DIMENSION IS POLYNOMIAL IN HEIGHT FOR POSETS WITH PLANAR COVER GRAPHS

JAKUB KOZIK, PIOTR MICEK, AND WILLIAM T. TROTTER

ABSTRACT. We show that height h posets that have planar cover graphs have dimension $\mathcal{O}(h^6)$. Previously, the best upper bound was $2^{\mathcal{O}(h^3)}$. Planarity plays a key role in our arguments, since there are posets such that (1) dimension is exponential in height and (2) the cover graph excludes K_5 as a minor.

1. Introduction

In this paper, we study finite partially ordered sets, *posets* for short, and we assume that readers are familiar with the basics of the subject, including chains and antichains; minimal and maximal elements; height and width; order diagrams (also called Hasse diagrams); and linear extensions. For readers who are new to combinatorics on posets, several of the recent research papers cited in our bibliography include extensive background information.

Following the traditions of the subject, *elements* of a poset are also called *points*. Recall that when P is a poset, an element x is *covered* by an element y in P when $x < y$ in P and there is no element z of P with $x < z < y$ in P. We associate with P an ordinary graph G, called the *cover graph* of P, defined as follows. The vertex set of G is the ground set of P, and distinct elements/vertices x and y are adjacent in G when either x is covered by y in P or y is covered by x in P.

Dushnik and Miller [1] defined the *dimension* of a poset P, denoted $\dim(P)$, as the least positive integer d such that there are d linear orders $L_1, \ldots, L_d$ on the ground set of P such that $x \leqslant y$ in P if and only if $x \leqslant y$ in L_i for each $i \in \{1, \ldots, d\}$. In general, there are many posets that have the same cover graph, and among them, there may be

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posets which have markedly different values of height, width and dimension. Indeed, it is somewhat surprising that we are able to bound any combinatorial property of a finite poset in terms of graph theoretic properties of its cover graph.

However, Streib and Trotter [11] proved that dimension is bounded in terms of height for posets that have a planar cover graph. This stands in sharp contrast with a number of well-known families of posets that have height 2 but unbounded dimension (e.g. the standard examples discussed below). The result from [11] prompted researchers to investigate in greater depth connections between dimension and graph theoretic properties of cover graphs. Subsequently, it has been shown that dimension is bounded in terms of height for posets whose cover graphs:

- Have bounded treewidth, bounded genus, or more generally exclude an apex-graph as minor [4];
- Exclude a fixed graph as a (topological) minor [17, 10];
- Belong to a fixed class with bounded expansion [7].

Moreover, the existence of bounds for dimension of posets with cover graphs in a fixed class can say something about the sparsity of the class. Joret, Micek, Ossona de Mendez, and Wiechert [3] proved that a monotone class of graphs is nowhere dense if and only if for every $h \geqslant 1$ and every $\varepsilon > 0$, posets of height h with n elements whose cover graphs are in the class have dimension $\mathcal{O}(n^{\varepsilon})$.

The best upper bound to date on dimension in terms of height for posets that have planar cover graphs is $2^{\mathcal{O}(h^3)}$. This result can be extracted from [3] via connections between dimension for posets and *weak-coloring numbers* of their cover graphs. We will give additional details on this work in the next section.

Our main theorem improves this exponential bound to one which is polynomial in h.

Theorem 1. *If P is a poset of height h and the cover graph of P is planar, then $\dim(P) = \mathcal{O}(h^6)$.*

Planarity plays a crucial role in the existence of a polynomial bound. In [6], Joret, Micek and Wiechert show that for each even integer $h \geqslant 2$, there is a height h poset P with dimension at least $2^{h/2}$ such that the cover graph of P excludes K_5 as a minor.

To discuss lower bounds, we pause to give the following construction which first appears in [1]. For each $n \geqslant 2$, let S_n be the height 2 poset with $\{a_1, a_2, \dots, a_n\}$ the set of minimal elements, $\{b_1, b_2, \dots, b_n\}$ the set of maximal elements, and $a_i < b_j$ in S_n if and only if $i \neq j$. Posets in the family $\{S_n : n \geqslant 2\}$ are now called *standard examples*, as $\dim(S_n) = n$ for every $n \geqslant 2$.

To date, the best lower bound for the maximum dimension of a height h poset with a planar cover graph is $2h-2$, and this bound comes from the "double wheel" construction given in [6], and illustrated here in Figure 1. To avoid clutter, we do not show arrowheads in our figures. Instead, we indicate directions using color and accompanying narrative.

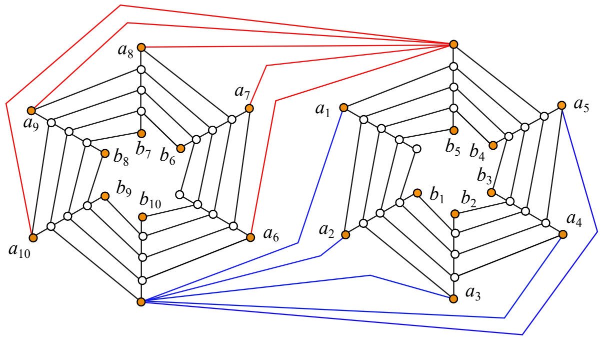

FIGURE 1. We illustrate the double wheel construction when $h = 5$. Note that the elements $a_1, \ldots, a_{10}$ and $b_1, \ldots, b_{10}$ induce a standard example, so the dimension of the depicted poset is at least 10. On the other hand, the height of P is 5.

In this figure, the black edges are oriented in each individual wheel from outside to inside. The elements of $\{a_1, \ldots, a_n\}$ are minimal elements so the red edges are oriented "left-to-right" and the blue edges are oriented "right-to-left."

Requiring that the diagram of a poset P is planar is a stronger restriction than requiring that the cover graph of P is planar. Recall that in the *diagram*: elements are drawn as distinct points in the plane, and each cover relation $a \leqslant b$ in P is represented by a curve from a to b going upwards. Accordingly among posets that have planar cover graphs, some but not all also have planar order diagrams. Among the class of posets with planar diagrams, Joret, Micek and Wiechert [6] showed that $\dim(P) \leqslant 192h + 96$ when P has height h.

The remainder of this paper is organized as follows. In the next section, we prove three reductions to simpler problems, and we give essential background material. The proof of Theorem 1 is given in the following two sections, and we close with brief comments on challenging open problems that remain.

2. Preliminary Reductions and Background Material

When P is a poset, we write $x \parallel_P y$ (also $x \parallel y$ in P) when x and y are incomparable. In general, we prefer the short form $x \parallel_P y$, except when subscripts or primes are involved. A similar remark applies to the relations $<, >, \leqslant, \geqslant$. Later in the proof, we will discuss a poset P and define linear orders T and S on subsets of the ground set of P. In that discussion, we will write $u <_T v$ or $u <_S v$, as appropriate.

We list below some elementary, and well known, properties of dimension.

(i) Dimension is monotonic, i.e., if Q is a subposet of P, then $\dim(Q) \leqslant \dim(P)$.
(ii) The dual of a poset P is the poset P' on the same ground set of P with $x < y$ in P' if and only $x >_P y$. Then $\dim(P) = \dim(P')$.

For the balance of this preliminary section, we fix a poset P. We let $\mathrm{Min}(P)$ and $\mathrm{Max}(P)$ denote, respectively, the set of minimal elements and the set of maximal elemets of P. Also, we let $\mathrm{Inc}(P)$ denote the set of all ordered pairs (x, y) with $x \parallel_P y$. We will assume $\mathrm{Inc}(P) \neq \emptyset$; otherwise P is a chain and $\dim(P) = 1$. When $(x, y) \in \mathrm{Inc}(P)$ and L is a linear extension of P, we say that L *reverses* (x, y) when $x > y$ in L. A set $I \subseteq \mathrm{Inc}(P)$ is *reversible* if there is a linear extension L of P which reverses every pair in I. Vacuously, the empty set is reversible. We then define $\dim(I)$ as the least $d \geqslant 1$ such that I can be covered by d reversible sets. It is easily seen that $\dim(P)$ is equal to $\dim(\mathrm{Inc}(P))$.

Given sets $A, B \subseteq P$, we let $\mathrm{Inc}(A, B)$ be the set of pairs $(a, b) \in \mathrm{Inc}(P)$ with $a \in A$ and $b \in B$. We use the abbreviation $\dim(A, B)$ for $\dim(\mathrm{Inc}(A, B))$. Again, $\dim(A, B) = 1$ when $\mathrm{Inc}(A, B) = \emptyset$. Typically, we will have $A \subseteq \mathrm{Min}(P)$ and $B \subseteq \mathrm{Max}(P)$.

A sequence $((x_1, y_1), \ldots, (x_k, y_k))$ of pairs from $\mathrm{Inc}(P)$ with $k \geqslant 2$ is an *alternating cycle of size* k if $x_i \leqslant_P y_{i+1}$ for all $i \in \{1, \ldots, k\}$, cyclically (so $x_k \leqslant_P y_1$ is required). Observe that if $((x_1, y_1), \ldots, (x_k, y_k))$ is an alternating cycle in P, then any subset $I \subseteq \mathrm{Inc}(P)$ containing all the pairs on this cycle is not reversible; otherwise we would have $y_i <_L x_i \leqslant_L y_{i+1}$ for each $i \in \{1, \ldots, k\}$ cyclically, which cannot hold.

A sequence $((x_1, y_1), \ldots, (x_k, y_k))$ of pairs from $\mathrm{Inc}(P)$ is a *strict* alternating cycle if for each $i, j \in \{1, \ldots, k\}$, we have $x_i \leqslant_P y_j$ if and only if $j = i + 1$ (cyclically). Note that in this case, $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_k\}$ are k-element antichains. Note that in alternating cycles, we allow that $x_i = y_{i+1}$ for some or even all values of i.

When a set S is not reversible and contains an alternating cycle, then an alternating cycle of mininum size in S is easily seen to be a strict alternating cycle. The converse is also true, a detail originally observed by Trotter and Moore [14]: A subset $I \subseteq \mathrm{Inc}(P)$ is reversible if and only if I contains no strict alternating cycle.

When $x <_P y$, a sequence $W = (u_0, u_1, \ldots, u_t)$ is called a *witnessing path* (*from* x *to* y) when $u_0 = x$, $u_t = y$ and u_i is covered by u_{i+1} in P for each $i \in \{0, 1, \ldots, t-1\}$.

The following elementary lemma allows us to concentrate our attention on incomparable pairs from $\mathrm{Inc}(\mathrm{Min}(P), \mathrm{Max}(P))$. See for instance [5, Observation 3] for a proof.

Lemma 2 (Reduction to min-max)**.** *For every poset P, there is a poset Q containing P as an induced subposet such that*

(i) *The height of P is the same as the height of Q;*
(ii) *The cover graph of Q is obtained from the cover graph of P by adding some degree 1 vertices; and*

$$\dim(P) \leqslant \dim(\mathrm{Min}(Q), \mathrm{Max}(Q)).$$

2.1. **Constrained Subsets and Weak-Coloring Numbers.** Let P be a poset. We say that a non-empty subset $I \subseteq \mathrm{Inc}(\mathrm{Min}(P), \mathrm{Max}(P))$ is *singly constrained* in P when there is an element $x_0 \in P$ such that $x_0 <_P b$ for every $(a,b) \in I$. To identify the element x_0, we will also say I is singly constrained *by* x_0.

The following lemma was used first in [11] for posets with planar cover graphs and in a more complex form in [5]. The underlying principle is the concept of *unfolding*, which is an analogue of breadth first search for posets.

Lemma 3 (Reduction to singly constrained)**.** *For every poset P, there exists a poset Q such that*

(i) *The height of Q is at most the height of P.*
(ii) *The cover graph of Q is a minor of the cover graph of P.*
(iii) *There is a minimal element x_0 in Q such that $x_0 \leqslant q$ for every $q \in \operatorname{Max}(Q)$, and*

$$\dim(\operatorname{Min}(P), \operatorname{Max}(P)) \leqslant 2 \dim(\operatorname{Min}(Q), \operatorname{Max}(Q)).$$

In particular, the set $I = \operatorname{Inc}(\operatorname{Min}(Q), \operatorname{Max}(Q))$ in the lemma is singly constrained by x_0. We point out that the lemma produces an element $x_0 \in \operatorname{Min}(Q)$, but later in this paper, we will be discussing sets $I \subseteq \operatorname{Inc}(\operatorname{Min}(Q), \operatorname{Max}(Q))$ such that I is singly constrained by an element x_0 which is not a minimal element in Q.

We say that a non-empty subset I of $\operatorname{Inc}(\operatorname{Min}(P), \operatorname{Max}(P))$ is *doubly constrained* in P when there is a pair (x_0, y_0) such that

(i) $x_0 <_P y_0$,
(ii) $x_0 <_P b$ for every $(a,b) \in I$, and
(iii) $a <_P y_0$ for every $(a,b) \in I$.

