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COMBINATORIAL PROBLEMS IN DIMENSION THEORY FOR PARTIALLY ORDERED SETS

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Résumé. — Nous présentons quelques résultats récents en théorie de la dimension des ensembles partiellement ordonnés. Parmi les sujets abordés nous considérons des problèmes de construction d'ensembles partiellement ordonnés irréductibles, doublement irréductibles et des inégalités concernant des opérations telles que coupures, amalgamations et produits cartésiens. Nous donnons également une liste de quelques problèmes irrésolus dans ce domaine des mathématiques combinatoires.

Abstract. — We present some recent results in the dimension theory of partially ordered sets. Among the topics discussed are construction problems for irreducible posets, doubly irreducible posets, and inequalities involving splits, amalgamations, and cartesian products. We also give a list of some unsolved problems in this area of combinatorial mathematics.

1. **Introduction and notation.** — A *partially ordered set* (poset) is a pair (X, P) where X is a set, always finite in this paper, and P is a reflexive, anti-symmetric, and transitive relation on X. The relation P is called a *partial order* on X. The notations $(x, y) \in P$, $x\,P\,y$, and $x \leqslant y$ in P are used interchangeably. We also write $x < y$ in P when $(x, y) \in P$ and $x \neq y$. When $x < y$ or $y < x$ in P, we say the distinct points x and y are *comparable*. When distinct points x and y are not comparable, we say x and y are *incomparable* and write $x\,I\,y$ in P.

If (X, P) and (Y, Q) are posets, then a function $f : X \to Y$ is called an *embedding* of (X, P) in (Y, Q) when $x_1 \leqslant x_2$ in P if and only if $f(x_1) \leqslant f(x_2)$ in Q for all $x_1, x_2 \in X$. The posets are said to be *isomorphic* and the embedding f is called an *isomorphism* when f is also a surjection. In this paper we do not distinguish between isomorphic posets.

If (X, P) is a poset and $Y \subset X$, then the poset (Y, Q) where $Q = P \cap (Y \times Y)$ is called a *subposet* of (X, P). Frequently we will find it convenient to use a single symbol to denote a poset. This notation is particularly useful in any discussion involving subposets. For example when Y is a subposet of X, we will say Y is contained in X and write $Y \subset X$.

A poset A is called an *antichain* if $a_1\,I\,a_2$ in A for each distinct pair of points from A. The width of a poset X, denoted $W(X)$, is the maximum number of points in an antichain contained in X. A poset C is called a *chain* if each distinct pair of points from C is comparable. The length of a poset X, denoted $L(X)$ is the maximum number of points in a chain contained in X. A chain (X, P) is also called a *totally ordered set* or *linearly ordered set*. In this case the partial order P is called a *total order* or *linear order*.

If P and Q are partial orders on a set X and $P \subset Q$, we say Q is an *extension* of P. If Q is also a linear order, then we say Q is a *linear extension* of P. A theorem of Szpilrajn [10] asserts that if P is a partial order on X and $\mathcal{C}$ is the collection of all linear extensions of P, then $\mathcal{C} \neq \varnothing$ and $\cap\, \mathcal{C} = P$. Dushnik and Miller [3] defined the dimension of a poset (X, P), denoted $\mathrm{Dim}\,(X, P)$, as the smallest positive integer n for which there exist linear extensions $L_1, L_2, \ldots, L_n$ of P so that $P = L_1 \cap L_2 \cap \ldots \cap L_n$.

We consider n-dimensional Euclidean space R^n as a poset with ordering

$$(a_1, a_2, \ldots, a_n) \leqslant (b_1, b_2, \ldots, b_n)$$

if and only if $a_i \leqslant b_i$ in R for $i = 1, 2, \ldots, n$. We then have the following alternate definition due to Ore [8]. $\mathrm{Dim}\,(X)$ is the smallest positive integer n for which $X \subset R^n$.

If P is a partial order on X, then the relation $\hat{P} = \{ (y, x) : (x, y) \in P \}$ is a partial order on X and is called the *dual* of P. We will sometimes write $\hat{X}$ to denote the dual of the poset X. A poset and its dual have the same width, length and dimension.

A poset X is said to be *irreducible* if

$$\mathrm{Dim}\,(X - x) < \mathrm{Dim}\,(X)$$

for each $x \in X$.

2. **Doubly irreducible posets.** — In 1955 Hiraguchi [4] proved the following result.

Theorem 1. — Dim $(X) \leqslant |X|/2$ *when* $|X| \geqslant 4$.

