x_1^{1/d})] + w_2 &= w_1x_1[x_1 - (1 - x_1^{1/d})] + 2 - w_1x_1^2 \\ &= -w_1x_1(1 - x_1^{1/d}) + 2 \\ &\geq -w_1r_1 + 2 \\ &\geq 1. \end{aligned}$$

With these observations, the proof of the upper bound is complete.

5 Posets of Bounded Degree

Let d be a non-negative integer. One of the long-standing problems in dimension theory is to determine as accurately as possible the quantity $\mathrm{MD}(d)$ which we define to be the maximum dimension among all posets in which each element is comparable with at most d other elements. Trivially, $\mathrm{MD}(0) = \mathrm{MD}(1) = 2$. With a little effort, one can show that $\mathrm{MD}(2) = 3$. On the other hand, it is not at all clear that $\mathrm{MD}(d)$ is well-defined when $d \geq 3$.

However, in 1983, Rödl and Trotter [20] proved that $\mathrm{MD}(d)$ is well-defined and satisfies $\mathrm{MD}(d) \leq 2d^2 + 2$. Subsequently, Füredi and Kahn [12] proved in 1986 that $\mathrm{MD}(d) = O(d \log^2 d)$, and it is historically significant that their proof was an early application of the Lovász local lemma.

From below, the standard examples show $\mathrm{MD}(d) \geq d+1$. However, in 1991, Erdős, Kierstead and Trotter [9] improved this to $\mathrm{MD}(d) = \Omega(d \log d)$. From 1991 to 2018, there were no improvements in either bound, but after a gap of nearly 30 years, Scott and Wood [18] have recently made the following substantive improvement in the upper bound.

Theorem 5.1 *If $d \geq 10^4$ and $d \to \infty$, then*

$$\mathrm{MD}(d) \leq (2e^3 + o(1))(d \log d)\left(e^{2\sqrt{\log\log d}}\right) \log\log d.$$

As a consequence, we know $\mathrm{MD}(d) \leq d \log^{1+o(1)} d$.

Of course, the analogous problem can be considered for any of the variants of dimension, and in some cases, the following phenomenon is then observed: There is a bound on the parameter for the class of height 2 posets in which each maximal element is comparable with at most d minimal elements *independent* of how many maximal elements are comparable with a single minimal element. As our results show, fractional local dimension is one such parameter. Fractional dimension and Boolean dimension are two others. Curiously, local dimension is not, since $\mathrm{ldim}(1, 2; n)$ goes to infinity with n.

For general posets, one can ask for the maximum value $\mathrm{MFLD}(d)$ of the fractional local dimension of a poset in which each element is comparable with at most d others. Taking the split of such a poset, the fractional local dimension cannot increase more than 1, so we have $\mathrm{FLD}(d) + 2$ as an upper bound. However, we have no feeling as to whether this is (essentially) best possible.

6 Closing Comments

One natural open problem for this new parameter is to determine the maximum fractional local dimension among all posets on n points. The answer to the analogous question for fractional dimension is $\lfloor n/2 \rfloor$ when $n \geq 4$, and the standard examples show this inequality is tight. On the other hand, it is shown in [15] that the maximum local dimension of a poset on n points is $\Theta(n/\log n)$, and we suspect that this is also the answer for fractional local dimension.

Other open questions revolve around the interplay between fractional local dimension and width. For any poset with width $w \geq 3$, the dimension is at most w [6]. This inequality is tight for fractional dimension precisely when P contains the standard example S_w [11]. However, we don't know if this inequality is tight for local dimension as standard examples have local dimension 3 [1]. This leaves open the analogous question: What is the maximum value of the fractional local dimension of a poset with width w?

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