As before, we will also say that I is doubly constrained *by* (x_0, y_0).

We would very much like to reduce to the case where we are bounding $\dim(I)$ when $I \subseteq \operatorname{Inc}(\operatorname{Min}(P), \operatorname{Max}(P))$ is doubly constrained. Unfortunately, Lemma 3 will not be of assistance. Instead, we will use a different reduction, one that will cost us an $\mathcal{O}(h^3)$-factor in the final bound.

The *length* of a path in a graph is the number of its edges. For two vertices u and v in a graph G, an *u–v path* is a path in G with ends in u and v. Let G be a graph and let σ be an ordering of the vertices of G. For $r \in \{0, 1, 2, \ldots\} \cup \{\infty\}$ and two vertices u and v of G, we say that u is *weakly r-reachable* from v in σ, if there exists an u–v path of length at most r such that for every vertex w on the path, $u \leqslant_\sigma w$. The set of vertices that are weakly r-reachable from a vertex v in σ is denoted by $\operatorname{WReach}_r[G, \sigma, v]$. The *weak r-coloring number* $\operatorname{wcol}_r(G)$ of G is defined as

$$\operatorname{wcol}_r(G) := \min_{\sigma} \max_{v \in V(G)} \left| \operatorname{WReach}_r[G, \sigma, v] \right|.$$

where σ ranges over the set of all vertex orderings of G. We call $\operatorname{wcol}_r(G)$ the *r-th weak coloring number* of G.

Weak coloring numbers were originally introduced by Kierstead and Yang [9] as a generalization of the degeneracy of a graph (also known as the coloring number). Since then, they have been applied in several novel situations (see Zhu [18] and Van den Heuvel et al. [16], for examples). We also have good bounds on weak coloring numbers. For planar graphs, van den Heuvel et al. [15] have shown that the r-th weak coloring number is at most $\binom{r+2}{2} \cdot (2r+1) = \mathcal{O}(r^3)$. See also a recent paper [2] with a lower bound in $\Omega(r^2 \log r)$.

Here is a lemma on weak coloring numbers from [3] that will play an important role in the reduction to the doubly constrained case.

Lemma 4. *Let P be a height h poset, let G be the cover graph of P, and let $c := \mathrm{wcol}_{4h-4}(G)$. Then there is an element $z_0 \in P$ such that the set $J = \{(a,b) \in I : a <_P z_0\}$ satisfies*

$$\dim(J) \geqslant \frac{\dim(I)}{c} - 2.$$

We then have the following immediate corollary.

Corollary 5. *Let P be a poset with a planar cover graph, and let x_0 be an element of P such that $x_0 < b$ in P for every $b \in \mathrm{Max}(P)$. Let I be a non-empty subset of $\mathrm{Inc}(\mathrm{Min}(P), \mathrm{Max}(P))$. Then there is a set $J \subseteq I$ such that J is doubly constrained in P and*

$$\dim(I) = \mathcal{O}(h^3) \cdot \dim(J).$$

Proof. Let G be the cover graph of P. Apply Lemma 4 with $c = \mathrm{wcol}_{4h-4}(G) = \mathcal{O}(h^3)$ to obtain the element z_0 and the set $J \subseteq I$. Let y_0 be any maximal element with $z_0 \leqslant y_0$ in P. Since $y_0 \in \mathrm{Max}(P)$ we have $x_0 < y_0$ in P. Evidently J is doubly constrained by the pair (x_0, y_0). The inequality from Lemma 4 becomes $\dim(I) \leqslant c \cdot (2 + \dim(J))$, and with this observation, the proof of the corollary is complete. □

2.2. **A Reduction to Doubly Exposed Posets.** Let P be a poset. We will say that a non-empty set $I \subseteq \mathrm{Inc}(\mathrm{Min}(P), \mathrm{Max}(P))$ is *doubly exposed* in P if the following conditions are met:

(i) I is doubly constrained by (x_0, y_0).
(ii) The cover graph G of P is planar, and there is a plane drawing of G with x_0 and y_0 on the same face.

Note that in the preceding definition, we could just as well have required that x_0 and y_0 be on the exterior face. The form of the definition allows us to determine that a set I is doubly exposed as evidenced by a plane drawing with x_0 and y_0 on the same face. If desired, we can then redraw the cover graph, without edge crossings, so that x_0 and y_0 are on the exterior face.

Our next goal is to prove a reduction to the doubly exposed case. The argument requires a technical detail regarding paths. When R is a tree in G and $u, v \in R$, we denote by

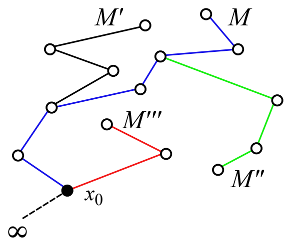

FIGURE 2. The black path M' leaves the blue path M from the left side while the green path M'' and the red path M''' leave M from the right side.

uRv the unique path in R from u to v. This notation is particularly convenient for discussing concatenation of paths, and it will be used extensively later in the paper.

Let P be a poset with a planar cover graph, and suppose that I is singly constrained by x_0. Then consider a plane drawing of the cover graph G of P with x_0 on the exterior face. Add to the drawing an extra edge linking x_0 from an "imaginary point" located in the outer face. Let M be a non-trivial path in G starting from x_0, and let v be the other endpoint of M. Now let M' be another path in G also starting from x_0, sharing some initial segment with M, say the portion from x_0Mu with $u \neq v$. Note that u could coincide with x_0. Suppose further that the portion of M' after u is non-empty. Since x_0 is on the exterior face in the drawing of G (and since we added the imaginary line), there is a natural notion of "sides", and we can say with precision that either M' *leaves* M from the *left side*, or M' leaves M from the *right side*. Note however that sides are not well defined when u is the last point of M. We illustrate these concepts in Figure 2.

With this technical detail in hand, we are ready for the reduction to the doubly exposed case.

Lemma 6. *Let P be a height h poset with a planar cover graph. If $I \subseteq \mathrm{Inc}(\mathrm{Min}(P), \mathrm{Max}(P))$ is doubly constrained in P, then there is a poset Q and a set $J \subseteq \mathrm{Inc}(\mathrm{Min}(Q), \mathrm{Max}(Q))$ such that*

(i) *the height of Q is at most h;*
(ii) *the cover graph of Q is a subgraph of the cover graph of P;*
(iii) *J is doubly exposed in Q; and*

$$\dim(I) \leqslant 2(h-1)\dim(J).$$

Proof. Let $I \subseteq \mathrm{Inc}(\mathrm{Min}(P), \mathrm{Max}(P))$ be a non-empty doubly constrained set by (x_0, z_0), and let $\mathbb{D}$ be a plane drawing of the cover graph G of P with x_0 on the exterior face.

We fix a chain from x_0 to z_0 and refer to this chain as the *spine*. Label the points on the spine as $\{u_0, u_1, \ldots, u_t\}$ with $x_0 = u_0$, $z_0 = u_t$ and u_i covered by u_{i+1} in P for each $i \in \{1, \ldots, t-1\}$. Note that $t \leqslant h-1$.

Let

$$A = \{a \in P \mid \text{there exists } b \text{ such that } (a, b) \in I\},$$
$$B = \{b \in P \mid \text{there exists } a \text{ such that } (a, b) \in I\}.$$

In particular, we have $I \subseteq A \times B$, so $\dim(I) \leqslant \dim(A, B)$.

For each $b \in B$, let $\tau(b)$ be the largest integer i so that $u_i <_P b$. Note that $0 \leqslant \tau(b) \leqslant t-1$. Let $W(b)$ be a witnessing path from x_0 to b such that $W(b)$ shares the initial segment $(u_0, u_1, \ldots, u_{\tau(b)})$ with the spine.

We partition B into B_{left} and B_{right} in such a way that b is assigned to the set B_{left} if $W(b)$ leaves the spine from the left side. Dually, we assign b to B_{right} if $W(b)$ leaves the spine from the right side.

For each $a \in A$, let $\tau(a)$ be the least integer i so that $a <_P u_i$. Now we have $2 \leqslant \tau(a) \leqslant t$. We partition the set A into $A_2 \cup A_3 \cup \cdots \cup A_t$ by assigning a to A_i when $\tau(a) = i$. Clearly,

$$\dim(I) \leqslant \dim(A, B) \leqslant \sum_{s \in \{2,\ldots,t\}} \sum_{\text{dir} \in \{\text{left},\text{right}\}} \dim(A_s, B_{\text{dir}}).$$

It follows that there is some $s \in \{2, \ldots, t\}$ and $\text{dir} \in \{\text{left}, \text{right}\}$ so that

$$\dim(A_s, B_{\text{dir}}) \geqslant \frac{\dim(I)}{2(h-1)}.$$

We assume that dir = right. From the details of the argument, it will be clear that the proof is symmetric in the other case.

We say that an edge $e = u_i v$ in the cover graph of P is *bad* if $0 \leqslant i < s$, v is not on the spine, and the path $\{u_0, u_1, \ldots, u_i, v\}$ leaves the spine from the left side. We then define a poset Q having the same ground set as P with $x \leqslant y$ in Q if and only if there is a witnessing path in P from x to y avoiding bad edges.

We claim that for every $a \in A_s$ and every $b \in B_{\text{right}}$, we have $a \leqslant b$ in Q if and only if $a \leqslant b$ in P. The forward implication is obvious. To see the backward one, let $a \in A_s$ and $b \in B_{\text{right}}$ with $a <_P b$. Then let W be a witnessing path from a to b in P. This path cannot use a bad edge as this would make $a < u_i$ in P for some $i \in \{1, \ldots, s-1\}$ contradicting $a \in A_s$. Therefore, the claim holds and also $\dim(A_s, B_{\text{right}})$ in Q is the same as $\dim(A_s, B_{\text{right}})$ in P.

Note that the diagram and the cover graph of Q are obtained simply by removing the bad edges from the diagram and cover graph, respectively, of P. It follows that the cover graph of Q is planar. Furthermore, x_0 and u_s are on the same face, and the set $\text{Inc}(A_s, B_{\text{right}})$ is doubly exposed by the pair (x_0, u_s). With this observation, the proof of the lemma is complete. □

Summarizing, we can combine Lemma 3, Corollary 5, and Lemma 6 to obtain:

Corollary 7. *Let P be a height h poset with a planar cover graph. Then there is a poset Q such that*

(i) *Q has height at most h;*
(ii) *Q has a planar cover graph;*
(iii) *There is a set $I \subseteq \text{Inc}(\text{Min}(Q), \text{Max}(Q))$ such that I is doubly exposed in Q and*

$$\dim(P) = \mathcal{O}(h^4) \cdot \dim(I).$$

We are now ready to begin the proof of our main theorem.

3. Large Standard Examples in Doubly Exposed Posets

We pause here to make the following important comment: The concept of height plays *no* role in the arguments given in this section.

Throughout this section, P will denote a poset with a planar cover graph. Also, I will denote a subset of $\text{Inc}(\text{Min}(P), \text{Max}(P))$ which is doubly exposed by (x_0, y_0). Let

$$A_I = \{a \in P \mid \text{there exists } b \text{ such that } (a, b) \in I\},$$
$$B_I = \{b \in P \mid \text{there exists } a \text{ such that } (a, b) \in I\}.$$

In particular, we have $I \subseteq A_I \times B_I$.

We will then fix a plane drawing $\mathbb{D}$ of G, the cover graph of P, with x_0 and y_0 on the exterior face. Next, we discuss a subgraph T of G associated with x_0 and the elements of B_I. Subsequently, this discussion will be repeated for y_0 and the elements of A_I.

It is easy to see that there is a subgraph T of G satisfying the following properties.

(i) The vertices and edges of T form a tree containing x_0 and all elements of B_I.
(ii) The leaves of T are the elements of B_I.
(iii) We consider x_0 as the *root* of T, and for each $b \in B_I$, we let x_0Tb denote the unique path in T from x_0 to b. We require that x_0Tb be a witnessing path from x_0 to b.

We choose and fix a tree T satisfying these properties. In the remainder of the discussion, we will refer to T as the *blue tree*. The vertices and edges of T are called blue vertices and blue edges respectively. The fact that x_0 is on the exterior face implies that T determines a clockwise linear order $<_T$ on the elements of B_I. We illustrate the notion of a blue tree in Figure 3 where we take $B_I = \{1, \ldots, 15\}$. The leaves have been labeled so that the clockwise order agrees with the natural order as integers. Note that in general, there are many elements of T that do not belong to $\{x_0\} \cup B_I$. Also, there are many elements of P that do not belong to T.