Much simpler proofs of this theorem have been provided by Bogart [1], Rabinovitch [9], and Trotter [12]. Trotter's proof is based on the following inequalities.

Theorem 2 [4]. — Dim $(X) \leqslant W(X)$.

Theorem 3 [12]. — *If A is an antichain of a poset X and $|X - A| \geqslant 2$, then* Dim $(X) \leqslant |X - A|$.

Rabinovitch's proof of theorem 1 is one of many simple proofs that can be fashioned by combining appropriate removal theorems together with some case work to show that the result holds for posets of small order. However, it is not known whether or not every poset with at least three points contains a pair of points whose removal decreases the dimension at most one. If this does not hold, then there exists an irreducible poset X so that $X - x$ is irreducible for every $x \in X$. Such posets are called doubly irreducible posets.

Problem 1. — Determine whether doubly irreducible posets exist.

We note that forbidden subposet characterizations have been provided for theorems 1 and 3 ([2], [6], [14]).

Problem 2. — Give a forbidden subposet characterization of theorem 2.

3. **Irreducible posets of large length.** — Trotter [13] proved that irreducible posets with arbitrarily large length exist. In [5], Kelly and Rival constructed 3-dimensional irreducible posets of length n for every $n \geqslant 2$.

Theorem 4. — *For every $m \geqslant 3$ and $n \geqslant 2$, there exists an m-dimensional irreducible poset with length $\geqslant n$.*

Proof. — We proceed by induction on m using Kelly and Rival's construction when $m = 3$ together with the well known result that if X and Y have universal bounds, then Dim $(X \times Y) =$ Dim $(X) +$ Dim (Y). Let $\underline{2}$ denote the 2-element chain $0 < 1$.

Now suppose that the theorem holds for $m = k$. Then choose a k-dimensional irreducible poset X having length at least $2n$. Let $\hat{X}$ denote the poset obtained from X by attaching universal bounds to X. Then let Y be a $k + 1$-dimensional irreducible subposet of the $k + 1$-dimensional poset $\hat{X} \times \underline{2}$. We now show that Y has length at least n.

Let $x_1 < x_2 < x_3 < \cdots < x_{2n}$ be a $2n$-element chain of X. Now let i be chosen so that $1 \leqslant i \leqslant 2n$. If neither $(x_i, 0)$ or $(x_i, 1)$ belongs to Y, then Y is a subposet of $\widehat{X - x_i} \times \underline{2}$ and thus has dimension at most k. The contradiction shows that Y contains at least one of the points $(x_i, 0)$ and $(x_i, 1)$. It follows that Y contains at least half of the points from

$$\{ (x_i, 0) : 1 \leqslant i \leqslant 2n \}$$

or at least half of the points $\{ (x_i, 1) : 1 \leqslant i \leqslant 2n \}$. In either case Y contains an n-element chain.

Problem 3. — For each $m \geqslant 3$ and $n \geqslant 2$, construct an irreducible poset with dimension m and length n.

4. **Splits and rooted posets.** — R. Kimble [7] defined the split of a poset X, denoted $S(X)$, as the poset of length two with maximal elements $\{ x' : x \in X \}$ and minimal elements $\{ x'' : x \in X \}$. The partial order on $S(X)$ is given by the rule $x'' < y'$ in $S(X)$ if and only if $x \leqslant y$ in X. Kimble proved the following result.

Theorem 5. —

$$\text{Dim}\,(X) \leqslant \text{Dim}\, S(X) \leqslant 1 + \text{Dim}\,(X)\,.$$

Proof. — Suppose $L_1, L_2, \ldots, L_t$ generate the partial order on $S(X)$. For each $i \leqslant t$, let

$$S_i = \{ (x, y) \in X : (x', y'') \in L_i \}\,.$$

It follows that there exists a linear order M_i on X so that $P \cup S_i \subset M_i$ where P denotes the partial order on X. Also it is easy to see that

$$P = M_1 \cap M_2 \cap \ldots \cap M_t$$

and thus Dim $(X) \leqslant$ Dim $S(X)$.

Now let $f : X \to R$ be an embedding of the chain $(X, \hat{M}_1)$ in R. Let x_0 denote the maximum element of $(X, \hat{M}_1)$ and let $f(x_0) = r_0$. Then let x_1 denote the minimum element of $(X, \hat{M}_1)$ and let $f(x_1) = r_1$. Now consider the function $g : S(X) \to X \times R$ defined by

$$g(x'') = (x, f(x)) \text{ and } g(x') = (x, f(x) + 1 + r_0 - r_1)\,.$$

It is easy to see that g is an embedding and therefore Dim $S(X) \leqslant$ Dim $(X \times R) \leqslant 1 +$ Dim (X).