In an entirely analogous manner, we determine a red tree S with y_0 as its root and the elements of A_I as its leaves. For each $a \in A_I$, we let aSy_0 denote the unique path in S

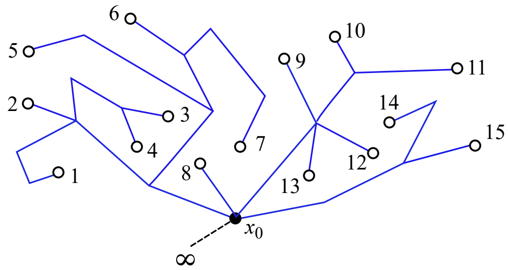

FIGURE 3. The leaves of T are $\{1, \ldots, 15\}$, the elements of B_I. They are ordered clockwise by $<_T$. Recall that we are working in a planar cover graph setup (and not necessarily planar diagram) so the poset relation does not have to go vertically upwards in the plane.

from a to y_0, and we require that aSy_0 be a witnessing path. Once the red tree S has been chosen, we have a clockwise order $<_S$ on the elements of S.

When $\mathcal{C}$ is a simple closed curve in the plane, it splits the points of the plane not on $\mathcal{C}$ into those that are in the interior of the region bounded by $\mathcal{C}$ and those in the exterior of this region. In the discussion to follow, we will abuse terminology slightly and say that a point not on $\mathcal{C}$ is either in the interior of $\mathcal{C}$ or it is in the exterior of $\mathcal{C}$, dropping the reference to the region bounded by $\mathcal{C}$.

We find it convenient to assume that in the plane drawing $\mathbb{D}$ of G, the vertices x_0 and y_0 are the lowest, respectively the highest, elements of P in the plane. The entire diagram will be enclosed in a simple closed curve $\mathcal{C}$ which intersects $\mathbb{D}$ only at x_0 and y_0. Note that x_0 and y_0 are on $\mathcal{C}$. All other vertices and edges of the cover graph G are in the interior of $\mathcal{C}$. If we start at x_0 and traverse the boundary of $\mathcal{C}$ in a clockwise direction, we refer to the portion of $\mathcal{C}$ between x_0 and y_0, as the *left side* of $\mathcal{C}$. Continuing on from y_0 to x_0, we are then on the *right side* of $\mathcal{C}$. If N is any path in G from x_0 to y_0, then N splits the region bounded by $\mathcal{C}$ into two subregions, called naturally, the *left half* and the *right half.*

With the set I fixed, with each pair $(a, b) \in A_I \times B_I$ such that $a <_P b$, we will associate a *separating path* $N = N(a, b)$ from x_0 to y_0 defined as follows: (1) $u = u(a, b)$ is the least element in P that is on the blue path x_0Tb and satisfies $a <_P u$; (2) $v = v(a, b)$ is the largest element of P that is on the red path aSy_0 and satisfies $u \leqslant_P v$; (3) $N = x_0TuWvSy_0$, where W is an arbitrary witnessing path from v to u. Strictly speaking, the path $N(a, b)$ depends on a and b as well as the choice for W. However, that detail can be safely ignored, as none of the results to follow depend on which choice is made for W.

The path x_0Tu will be called the *blue* part of N; the path uWv will be called the *black* part of N; and the path vSy_0 will be called the *red* part of N. We note that the red and black parts share a point, as do the blue and black parts. The black part may consist of a single point, but the red part and the blue part are always non-trivial. We also note that a point on the red part of N may be an element of the blue tree. Analogous comments hold for the other parts of N. In general, the vertices a and b do not have to

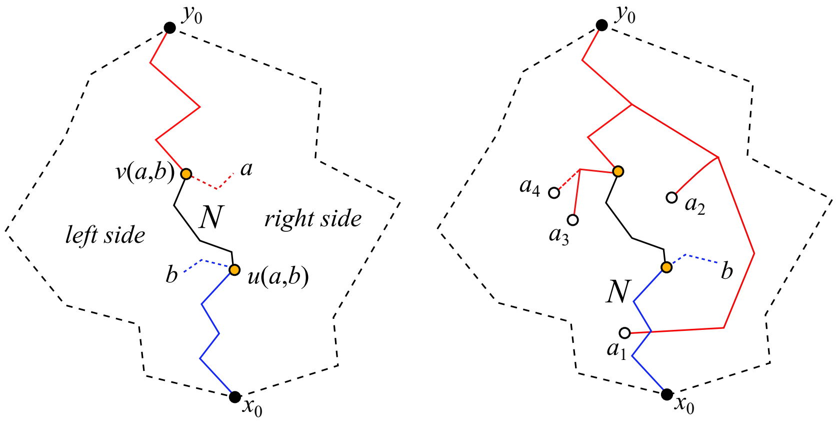

FIGURE 4. In the figure on the left, we illustrate a separating path $N = N(a, b)$, with its blue, red, and black parts colored appropriately. In the figure on the right, we show a separating path $N = N(a_4, b)$ and four points in S with $a_1 <_S a_2 <_S< a_3 <_S a_4$. Since a_1 and a_3 are left of N, Proposition 8 implies that each of the paths y_0Sa_1 and y_0Sa_3 contains a point from the union of the black part of N and the blue part of N.

belong to the path N. However, if z is on the path N, and z is either on the red part or the black part, then $a \leqslant_P z$. Symmetrically, if z is on the blue part or the black part, then $z \leqslant_P b$.

When $N = N(a, b)$ is a separating path, we consider the two halves of the region $\mathcal{C}$ determined by N. When u is a point of P, and u is not on N, then we will simply say u *is left of* N when u is in the left half of $\mathcal{F}$; analogously, we will say that u *is right of* $N(a, b)$ when u is in the right half of $\mathcal{C}$. Our convention regarding the labeling of the two halves is illustrated in first figure in 4.

The following elementary proposition has four symmetric statements: two for the tree S and two for the tree T.

Proposition 8. *Let $N = N(a, b)$ be a separating path. If $a' \in A_I$, $a' <_S a$ and a' is left of N, then y_0Sa' contains a point of N from the union of the blue part and the black part of N.*

Proof. If $v(a, b) \in y_0Sa'$, then the proposition holds, since $v(a, b)$ belongs to the black part of N. So we may assume $v(a, b) \notin y_0Sz'$. Let v' be the least point of P common to y_0Sa and y_0Sa'. Then $v(a, b) <_P v'$. Let w be the first vertex on y_0Sa' after v'. Since $a' <_S a$, we know w is right of N. Since a' is left of N, the path wSa' must intersect N. Since S is a tree, any point common to N and wSa' belongs to the union of the blue and black parts of N. □

The next proposition is actually an immediate corollary of Proposition 8. It is stated for emphasis. Note that that there is dual form.

Proposition 9. *Let $N = N(a,b)$ be a separating path . If $a' \in A_I$ and $a' \parallel_P b$, then a' is right of N if and only if $a' <_S a$. Also, if $b' \in B_I$ and $b' \parallel_P a$, then b' is left of N if and only if $b' <_T b$.*

Proposition 10. *Let $N = N(a,b)$ be a separating path. If $w <_P z$, w is on one side of N and z is on the other, then either $w <_P b$ or $a <_P z$.*

Proof. Let W be a witnessing path from w to z. Then W and N must intersect. Let q be a common point. If q is on the blue part of N, then $w <_P b$. If q is on the red part of N, then $a <_P z$. If q is on the black part of N, then both $w <_P b$ and $a <_P z$ hold. □

Let b and b' be distinct elements of B_I. We say that b is *enclosed by* b' if there is a cycle $\mathcal{D}$ in G such that (1) all points of $\mathcal{D}$ belong to $D_P[b']$; and (2) b is in the interior of $\mathcal{D}$. Dually, if a and a' are distinct points of A_I, we say that a is *enclosed by* a' if there is a cycle $\mathcal{D}$ in G such that (1) all points of $\mathcal{D}$ belong to $U_P[a']$; and (2) a is in the interior of $\mathcal{D}$.

Proposition 11. *Let $k \geqslant 2$ and let $((a_1,b_1),\dots,(a_k,b_k))$ be a strict alternating cycle of pairs from I. If i,j are distinct integers from $[k]$, then b_i is not enclosed by b_j, and a_i is not enclosed by a_j.*

Proof. We prove that if i,j are distinct integers from $[k]$, then b_i is not enclosed by b_j. The proof of the second assertion is symmetric.

Let $\mathcal{D}$ be a cycle in G evidencing that b_i is enclosed in b_j. Since y_0 is on the exterior face, y_0 is not in the interior of $\mathcal{D}$. Note that $a_{i-1} <_P b_i$, and $a_{i-1} \parallel_P b_j$. If a_{i-1} is in the interior of $\mathcal{D}$, then a witnessing path $W = W(a_{i-1}, y_0)$ contains a point w from $\mathcal{D}$. This implies $a_{i-1} <_P w <_P b_j$. In turn, this implies $a_{i-1} <_P b_j$, which is false. The contradiction shows that a_{i-1} is not in the interior of $\mathcal{D}$.

Now consider a witnessing path $W' = W'(a_{i-1}, b_i)$. Since b_i is in the interior of $\mathcal{D}$, W' contains a point w' from $\mathcal{D}$. This implies $w' \leqslant_P b_j$. In turn, we have $a_{i-1} \leqslant_P w' \leqslant_P b_j$. Again, this implies $a_{i-1} <_P b_j$. The contradiction completes the proof. □

Proposition 12. *If $((a_1,b_1),(a_2,b_2))$ is an alternating cycle of pairs from I, then $a_1 <_S a_2$ if and only if $b_1 <_T b_2$.*

Proof. We assume that $a_1 <_S a_2$ and $b_2 <_T b_1$ and show that this leads to a contradiction. Let $N = N(a_1, b_2)$ be a separating path. Since we are assuming $b_2 <_T b_1$ and $a_1 \parallel_P b_1$, it follows by Proposition 9 that b_1 is right of N. Since $a_2 >_S a_1$ and $a_2 \parallel_P b_2$, again by Proposition 9, we know a_2 is left of N. Applying Proposition 10 for $w = a_2$ and $z = b_1$ we conclude that either $a_1 <_P b_1$ or $a_2 <_P b_2$, but both of these statements are false. The contradiction completes the proof. □

We illustrate the implications of Proposition 12 on the left side of Figure 5.

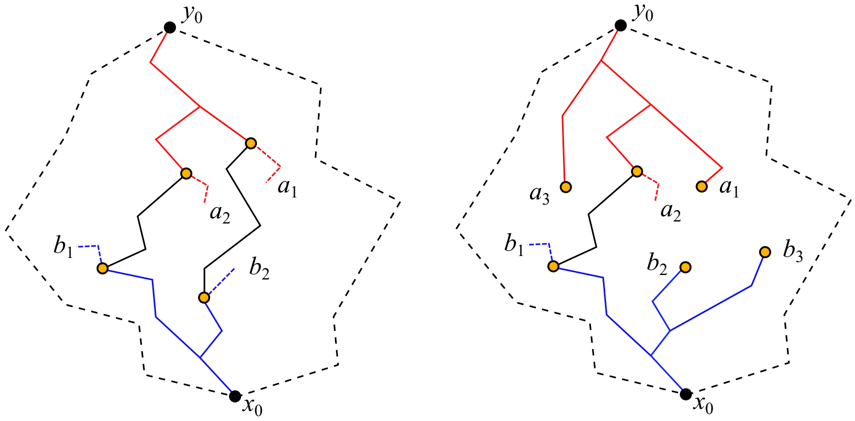

FIGURE 5. On the left side, we show the typical appearance of an alternating cycle of size 2. On the right side, we show the first case in the proof of Proposition 13.

For a non-empty subset $I \subseteq \text{Inc}(\text{Min}(P), \text{Max}(P))$, we define an auxiliary digraph H_I whose vertex set is $\text{Inc}(A_I, B_I)$. In H_I, we have a directed edge from (a, b) to (a', b') when these two pairs form an alternating cycle and $a <_S a'$ (therefore, $b <_T b'$ by Proposition 12). The next proposition implies a notion of transitivity for directed paths in H_I, and this concept will prove to be fundamentally important.

Proposition 13. *Let $n \geqslant 3$ and let $((a_1, b_1), \ldots, (a_n, b_n))$ be a directed path in H_I. Then $((a_i, b_i), (a_j, b_j))$ is an edge in H_I for all i, j with $1 \leqslant i < j \leqslant n$. In particular, these pairs form a copy of the standard example S_n.*

Proof. Using induction, it is clear that the lemma holds in general if it holds when $n = 3$. Since $a_1 <_S a_3$ (and $b_1 <_T b_3$), it suffices to show that $a_1 <_P b_3$ and $a_3 <_P b_1$.

We suppose first that $a_1 \parallel_P b_3$ and argue to a contradiction. A symmetric argument shows that the assumption that $a_3 \parallel_P b_3$ leads to a contradiction.

Let $N = N((a_2, b_3)$ be a separating path. Since $b_2 \parallel_P a_2$ and $b_2 <_T b_3$, Proposition 9 implies that b_2 is left of N. Since $a_1 <_S a_2$, and $a_2 \parallel_P b_3$, Proposition 9 also implies a_1 is right of N. Since $a_1 <_P b_2$, Proposition 10 implies that either $a_2 < b_2$ or $a_1 <_P b_3$. Since the first option is false, and the second is assumed to be false, we have reached a contradiction. ☐

The argument for the preceding proposition is illustrated on the right side of Figure 5.

For a non-empty subset $I \subseteq \text{Inc}(\text{Min}(P), \text{Max}(P))$, we define $\rho(I)$ to be the maximum size (number of vertices) of a directed path in H_I. The proof of the following lemma is (essentially) the same as the argument given for Lemma 5.9 in [11], although we are working here in a more general setting.

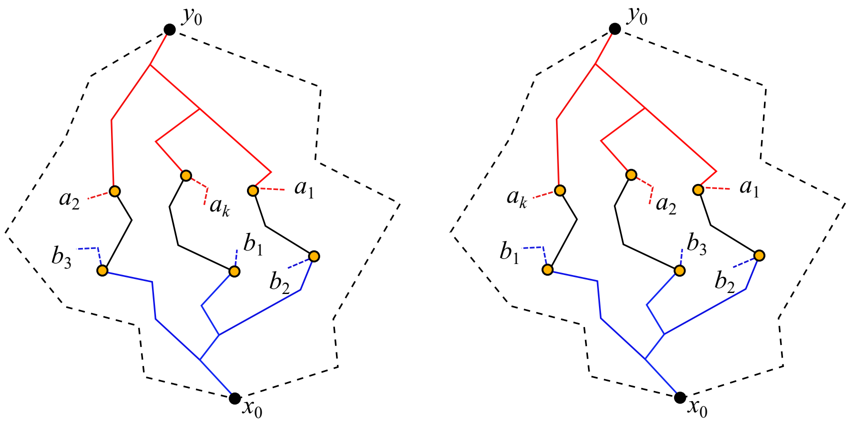

FIGURE 6. Two cases of the $<_S$-ordering of a_1, a_2, a_k.

Lemma 14. *Let P be a poset with a planar cover graph, and let $I \subseteq \operatorname{Inc}(\operatorname{Min}(P), \operatorname{Max}(P))$ be doubly exposed. Then*

$$\dim(I) \leqslant \rho(I)^2.$$

In particular, if k is the largest size of a standard example in P, then $\dim(I) \leqslant k^2$.

Proof. We show $\dim(I) \leqslant \rho(I)^2$ by exhibiting a partition of I into $\rho(I)^2$ reversible sets. These sets will have the form $I(m, n)$ where $1 \leqslant m, n \leqslant \rho(I)$. A pair $(a, b) \in I$ belongs to $I(m, n)$ if

(i) the longest directed path in H_I starting from (a, b) has size m, and
(ii) the longest directed path in H_I ending at (a, b) has size n.

To complete the proof, it suffices to show that each $I(m, n)$ is reversible. We argue by contradiction.

Suppose that for some pair (m, n), the set $I(m, n)$ is not reversible. Therefore there is a strict alternating cycle $((a_1, b_1), \ldots, (a_k, b_k))$ of size $k \geqslant 2$ with all pairs from $I(m, n)$. Without loss of generality, $a_1 <_S a_i$ for each $i \in \{2, \ldots, k\}$.

If $k = 2$, then there is a directed edge from (a_1, b_1) to (a_2, b_2) in H_I. It follows that any directed path in H_I starting at (a_2, b_2) can be extended by prepending (a_1, b_1). Thus (a_1, b_1), (a_2, b_2) cannot both belong to $I(m, n)$. We conclude that $k \geqslant 3$.

The balance of the proof divides into two cases. In view of our assumptions regarding the labeling of the pairs in the alternating cycle, exactly one of the following two cases is applicable (see Figure 6):

$$a_1 <_S a_k <_S a_2 \qquad \text{or} \qquad a_1 <_S a_2 <_S a_k.$$

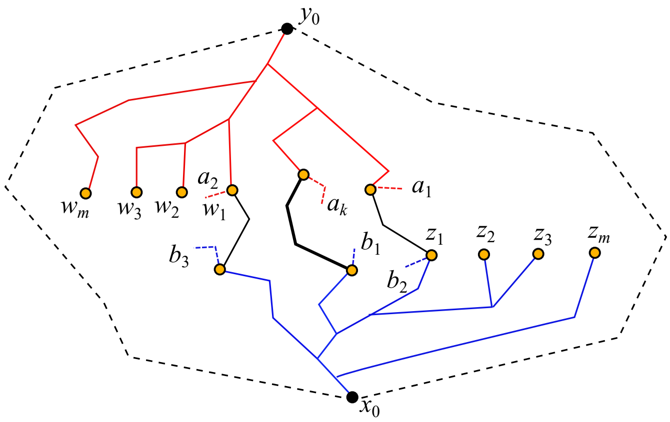

FIGURE 7. The argument shows that a_2 is left of N and z_2 is right of N. Therefore, a witnessing path for $a_2 <_P z_2$ has to cross N, and this forces $a_k <_P z_2$.

In the first case, we will show that there is a directed path in H_I of size $m+1$ starting at (a_1, b_1). In the second case, we will show that there is a directed path in H_I of size $n+1$ ending at (a_2, b_2). Both implications are contradictions. We will give details of the proof for the first case. It will be clear that the argument for the second case is symmetric.

Therefore we assume $a_1 <_S a_k <_S a_2$. Since the pairs $(a_1, b_1), (a_k, b_2) \in \text{Inc}(A_I, B_I)$ form an alternating cycle of size 2 and $a_1 <_S a_k$, we have an edge in H_I from (a_1, b_1) to (a_k, b_2). Since (a_1, b_1) is the first vertex on this edge, we know $m \geqslant 2$. By Proposition 12, we have $b_1 <_T b_2$. Similarly, there is a directed edge in H_I from (a_k, b_3) to (a_2, b_1), and $b_3 <_T b_1$. Therefore,

$$b_3 <_T b_1 <_T b_2.$$

Fix a directed path $((w_1, z_1), (w_2, z_2), \ldots, (w_m, z_m))$ in H_I with $(w_1, z_1) = (a_2, b_2)$. (Recall that $m \geqslant 2$.) Now consider the sequence

$$((a_1, b_1), (a_k, b_2), (w_2, z_2), \ldots, (w_m, z_m)).$$

We claim that this sequence is a directed path in H_I. Since it has size $m+1$ and it starts at (a_1, b_1), this will be a contradiction.

We have already noted that $((a_1, b_1), (a_k, b_2))$ is an edge in H_I and since all $((w_i, z_i), (w_{i+1}, z_{i+1}))$ are edges in H_I as well, it remains only to show that there is an edge from (a_k, b_2) to (w_2, z_2) in H_I. Note that $a_k <_S a_2 = w_1 <_S w_2$, and $w_2 <_P z_1 = b_2$. Therefore, we only need to show that $a_k <_P z_2$. We assume that $a_k \parallel_P z_2$ and show that this leads to a contradiction. Let $N = N(a_k, b_1)$ be a separating path (see Figure 7). Since $a_2 \parallel_P b_1$, we know from Proposition 9 that a_2 is left of N. Note also that $b_1 <_T b_2 = z_1 <_T z_2$. With our assumption that $a_k \parallel_P z_2$, we know from Proposition 9 that z_2 is right of N. Since $a_2 = w_1 <_P z_2$, it follows from Proposition 10 that either $a_2 <_P b_1$ or $a_k <_P z_2$. The first option is false, and the second is assumed false. Again, we have reached a contradiction. □

When I is doubly exposed, we now have $\dim(I)$ bounded in terms of $\rho(I)$, *independent* of the height h of P. Now we turn our attention to bounding $\rho(I)$ in terms of h.

4. Restrictions Resulting from Bounded Height

This section is devoted to proving the following lemma.

Lemma 15. *Let P be a height h poset with a planar cover graph. Let $I \subseteq \mathrm{Inc}(\mathrm{Min}(P), \mathrm{Max}(P))$ be doubly exposed in P. Then*

$$\rho(I) \leqslant 34h + 11.$$

Once this lemma has been proven, the proof of our main theorem will be complete. To see this, recall that using Corollary 7, we paid a price of $\mathcal{O}(h^4)$ to reduce to the case where we need to bound $\dim(I)$ for I doubly exposed in P. Lemma 14 asserts that $\dim(I) \leqslant \rho(I)^2$. Combining this with Lemma 15, we obtain the bound $\mathcal{O}(h^6)$.

Our final bound on $\rho(I)$ will emerge from a series of preliminary results. In their presentation, we aim for reasonable multiplicative constants and consciously tolerate less than optimal additive constants.

As we did in the last section, we will present a series of small propositions all working within the following context. We fix an integer $h \geqslant 2$ and assume that we have a poset P whose height is at most h. We let G denote the cover graph of P. We assume that we have a set $I \subseteq \mathrm{Inc}(\mathrm{Min}(P), \mathrm{Max}(P))$ which is doubly exposed by (x_0, y_0), and we have a plane drawing of G with x_0 and y_0 on the exterior face. Finally, we have an integer $n \geqslant 2$ and a directed path $((a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n))$ in H_I.

As before, we choose a blue tree T and a red tree S. We then have linear orders $<_T$ and $<_S$ on B_I and A_I, respectively.

The next proposition is concerned with how a witnessing path W can intersect paths of the form aSy_0, where a is an element of the red tree S. There is an analogous version for paths of the form x_0Tb where b is a point in the blue tree T. When $i \in [n]$, we will say that a witnessing path W *cuts* a_i when W intersects the path a_iSy_0. Also, when s_i is a point common to W and a_iSy_0, we will say W *cuts* a_i *at* s_i. When $1 \leqslant i < j < k \leqslant n$, W cuts a_i at s_i, W cuts a_k at s_k, and $s = s_i = s_k$, then the fact that S is a tree implies that W cuts a_j at s. When $s_i \neq s_k$, the path W may cut each of a_iSy_0 and a_kSy_0 at several other points. Subsequently, there may be several regions in the plane formed by portions of a_iSa_k and W.

The following elementary proposition will play a key role in subsequent arguments. There are actually two versions, one for the red tree S and one for the blue tree T. The impact of the proposition is illustrated in Figure 8.

Proposition 16. *Let i and k be integers with $1 \leqslant i \leqslant k \leqslant n$. If W is a witnessing path, and W cuts a_i and a_k, then W cuts a_j for every j with $i < j < k$. Suppose further*

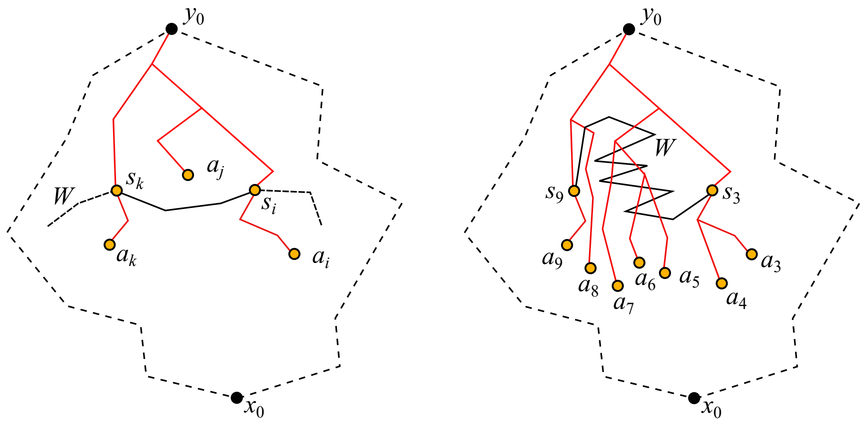

FIGURE 8. In the figure on the left, a_j is in the interior of a region whose boundary is the union of z_iWz_k and z_iSz_k. This configuration leads to the conclusion that either a_j is enclosed by a_i or a_j is enclosed by a_k. Either of these outcomes violates Proposition 11. In the figure on the right, we show a specific case with $i = 3$ and $k = 9$. Some care must be taken in choosing $s_4, \ldots, s_k$ to insure that the desired monotonic conditions are satisfied.

that W cuts a_i at s_i, W cuts a_k at s_k, and $s_i \leqslant_P s_k$ ($s_i \geqslant_P s_k$, respectively). Then for every j with $i < j < k$, there is an element s_j of P such that W cuts a_j at s_j, and $s_i \leqslant_P s_{i+1} \leqslant_P \cdots \leqslant_P s_k$ ($s_i \geqslant_P s_{i+1} \geqslant_P \cdots \geqslant_P s_k$, respectively). Also, if the paths s_iWs_k and s_iSs_k form a simple closed curve $\mathcal{R}$. Then a_j is in the exterior of $\mathcal{R}$ for every j with $i < j < k$.

Proof. If $s_i = s_k$, then s_i belongs to a_jSy_0 for every j with $i \leqslant j \leqslant k$. In this case, the final statement of the proposition holds vacuously. Now supose that $s_i \neq s_k$ and say $s_i < s_k$ in P. When $s_i > s_k$ in P the argument is symmetric.

It is easy to see that there are points s'_i and s'_k on W such that $s_i \leqslant_P s'_i <_P s'_k \leqslant_P s_k$; and the paths $s'_iWs'_k$ and $s'_iSs'_k$ form a simple closed curve $\mathcal{R}$ in the plane. First, let j be any integer with $i < j < k$. Since $a_i <_S a_j <_S< a_k$, either (1) a_j is in interior of $\mathcal{R}$; or (2) a_j is in the exterior of $\mathcal{R}$ and W cuts a_j at a point which belongs to $s'_iWs'_k$. Now assume that option (1) holds. Since $a_i \leqslant_P s_i \leqslant_P s'_i <_P s'_k$, all points on the boundary of $\mathcal{R}$ belong to $U_P[a_i]$. This implies that a_j is enclosed by a_i, which contradicts Proposition 11. We conclude that option (2) must hold. Note that this proof verifies the final statement of the proposition in the special case where $s_i = s'_i$ and $s'_k = s_k$.

The remaining part of the proof is a simple inductive argument. There is nothing to prove if $k = i + 1$. When $k = i + 2$, there is only one valid choice for j, i.e. $j = i + 1$, and the previous paragraphs proves the statement. Now suppose that $k > i + 2$. The path W can cut a_{i+1} in many different places, but we simply fix such an element s_{i+1} common to W and $a_{i+1}Sy_0$. With this choice, we have $s_i \leqslant_P s_{i+1} \leqslant_P s_k$. If $s_{i+1} = s_k$, we simply take $s_j = s_k$ for all j with $i + 1 < j < k$. If $s_{i+1} \neq s_k$, then $s_{i+1} <_P s_k$, and we apply the proposition to the witnessing path $s_{i+1}Ws_k$ which cuts a_{i+1} and a_k. □

For a non-empty subset $X \subseteq [n]$, we let $A(X) = \{a_i : i \in X\}$ and $B(X) = \{b_i : i \in X\}$. Note that the pairs in $\{(a_i, b_i) : i \in X\}$ determine a directed path of size $|X|$ in H_I. Let $\alpha, \beta \in [n] - X$ with $\alpha \neq \beta$, and let $N = N(a_\alpha, b_\beta)$ be a separating path. We will say that N *separates* $A(X)$ from $B(X)$ if all points of $A(X)$ are on one side of N and all points of $B(X)$ are on the other side.

We present the first of three key results bounding $\rho(I)$ in terms of the height of P.

Proposition 17. *Let X be a non-empty subset of $[n]$, let α ,β be two distinct integers in $[n] - X$, and let $N = N(a_\alpha, b_\beta)$. Suppose further that N separates $A(X)$ from $B(X)$.*

(i) *If the black portion of N is trivial, then $|X| = 1$.*
(ii) *If the black portion of N is non-trivial, then $|X| \leqslant 2h - 1$.*

Proof. We give the argument when the points of $A(X)$ are left of N and the points of $B(X)$ are right of N. The argument when the sides are reversed is symmetric.

Let W be the black portion of N. We assume first that W is trivial. Then N is a witnessing path from x_0 to y_0, so that the elements on N form a chain in P. Now assume that $|X| \geqslant 2$, and let a_i and a_j be distinct elements of $A(X)$. Then let $W(a_i, b_j)$ and $W(a_j, b_i)$ be arbitrary witnessing paths. Since a_i and a_j are left of N, while b_i and b_j are right of N, there must be a point z common to $W(a_i, b_j)$ and N and a point z' common to $W(a_j, b_i)$ and N. However, N is a chain, so z and z' are comparable in P. If $z \leqslant_P z'$, then $a_i < z \leqslant z' < b_i$ in P, which is false. A similar contradiction is reached if $z' <_P z$. We conclude that $|X| = 1$. This observation completes the proof of the first assertion.

We now assume that W, the black portion on N, is non-trivial. Consider the red portion of N. Clearly, it is a chain on at most h elements from $v(a_\alpha, b_\beta)$ to y_0. For each $a \in A(X)$ let $\tau(a)$ be the lowest element of this chain such that $a <_P \tau(a)$. Consider also the blue portion of N which is a chain on at most h elements from x_0 to $u(a_\alpha, b_\beta)$. For each $b \in B(X)$ let $\tau(b)$ be the highest element of this chain such that $\tau(b) <_P b$.

We claim that when $i, j \in X$ and $i < j$, then $\tau(a_i) \leqslant_P \tau(a_j)$. Assume the contrary, $\tau(a_i) >_P \tau(a_j)$. Consider a witnessing path from a_j to $\tau(a_j)$. By our assumption this path must avoid $a_i S y_0$. Thus, we have $\alpha < i < j$ and a witnessing path from $a_j S y_0$ to $a_\alpha S y_0$ avoiding $a_i S y_0$. This is a contradiction with the statement of Proposition 16. Similarly, when $i, j \in X$ and $i < j$, we have $\tau(b_i) \geqslant_P \tau(b_j)$.

We claim that at least one of these two inequalities $\tau(a_i) \leqslant_P \tau(a_j)$, $\tau(b_i) \geqslant_P \tau(b_j)$ must be strict. To see this assume that $\tau(a_i) = \tau(a_j)$ and $\tau(b_i) = \tau(b_j)$. Consider a witnessing path W' from a_j to b_i. Since a_j is left of N and b_i is right of N, we know that W' intersects N. Let z' be a common point of W' and N. If z' lies on the red portion of N, then $b_i > z' \geqslant \tau(a_j) = \tau(a_i) \geqslant a_i$ in P which is a contradiction. If z' lies on the blue portion of N, then $a_j < z' \leqslant \tau(b_i) = \tau(b_j) \leqslant b_j$ in P which is a contradiction. Thus z' is black. Similarly a witnessing path W'' from a_i to b_j must intersect N at a point z'' which is black. Since the black portion is a chain, z' and z'' are comparable in P. If

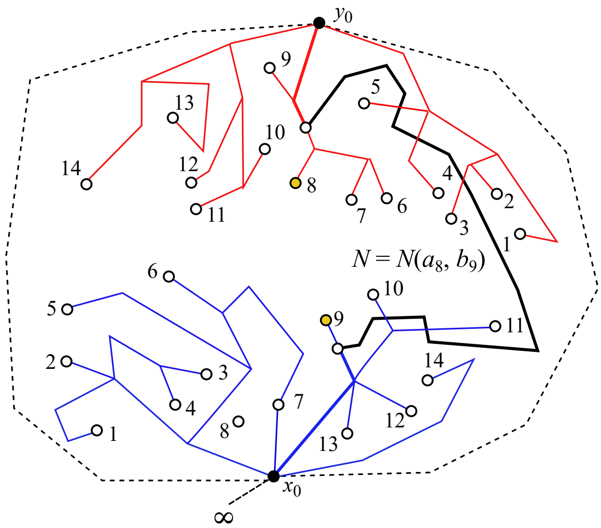

FIGURE 9. Consider $N = N(a_8, b_9)$. The figure suggests that the black part of N intersects a_3Sy_0. This forces a_4, a_5, a_6, a_7 to be left of N. Similarly, the black part of N intersects x_0Tb_{11}. This forces b_{10} to be left of N.

$z' \leqslant_P z''$, then $a_j \leqslant z' \leqslant z'' \leqslant b_j$ in P. If $z' \geqslant_P z''$, then $a_i \leqslant z'' \leqslant z' \leqslant b_i$ in P. Both statements are false. This observation confirms our claim.

Consider the following two sets $\{\tau(a) \mid a \in A(X)\}$, $\{\tau(b) \mid b \in B(X)\}$. Each of these can be considered as a sequence sorted by the linear order on X as a set of integers. The first sequence is non-decreasing on the red chain in N. The second sequence is non-increasing on the blue chain in N. For each consecutive pair $i < j$ of integers in X (i.e. there is no $i' \in X$ with $i < i' < j$), we have a change in at least one of the two sequences. Therefore,

$$2h \geqslant |\{\tau(a) \mid a \in A(X)\}| + |\{\tau(b) \mid b \in B(X)\}| \geqslant 2 + (|X| - 1).$$

With this observation, the proof is complete. □

Proposition 18. *If T and S have no common vertices, then $n \leqslant 6h + 1$.*

Proof. We assume that $n \geqslant 6h + 2$ and argue to a contradiction. Let $N = N(a_{4h}, b_{4h+1})$ be a separating path. Then set $u = u(a_{4h}, b_{4h+1})$, $v = v(a_{4h}, b_{4h+1})$. We note that v belongs to $a_{4h}Sy_0$ but not $a_{4h+1}Sy_0$. Dually, $u \in x_0Tb_{4h+1}$ but not x_0Tb_{4h}. Let W be the black portion of N. Note that W is non-trivial. We split the elements of the pairs into $A_1 = \{a_1, a_2, \ldots, a_{4h-1}\}$, $A_2 = \{a_{4h+2}, a_{4h+3}, \ldots, a_{6h+2}\}$, $B_1 = \{b_1, b_2, \ldots, b_{4h-1}\}$ and $B_2 = \{b_{4h+2}, b_{4h+3}, \ldots, b_{6h+2}\}$.

Proposition 9 implies that both a_{4h+1} and b_{4h} are left of N. Propositions 8 and 16 together imply that all elements of $A_2 \cup B_1$ are left of N. On the other hand, elements of $A_1 \cup B_2$ may be on either side of N.

We illustrate the path N and possible intersections with paths in T and S in Figure 9.

We partition the set $\{1, 2, \ldots, 4h-1\}$ as $X_1 \cup X_2$, where $i \in X_1$ if and only if a_i is left of N. Since N separates $A(X_2)$ from $B(X_2) \subseteq B_1$, it follows from Proposition 17 that $|X_2| \leqslant 2h-1$. Therefore $|X_1| \geqslant 4h-1-(2h-1) = 2h$. Similarly, we partition $\{4h+2, 4h+3, \ldots, 6h+2\}$ as $Y_1 \cup Y_2$, where $i \in Y_1$ if and only if b_i is left of N. Now we conclude that $|Y_2| \leqslant 2h-1$ and therefore $|Y_1| \geqslant 2h+1-(2h-1) = 2$. Set $m = 2h$. Now we are going to discard excess elements and relabel those that remain. Let A' be a subset of $A(X_1)$ of size m with elements relabeled as $\{w_1, \ldots, w_m\}$ so that $w_1 <_S \cdots <_S w_m$. Let B' be the corresponding subset of elements of $B(X_1)$ with elements relabeled correspondingly as $\{z_1, \ldots, z_m\}$. Let $\{z_{m+1}, z_{m+2}\}$ be a subset of $B(Y_1)$ of size 2 so that $z_{m+1} <_T z_{m+2}$. Let $\{w_{m+1}, w_{m+2}\}$ be the corresponding subset of elements of $A(Y_1)$. Note that we have

$$w_1 <_S \cdots <_S w_m <_S a_{4h} <_S a_{4h+1} <_S w_{m+1} <_S w_{m+2},$$
$$z_1 <_T \cdots <_T z_m <_T b_{4h} <_T b_{4h+1} <_T z_{m+1} <_T z_{m+2}.$$

Let $N' = N(w_{m+1}, z_{m+2})$ be a separating path. Then set $u' = u(w_{m+1}, z_{m+2})$ and $v' = v(w_{m+1}, z_{m+2})$. Also, let W' denote the black part of N'. Proposition 9 implies that z_{m+1} is left of N'. Propositions 8 and 16 then imply that all elements of B' are left of N'.

Claim. All elements of A' are right of N'.

Proof. Considet an element $a \in A'$. Since a is left of N, W cuts a. Let p be the largest point of W that is also on aSy_0. Since z_{m+2} is left of N, we know that W cuts z_{m+2}. Let q be the least element of W that is also on x_0Tz_{m+2}. Since the red and blue trees are disjoint, we know $v \leqslant p < q \leqslant u$ in P.

By the planarity of the drawing, we have that q is also the *first* point of x_0Tz_{m+2} that lies in W. Also, the path qWp leaves x_0Tz_{m+2} from the right side.

Proposition 16 implies that there is a point $r \in qWu$ such that W cuts z_{m+1} at r. In particular, we have $q \leqslant r \leqslant z_{m+1}$ in P.

In our research, we found it convenient to view the path $M = x_0TqWpSa$ as a *pillar*. M is not a witnessing path, and it is not a separating path. Nevertheless, it has a useful property

$$z \leqslant q \leqslant r < z_{m+1} \text{ in } P, \text{ for every } z \text{ in } M.$$

In particular, this implies that W' does not intersect M as otherwise if w is an intersection point of W' and M, we would $w_{m+1} \leqslant w \leqslant z_{m+1}$ in P, a clear contradiction.

Recall that W' hits the branch x_0Tz_{m+2} at element u'. There are two options: either u' lies in the section x_0Tq (including q) or u' lies in the section qTz_{m+2} (excluding q). We want to exclude the the first option. Indeed, if u' is on the path x_0Tq, then

$$w_{m+1} < u' \leqslant q \leqslant r < z_{m+1} \text{ in } P,$$

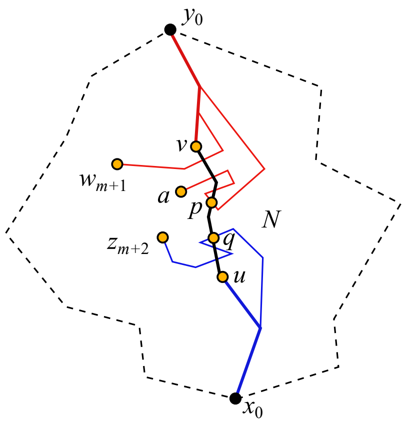

FIGURE 10. On the left side, we show the separating path N with thickened lines. We also show the points w_{m+1}, z_{m+2}, a that have crossed over to the left of N. On the right side, we show the pillar M with thickened lines. The remaining lines are part of the separating path N'. We note that a must be right of N', which is a contradiction.

which is a contradiction. We conclude that u' is an element of x_0Tz_{m+2} that occurs *after* q. This implies that the pillar M leaves the path x_0Tu' from the right side, as illustrated in Figure 10.

Now consider the red path $v'Sy_0$. We observe that there is no point on $v'Sy_0$ that also belongs to the pillar M. To see this, suppose that w is a common point. Then

$$w_{m+1} \leqslant w \leqslant q \leqslant r < z_{m+1} \text{ in } P,$$

which is again clearly false.

To complete the proof, we simply recall that the pillar leaves the path x_0Tu' from the right side and never touches N' again. This means that a must be right of N', as desired. The final statement in the proof of the preceding claim is also illustrated in Figure 10. □

We have now reached a contradiction since we have shown that N' separates A' and B' with $|A'| = |B'| = m = 2h$, contradicting Proposition 17. This completes the proof of Proposition 18. □

4.1. **Separating the Red and Blue Trees.** To illustrate the challenges we face in separating the blue and red trees, we show on the left side of Figure 11 how it can happen that $x_0Tb_i \cap y_0Sa_j$ can intersect for at least half the comparable pairs in $A_I \times B_I$. In this example, there is one "essential" crossing of the red and blue trees. On the right side of this figure, we show a small example with two essential crossings. With these examples in mind, it is conceivable that a more complex example might have arbitrarily many different essential crossings. Accordingly, it will take some effort to show that this cannot happen.

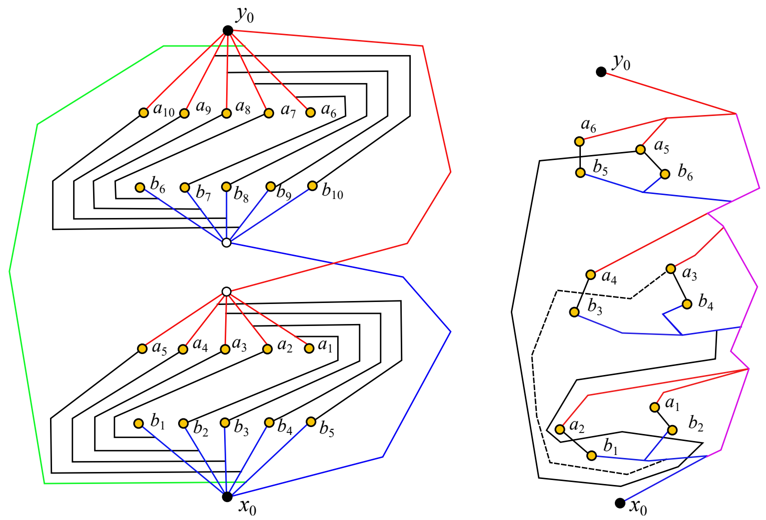

FIGURE 11. On the left side, we show two copies of a wheel stacked vertically. In this figure, black and green paths are oriented downwards. On the right side, we show three copies of S_2 stacked vertically. In this figure, edges common to the red tree and the blue tree are shown in purple. The black paths (solid and dotted) are oriented downwards.

Now we begin the material necessary to separate the red and blue trees. Let Z be the non-empty subposet of P consisting of all elements of P that belong to a witnessing path from x_0 to y_0. If we restrict our drawing of G to the induced subgraph determined by the elements of Z, we obtain a drawing without edge crossings of the cover graph of Z. Furthermore, x_0 is the unique minimal element of Z, and y_0 is the unique maximal element of Z. We note that no element of $A_I \cup B_I$ belongs to Z.

Although we are working with a drawing of a cover graph, and not an order diagram, the fact that x_0 and y_0 are on the exterior face implies that when we restrict to the cover graph of Z, and we take an element z of Z, then like in an order diagram, the up covers of z must appear in a block as do the down covers of z. Accordingly, when $z \neq y_0$, among the up covers of z, there is a well defined left-to-right order (clockwise). And dually, when $z \neq x_0$, there is a left-to-right order on down covers of z (counterclockwise). See Figure 12.

It is easy to see that Z has dimension at most 2. We carry out a depth-first search of Z, starting at x_0, with a local left-to-right preference, to obtain a linear extension L of Z. Dually, we carry out a depth-first search of Z using a local right-to-left preference, to obtain a linear extension R of Z. These two linear extensions form a realizer of Z as $z < z'$ in Z if and only if $z <_L z'$ and $z <_R z'$. When $u, v \in Z$ and $u \parallel_P v$, we we will say that u is *left of* v (also v is *right of* u) when $u <_L v$ and $v <_R u$. Note that these two terms are transitive, e.g., if u is left of v and v is left of w, then u is left of w. We illustrate these concepts in Figure 12.

When W is a witnessing path from x_0 to y_0, then W is also a maximal chain in Z. Therefore, if $u \in Z$ and u is not on W, then there is an element $v \in W$ such that $u \parallel_P v$.

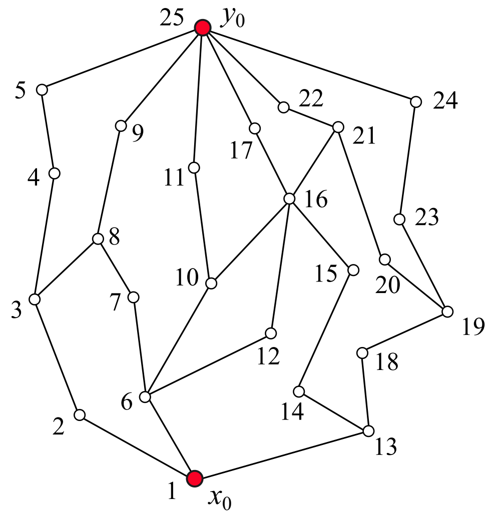

FIGURE 12. The subposet Z consists of all points of P that are on witnessing paths from x_0 to y_0. The element 16 has two up covers 17, 21, and three down covers 10, 12, 15 (both lists sorted from left to right). The elements of Z are labeled with the integers in [25] according to the depth-first search linear extension $<_L$ that uses a local left-to-right preference. Note that 7 is left of 16 and 16 is left of 18. This implies 7 is left of 18. Also, 10 is right of 5.

We will say that u is *left of* W if u is left of v. In a symmetric manner, we will say that a point $u \in Z$ that is not on W is *right of* W if there is a point w on W such that u is right of w. It is easy to verify that when $u \in Z$ and u is not on W, then either u is left of W or right of W, and these two options are mutually exclusive. Naturally, when $u \in Z$, the phrase u is *not left* of W means that either u is on W or u is right of W.

The next two elementary propositions highlight the interplay of the terms left and right as applied to a pair of elements, and an element vs. a witnessing path.

Proposition 19. *Let W be a witnessing path from x_0 to y_0 and let u and v be a pair of points from Z with $u \parallel_P v$. If u is not right of W and v is not left of W, then u is left of v.*

Proof. Since u is not right of W, we know that either u is left of W or u is on W. In both cases, there exists u' in W such that $u \leqslant_L u'$. Analogously, since v is not left of W, we know that either v is right of W or v is on W. In both cases, there exists v' in W such that $v \leqslant_R v'$.

Since u' and v' both belong to W, they are comparable in P, i.e., $u' \leqslant v'$ or $v' \leqslant u'$ in P. In the first case, we conclude $u \leqslant_L u' \leqslant_L v' \leqslant_L v$ and since $u \parallel_P v$ in P, at least one of these inequalities must be strict. This proves that u is left of v. In the second case, we conclude $v \leqslant_R v' \leqslant_R u' \leqslant_R u$, and since $u \parallel_P v$ in P, at least one of these inequalities must be strict. This proves that v is right of u (so u is left of v). These observations complete the proof of the proposition. □

Proposition 20. *Let W be a witnessing path from x_0 to y_0, and let u and v be points of Z with u left of v. If v is not right of W, then u is left of W. Also, if u is not left of W, then v is right of W.*

Proof. We prove the first assertion. The argument for the the second is symmetric. If v is on W, then the fact that u is left of v implies u is left of W. If v is not on W, then our assumption forces that v left of W. Choose v' on W such that v is left of v'. Thus by transitivity, u is left of v' and so u is left of W. □

Our fixed drawing of the cover graph of Z splits the plane into regions: some number of bounded regions and one unbounded. We call such a bounded region a *Z-face*. Each element of P that is not in Z is in the interior of one of the regions. After adding two dummy element z', z'' into P (and Z) such that (1) $x_0 < z' < y_0$, $x_0 < z'' < y_0$ in P and all these relations are covers; (2) z' is the leftmost up (down) cover of x_0 (y_0); (3) z'' is the rightmost up (down) cover of x_0 (y_0); we can assume that any element of P that is not in Z is in the interior of one of the (bounded) Z-faces.

Each Z-face $\mathcal{F}$ is bounded by two distinct witnessing paths that have only their starting and ending points in common. We let $x(\mathcal{F})$ denote the common starting point, and we let $y(\mathcal{F})$ denote the common ending point of these two witnessing paths. When we start at $x(\mathcal{F})$ and traverse the boundary of $\mathcal{F}$ in a clockwise manner, we follow the *left side* of $\mathcal{F}$ until we reach $y(\mathcal{F})$. Then we traverse the *right side* of $\mathcal{F}$ backwards until we arrive back at $x(\mathcal{F})$.

When $\mathcal{F}$ is a Z-face, no element u of P that is in the interior of $\mathcal{F}$ satisfies $x(\mathcal{F}) <_P u <_P y(\mathcal{F})$; otherwise this region would be split into smaller Z-faces. Also, a Z-face has no chords.

When t, t' are elements of Z, we let $[t, t']$ consist of all elements $s \in Z$ with $t \leqslant s \leqslant t'$ in Z. However, we also consider $[t, t']$ as a region in the plane, i.e, we consider all points in the plane that are on witnessing paths from t to t' as well as all points in the plane that are in the interior of regions bounded by portions of two witnessing paths from t to t'. Thus, there is a well defined *left side* and *right side* of $[t, t']$. Note that $[t, t']$ is a union of Z-faces and witnessing paths. It might be that the left side and the right side of $[t, t']$ can share points and even edges of witnessing paths. We illustrate these concepts in Figure 13.

When $\mathcal{F}$ is a Z-face, we define the *left side path of $\mathcal{F}$* formed by concatenating the following three paths: (1) the left side of $[x_0, x(\mathcal{F})]$; (2) the left side of $\mathcal{F}$; and (3) the left side of $[y(\mathcal{F}), y_0]$. The *right side path of $\mathcal{F}$* is defined symmetrically.

Let $\mathcal{F}$ and $\mathcal{F}'$ be two distinct Z-faces. We say that $\mathcal{F}$ is *under* $\mathcal{F}'$ if $y(\mathcal{F}) \leqslant_P x(\mathcal{F}')$. Dually, we say that $\mathcal{F}$ is *over* $\mathcal{F}'$ if $x(\mathcal{F}) \geqslant_P y(\mathcal{F}')$. We say that $\mathcal{F}$ is *left of* $\mathcal{F}'$ if no point on the boundary of $\mathcal{F}$ is right of the left side path of $\mathcal{F}'$. Symmetrically, we say that $\mathcal{F}$ is *right of* $\mathcal{F}'$ if no point on the boundary of $\mathcal{F}$ is left of the right side path of $\mathcal{F}'$. Proposition 20 implies that when $\mathcal{F}$ is left of $\mathcal{F}'$, there is a point u on the boundary of

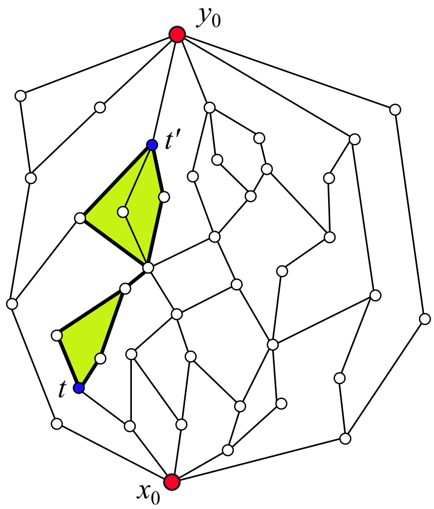

FIGURE 13. An interval $[t, t']$ with its two sides bolded.

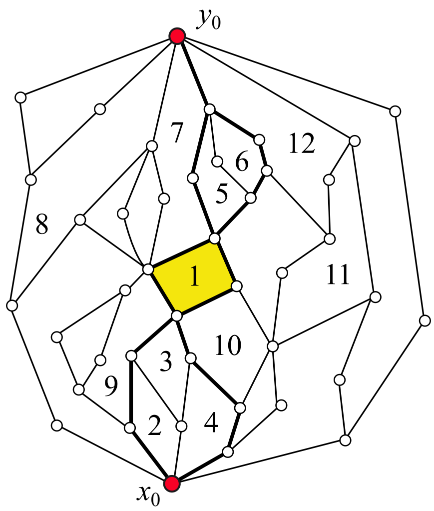

FIGURE 14. The left side path and the right side path for the face 1 are shown using thick lines. Shadow face 1 is over 2, 3 and 4. It is under 5 and 6. All other Z-faces are either left or right of 1. In particular, $7, 8, 9$ are left of 1, while $10, 11, 12$ are right of 1.

$\mathcal{F}$ that is left of the left side path of $\mathcal{F}'$. Symmetrically, when $\mathcal{F}$ is right of $\mathcal{F}'$, there is a point v on the boundary of $\mathcal{F}$ that is right of the right side path of $\mathcal{F}'$. These observations give a formal argument for the natural conclusion that if $\mathcal{F}$ and $\mathcal{F}'$ are distinct Z-faces, then $\mathcal{F}$ is either over, under, left of, or right of $\mathcal{F}'$. Furthermore, the four options are mutually exclusive. We illustrate the concept of over, under, left and right for Z-faces in Figure 14.

When $u \in P$ and u is not in Z, there is a unique Z-face $\mathcal{F}_u$ containing u in its interior. We let $y_u = y(\mathcal{F}_u)$ and $x_u = x(\mathcal{F}_u)$. We note that when $(a, b) \in I$, then $a \notin Z$ and $b \notin Z$. A witnessing path from a to y_0 has to leave the interior of $\mathcal{F}_a$, and this implies $a < y_a$ in

P. Dually, a witnessing path from b to x_0 (going backward) has to leave the interior of $\mathcal{F}_b$, and this implies $x_b < b$ in P.

A pair $(a, b) \in I$ will be called a *same-face pair* if $\mathcal{F}_a = \mathcal{F}_b$; otherwise we will say that (a, b) is a *diff-face pair*. When (a, b) is a diff-face pair, then $\mathcal{F}_a$ is not under $\mathcal{F}_b$, as this would imply $a < y_a \leqslant x_b < b$ in P. It follows that $\mathcal{F}_a$ is either over, left of or right of $\mathcal{F}_b$.

Lemma 21. *The set of all diff-face pairs in I can be covered by two reversible sets.*

Proof. Let M_1 be the set consisting of all diff-face pairs $(a, b) \in I$ such that either $\mathcal{F}_a$ is left of $\mathcal{F}_b$ or $\mathcal{F}_a$ is over $\mathcal{F}_b$. Let M_2 be the set consisting of all diff-face pairs $(a, b) \in I$ such that either $\mathcal{F}_a$ is right of $\mathcal{F}_b$ or $\mathcal{F}_a$ is over $\mathcal{F}_b$. Clearly, the two sets M_1 and M_2 cover the set of all diff-face pairs in I. We now show that M_2 is reversible. The argument to show that M_1 is reversible is symmetric. Suppose to the contrary that $((a_1, b_1), \ldots, (a_k, b_k))$ is an alternating cycle of diff-face pairs from M_2. For each $i \in [k]$, let z_i be the $<_L$-largest point on the boundary of $\mathcal{F}_{b_i}$ with $z_i <_P b_i$. There is such an element since $x_{b_i} <_P b_i$.

Claim. $z_i <_L z_{i+1}$ for all $i \in [k]$ (cyclically).

Proof. Let $i \in [k]$. Since (a_i, b_i) is a diff-face pair, $\mathcal{F}_{a_i}$ and $\mathcal{F}_{b_i}$ are distinct Z-faces. We know that $a_i <_P b_{i+1}$. Let W_i be a witnessing path from a_i to b_{i+1}. Then let u_i be the first point of W_i that is on the boundary of $\mathcal{F}_{a_i}$, and let v_{i+1} be the last point of W_i that is on the boundary of $\mathcal{F}_{b_{i+1}}$. Then $u_i \leqslant_P v_{i+1}$, and $v_{i+1} \leqslant_L z_{i+1}$. Since $\leqslant_L$ is a linear extension of P, this implies $u_i \leqslant_L z_{i+1}$.

If $u_i \leqslant_P z_i$, then $a_i <_P u_i \leqslant_P z_i <_P b_i$ so $a_i <_P b_i$, which is false. We conclude that $u_i \nleqslant_P z_i$. If $z_i <_P u_i$, then we have $z_i <_L u_i \leqslant_L z_{i+1}$. This implies $z_i <_L z_{i+1}$ as desired.

So we may assume that $u_i \nleqslant z_i$ and $z_i \nless u_i$ in P. In other words, $u_i \parallel z_i$ in P. Since $(a_i, b_i) \in M_2$, either $\mathcal{F}_{a_i}$ is over $\mathcal{F}_{b_i}$ or $\mathcal{F}_{a_i}$ is right of $\mathcal{F}_{b_i}$. If $\mathcal{F}_{a_i}$ is over the face $\mathcal{F}_{b_i}$, then we have $u_i \leqslant y_{a_i} \leqslant x_{b_i} \leqslant z_i$ in P but we are now assuming that $u_i \parallel_P z_i$, so this option cannot hold. We conclude that $\mathcal{F}_{a_i}$ is right of $\mathcal{F}_{b_i}$. Let W be the left side path of $\mathcal{F}_{a_i}$. Then no point on the boundary of $\mathcal{F}_{b_i}$ is right of W. In particular, z_i is not right of W. On the other hand, the point u_i is on the boundary of $\mathcal{F}_{a_i}$. Therefore, u_i is not left of W. Now Proposition 19 implies z_i is left of u_i. Altogether we have $z_i < u_i \leqslant z_{i+1}$ in L. This completes the proof of the claim. □

To complete the proof of the lemma, we simply note that the statement of the claim cannot hold for all $i \in [k]$ cyclically. □

Lemma 22. *For every strict alternating cycle $((a_1, b_1), \ldots, (a_k, b_k))$ of same-face pairs in I, there is a Z-face $\mathcal{F}$ such that all elements $a_1, \ldots, a_k, b_1, \ldots, b_k$ are in the interior of $\mathcal{F}$.*

Proof. We assume to the contrary that $((a_1, b_1), \ldots, (a_k, b_k))$ is a strict alternating cycle of same-face pairs from I, and there is no Z-face that contains all elements of the cycle in its interior. Of all such cycles, we assume further that k is minimum.

Claim 1. There do not exist distinct integers $i, j \in [k]$ such that the pairs (a_i, b_i) and (a_j, b_j) are in the same Z-face.

Proof. Suppose that for some $i \neq j$ all four elements involved in (a_i, b_i), (a_j, b_j) lie in the same Z-face. Since our alternating cycle is a counterexample, we do not have all the pairs lying in the same Z-face, so we know that $k \geqslant 3$. After a relabeling, we may assume that $j = k$ and $2 \leqslant i \leqslant k-1$. However, this implies that

$$((a_1, b_1), \ldots, (a_{i-1}, b_{i-1}), (a_k, b_i))$$

is an alternating cycle of same-face pairs from I. This is a contradiction unless all the pairs on this cycle belong the same Z-face. In this case, we consider the alternating cycle

$$((a_i, b_1), (a_{i+1}, b_{i+1}), \ldots, (a_k, b_k)).$$

Again, we have a strict alternating cycle. However, now it is clear that not all the pairs on this cycle belong to the same Z-face. Furthermore, the length of this cycle is less than k. The contradiction completes the proof of the claim. □

For each $i \in [k]$, let $\mathcal{F}_i$ be the common Z-face $\mathcal{F}_{a_i} = \mathcal{F}_{b_i}$, let $x_i = x_{b_i}$, and let $y_i = y_{a_i}$. Let $W_i = W_i(a_i, b_{i+1})$ be a witnessing path. Then let u_i be the lowest point of W_i that is on the boundary of $\mathcal{F}_i$, and let v_{i+1} be the highest point of W_i that is on the boundary of $\mathcal{F}_{i+1}$. We note that $a_i <_P u_i \leqslant_P v_{i+1} <_P b_{i+1}$.

Claim 2. For all $i, j \in [k]$, $u_i \leqslant_P v_j$ if and only if $j = i + 1$ (cyclically).

Proof. We already know that $u_i \leqslant_P v_{i+1}$ for all $i \in [k]$. Now suppose $j \neq i + 1$ and $u_i \leqslant v_j$. Then $a_i <_P u_i \leqslant_P v_j < b_j$. This implies $a_i < b_j$. Now we have contradicted the assumption that our original cycle is strict. With this observation, the proof of the claim is complete. □

Claim 2 implies that $((u_1, v_1), \ldots, (u_k, v_k))$ is a strict alternating cycle of incomparable pairs in Z. Let $i \in [k]$. Since $u_i \parallel_P v_i$, and both u_i and v_i are on the boundary of $\mathcal{F}_i$, it implies that they are on opposite sides of $\mathcal{F}_i$. Also, $\{u_i, v_i\} \cap \{x_i, y_i\} = \emptyset$. Furthermore, the statement u_i is left of v_i means the same as saying u_i is on the left side of $\mathcal{F}_i$ and v_i is on the right side of $\mathcal{F}_i$. A symmetric statement holds when u_i is right of v_i.

Claim 3. For each $i \in [k]$, the following statements hold.

(i) If $u_i <_L v_i$, then $u_{i+1} <_L v_{i+1}$ and $u_{i+1} <_L u_i$.
(ii) If $u_i <_R v_i$, then $u_{i+1} <_R v_{i+1}$ and $u_{i+1} <_R u_i$.

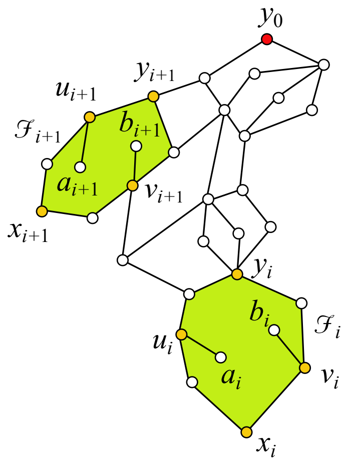

FIGURE 15. The point v_{i+1} must be on the left side of $[u_i, y_0]$. This forces v_{i+1} to be on the right side of $\mathcal{F}_{i+1}$. In turn, this forces u_{i+1} to be on the left side of $\mathcal{F}_{i+1}$.

Proof. We prove the first statement. The proof of the second is symmetric. Let $i \in [k]$. Then u_i is on the left side of $\mathcal{F}_i$, and v_i is on the right side of $\mathcal{F}_i$.

Since u_{i+1} and v_{i+1} lie on the boundary of $\mathcal{F}_{i+1}$, and $u_i \leqslant_P v_{i+1}$, $u_i \parallel_P u_{i+1}$, we conclude that v_{i+1} must be on the boundary (left side or right side) of $[u_i, y_0]$ while the point u_{i+1} as well as all points in plane that are in the interior of $\mathcal{F}_{i+1}$ are in the exterior of the interval $[u_i, y_0]$.

First, we assume that v_{i+1} is on the right side of $[u_i, y_0]$. Clearly, the right side of this interval is the portion of the left side of $\mathcal{F}_i$ from u_i to y_i concatenated with the right side of the interval $[y_i, y_0]$. If v_{i+1} lies on the left side of $\mathcal{F}_i$ (somewhere starting from u_i but before y_i), then since $\mathcal{F}_i$ and $\mathcal{F}_{i+1}$ are distinct faces, this would force v_{i+1} to be on the right side of $\mathcal{F}_{i+1}$. In turn, this would imply that u_{i+1} is on the left side of $\mathcal{F}_{i+1}$ and therefore $u_{i+1} <_L v_{i+1}$. Also when we extend any witnessing path from u_i to v_{i+1} to a witnessing path W from x_0 to y_0, we clearly have u_i on W and u_{i+1} left of W. Now by Proposition 19, we conclude that u_{i+1} is left of u_i as desired. If v_{i+1} lies on the right side of $[y_i, y_0]$, then $v_i < y_i \leqslant v_{i+1}$ in P, which cannot hold in our strict alternating cycle.

It remains only to consider the case that v_{i+1} is on the left side of the interval $[u_i, y_0]$. This forces v_{i+1} to be on the right side of $\mathcal{F}_{i+1}$ and therefore u_{i+1} is on the left side of $\mathcal{F}_{i+1}$. Thus we get $u_{i+1} <_L v_{i+1}$. This is the situation illustrated in Figure 15. Again when we extend any witnessing path from u_i to v_{i+1} to a witnessing path W from x_0 to y_0, we clearly have u_i on W, and u_{i+1} left of W. Now by Proposition 19, we conclude that u_{i+1} is left of u_i as desired. With this observation, the proof of the claim is complete. □

To complete the proof of the lemma, we simply note that the statement of the claim cannot hold for all $i \in [k]$ cyclically. □

When $\mathcal{F}$ is a Z-face, let $I(\mathcal{F})$ consist of those pairs $(a,b) \in I$ such that $\mathcal{F} = \mathcal{F}_a = \mathcal{F}_b$. Then Lemma 21 and Lemma 22 imply

$$\rho(I) \leqslant 2 + \max_{\mathcal{F}} \ \rho(I(\mathcal{F})).$$

With this material as background, we may assume that:

(i) The boundary of G is the boundary of a Z-face $\mathcal{F}$.
(ii) $I = I(\mathcal{F})$, $x_0 = x(\mathcal{F})$, and $y_0 = y(\mathcal{F})$.
(iii) If $(a,b) \in I$, then $x_0 <_P b$ and $a <_P y_0$.

Let $((a_1,b_1),\ldots,(a_n,b_n))$ be a directed path in I. Suppose that (i,j) is a pair of distinct integers in $[n]$ such that x_0Tb_j intersects y_0Sa_i. Let $u = u(a_i,b_j)$ and $v = v(a_i,b_j)$ be, respectively the least and greatest element of P common to x_0Tb_j and y_0Sa_i. Then $x_0 <_P u \leqslant_P v <_P y_0$. Since $\mathcal{F}$ is a Z-face, no point in the interval $[u,v]$ is in the interior of $\mathcal{F}$. It follows that $[u,v]$ is a portion of one of the two sides of $\mathcal{F}$. Furthermore, all points and edges of x_0Tb_j after v are in the interior of $\mathcal{F}$, and all points and edges of a_iSy_0 before u are in the interior of $\mathcal{F}$.

Now suppose that $i < j$. It is easy to see that the following three statements hold:

$S(1)$: x_0Tv and uSy_0 are portions of the right side of $\mathcal{F}$.
$S(2)$: If $1 \leqslant i' < i$, then uSy_0 is a terminal portion of $a_{i'}Sy_0$.
$S(3)$: If $j < j' \leqslant n$, then x_0Tv is an initial portion of $x_0Tb_{j'}$.

If $i > j$, then there is a symmetric set of three statements for the left side of $\mathcal{F}$.

The proof of the following proposition completes the proof of our Theorem.

Proposition 23. *If $((a_1,b_1),\ldots,(a_n,b_n))$ is a directed path in H_I, then $n \leqslant 34h+9$.*

Proof. We argue by contradiction and assume that there is a directed path $((a_1,b_1),\ldots,(a_n,b_n))$ in H_I with $n = 34h+10$. Set $s = 6h+2$, and note that $n = 5s+4h$.

If X is any subset of $[n]$ with $|X| = s$, then Proposition 18 implies that there are distinct integers $i,j \in X$ such that x_0Tb_j intersects a_iSy_0. We apply this observation to the set $X = [s]$. We give the balance of the argument under the assumption that $i < j$. From the details of the argument, it will be clear that the proof when $i > j$ is symmetric.

Since $1 \leqslant i < j \leqslant s$, we know that statements $S(1), S(2), S(3)$ hold. For each α with $i \leqslant \alpha \leqslant n$, let u_α be the lowest point on the right side of $\mathcal{F}$ that belongs to $a_\alpha Sy_0$. If $\alpha < n$, then $u_\alpha \leqslant_P u_{\alpha+1}$. For each β with $j \leqslant \beta \leqslant n$, let v_β be the highest point on the right side of $\mathcal{F}$ that belongs to x_0Tb_β. Since $a_\beta \parallel_P b_\beta$, we know $v_\beta <_P u_\beta$. Furthermore, if $\beta < n$, then $v_\beta \leqslant_P v_{\beta+1}$.

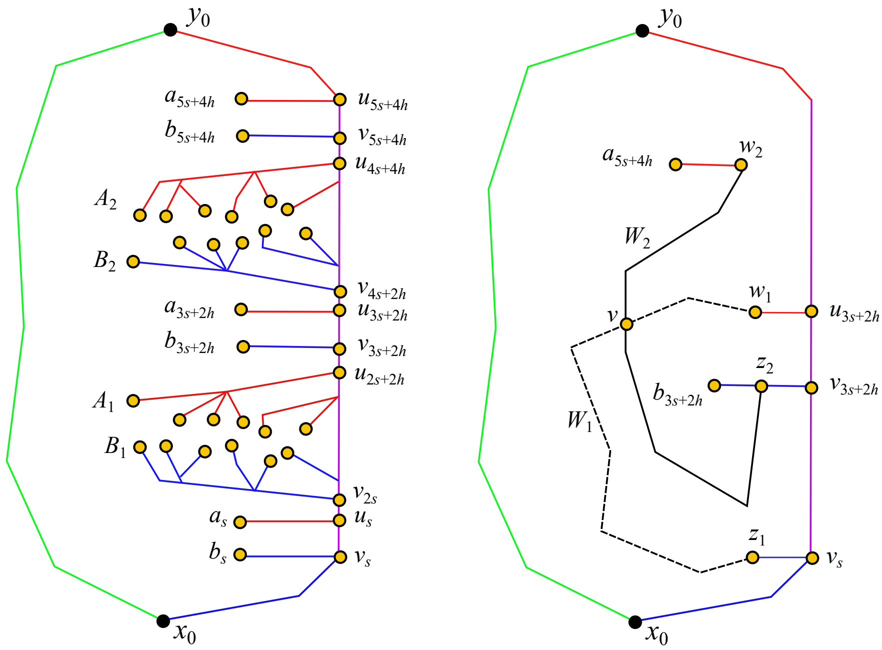

FIGURE 16. On the left, we show an intersection between the red and blue trees in the Z-face $\mathcal{F}$. On the right, we show a forced intersection, and the resulting contradiction completes the proof.

An important consequence of the previous paragraph is that for every β with $j \leqslant \beta < n$, the path x_0Tb_β contains a non-trivial portion of the right side of $\mathcal{F}$. Since we are working with a single Z-face x_0Tb_β cannot hit both the left side and the right side of $\mathcal{F}$. Therefore, x_0Tb_β stays disjoint from the left side of $\mathcal{F}$. This implies that whenever x_0Tb_β intersects $a_\alpha Sy_0$ m, we have $\alpha < \beta$.

Claim 1. If $j \leqslant k \leqslant n - s$, then $u_k \leqslant_P v_{k+s}$.

Proof. Consider the set $X = \{k, k+1, \dots, k+s\}$. Since this set has size $s+1$, by Proposition 18 there is some distinct pair α, β of elements of this set such that x_0Tb_β intersects $a_\alpha Sy_0$. Our remarks just above require $\alpha < \beta$, and this implies $u_\alpha \leqslant v_\beta$. We conclude that $u_k \leqslant u_\alpha \leqslant v_\beta \leqslant v_{k+s}$ in P. □

We note that if $j \leqslant k \leqslant m \leqslant n$, then $v_j <_P u_j \leqslant_P u_m$, so $v_j <_P u_m$. As illustrated on the left side of Figure 16, we then have the following comprehensive inequality:

$$\begin{aligned} v_s < u_s \leqslant v_{2s} < u_{2s+2h} \leqslant v_{3s+2h} < u_{3s+2h} \\ \leqslant v_{4s+2h} < u_{4s+4h} \leqslant v_{5s+4h} < u_{5s+4h} \text{ in } P. \end{aligned}$$

Let $N_1 = N(a_{3s+2h}, b_s)$ be a separating path. We note that x_0Tb_s and $a_{3s+2h}Sy_0$ are disjoint. Let $W_1 = W(w_1, z_1)$ be the (necessarily non-trivial) witnessing path that forms the black part of N_1. Proposition 9 implies that b_{3s+2h} is right of N_1.

Referring to the right side of Figure 16, let $\mathcal{R}$ be the region in the plane formed by $v_sTz_1W_1w_1Su_{3s+2h}$ and the portion of the right side of $\mathcal{F}$ between v_s and u_{3s+2h}. Clearly, when u is an element of $A_I \cup B_I$, we have u is right of N_1 if and only if u is in the interior of $\mathcal{R}$.

We assert that there is no point v in the blue tree with $v \geqslant_P u_s$ such that $v \in W_1$. To see this, the existence of v would imply:

$$a_s <_P u_s \leqslant_P v \leqslant z_1 \leqslant b_s.$$

This would imply $a_s <_P b_s$, which is false. Therefore, our assertion holds. From this, it follows that all points of $B_2 = \{b_{4s+2h}, \ldots, b_{4s+4h}\}$ are left of N_1.

Claim 2. W_1 does not intersect $a_{5s+4h}Sy_0$.

Proof. If the claim fails, then Proposition 16 implies that W_1 intersects aSy_0 for every $a \in A_2 = \{a_{4s+2h}, \ldots, a_{4s+4h}\}$. We assert that in fact, a is right of N_1 for every $a \in A_2$. If this assertion does not hold, then it is easy to see that a is enclosed by a_{3s+2h}. Therefore, our assertion holds. However, this now implies that N_1 separates A_2 from B_2. This is a contradiction with Proposition 17 since these two sets have size larger than $2h - 1$. □

As a consequence of Claim 2, we know that a_{5s+4h} is left of N_1. Now let $N_2 = N(a_{5s+4h}, b_{3s+2h})$ be a separating path, and let $W_2 = W(w_2, z_2)$ be the non-trivial witnessing path that forms the black part of N_2. Using symmetric arguments, the following statements hold: (1) b_{5s+4h} is right of N_2; (2) there is no point w of the red tree with $w \leqslant_P v_{5s+4h}$ such that $w \in W_2$; (3) all elements of A_2 are right of N_2; and (4) W_2 does not intersect x_0Tb_s.

Since a_{5s+4h} is left of N_1, and b_{3s+2h} is right of N_1, the witnessing path $W = a_{5s+4h}Sw_2W_2z_2Tb_{3s+2h}$ must intersect the boundary of $\mathcal{R}$. Clearly, this requires that there is a point v common to W_2 and W_1. This implies

$$a_{3s+2h} \leqslant_P w_1 \leqslant_P v \leqslant_P z_2 \leqslant_P b_{3s+2h}.$$

In turn, this implies $a_{3s+2h} <_P b_{3s+2h}$, which is false. The contradiction completes the proof of the proposition. □

And as noted previously, this completes the proof of Lemma 15, as well as the principal theorem of the paper.

5. Closing Comments

Since we have not been able to disprove that $\dim(P) = \mathcal{O}(h)$ we comment that our proof that for $\mathcal{O}(h^6)$ has three steps where improvements might be possible. Do we really need the $\mathcal{O}(h^3)$ factor in the transition from singly constrained to doubly constrained set of incomparable pairs? When I is a set of doubly constrained pairs, did we need another factor of h to transition to the doubly exposed case? Could $\dim(I)$ be linear in $\rho(I)$ when I is doubly exposed in P?

Although we believe the establishment of a polynomial bound for dimension in terms of height for posets with planar cover graphs is intrinsically interesting, we find the results of Section 3, where height plays no role, particularly intriguing. Indeed, we hope that insights from this line of research may help to resolve the following long-standing conjecture.

Conjecture 24. *For every $n \geqslant 2$, there is a least positive integer d so that if P is a poset with a planar cover graph and $\dim(P) \geqslant d$, then P contains the standard example S_n.*

Apparently, the first reference in print to Conjecture 24 is in an informal comment on page 119 of [13], published in 1991. However, the problem goes back at least 10 years earlier. In 1978, Trotter [12] showed that there are posets that have large dimension and have planar cover graphs. In 1981, Kelly [8] showed that there are posets that have large dimension and have planar order diagrams. In both of these constructions, the fact that the posets have large dimension is evidenced by large standard examples that they contain. The belief that large standard examples are necessary for large dimension among posets with planar cover graphs grew naturally from these observations.

To attack Conjecture 24, it is tempting to believe that we can achieve a transition from a singly constrained poset to a doubly exposed poset, independent of height, by allowing a considerable reduction in the dimension d.

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