Problem 4. — Determine conditions on X which insure that Dim $S(X) = 1 +$ Dim (X).

The question arose immediately as to whether the repeated splitting of a poset could increase the dimension without bound. Let us denote the n-th split of a poset by $S^n(X)$, i.e.

$$S^0(X) = X\,,\quad S^1(X) = S(X)\,,$$
$$S^2(X) = S(S(X))\,,\quad \text{etc.}$$

The question then becomes whether there exists a constant k so that Dim $S^n(X) \leqslant k +$ Dim (X) for every X and every $n \geqslant 0$.

If a one point poset is split four times, a 3-dimensional poset is obtained. Note that the Hasse diagram of $S^n(X)$ is always a tree in the graph theoretic sense when X is a one point poset. It was this observation and its application in the proof of theorem 7 that led Trotter and Moore [15] to investigate the relationship between dimension and planarity. One of the results of this investigation was the following theorem (see [16] for additional results).

Theorem 6. — *If the Hasse diagram of a poset X is a tree in the graph theoretic sense, then* $\mathrm{Dim}(X) \leqslant 3$.

We call a poset a *rooted poset* when a single point from the poset has been designated as a root. Now consider the following general situation. Suppose we have a poset X and for each $x \in X$, suppose we also have a rooted poset Y_x with root r_x. We form a poset Z called the amalgamation of $\{ Y_x : x \in X \}$ by identifying the points x and r_x for each $x \in X$. For example, consider an arbitrary poset X. For each $x \in X$, let Y_x be a 3-element chain $x'' < r_x < x'$. Then the amalgamation of $\{ Y_x : x \in X \}$ has $S(X)$ as a subposet.

Theorem 7. — *Let Z be the amalgamation of $\{ Y_x : x \in X \}$. If*

$$t_1 = \mathrm{Dim}(X), \quad t_2 = \max \{ \mathrm{Dim}(Y_x) : x \in X \},$$

and $t = \max \{ t_1, t_2 \}$, *then* $t \leqslant \mathrm{Dim}(Z) \leqslant t + 2$.

Proof. — The inequality $t \leqslant \mathrm{Dim}(Z)$ is trivial since X and each Y_x are subposets of Z. On the other hand, let $L_1, L_2, \ldots, L_t$ be linear orders which generate the partial order on X and for each $x \in X$, let $M_1^x, M_2^x, \ldots, M_t^x$ be linear orders which generate the partial order on Y_x. For each $i \leqslant t$, define a linear order K_i on Z by the following rules. If $w_1 \in Y_{x_1}$ and $w_2 \in Y_{x_2}$ with $x_1 \neq x_2$, then $w_1 < w_2$ in K_i if and only if $x_1 < x_2$ in L_i. If $w_1, w_2 \in Y_x$ with $w_1 \neq w_2$, then $w_1 < w_2$ in K_i if and only if $w_1 < w_2$ in M_i^x.

The remainder of our proof relies heavily of the concept of TM-cycles as introduced in [15]. We refer the reader to [17] and [18] for similar applications of this concept.

Let $S_1 = \{ (w_1, w_2) : w_1 \in Y_x, w_2 \in Y_y, w_1 \, I \, w_2$ in $Z, x > y$ in $X, w_2 \leqslant y$ in $Y_y \}$ and $S_2 = \{ (w_1, w_2) : w_1 \in Y_x, w_2 \in Y_y, w_1 \, I \, w_2$ in $Z, x > y$ in $X, w_2 \nleqslant y$ in $Y_y \}$. We next show that neither S_1 nor S_2 contains any TM-cycles. Suppose first that S_1 contains a TM-cycle $\{ (a_i, b_i) : 1 \leqslant i \leqslant m \}$. We choose for each $i \leqslant m$ points $x_i, y_i \in X$ with $a_i \in Y_{x_i}, b_i \in Y_{y_i}, x_i > y_i$ in X, and $b_i \leqslant y_i$ in Y_{y_i}. Now choose an arbitrary integer $i \leqslant m$. Since $a_i \, I \, b_i$ in Z, we conclude that $a_i \ngeqslant x_i$ in Y_{x_i} and $w \leqslant a_i$ in Z implies that

$$w \in Y_{x_i} - \{ x_i \}$$

and that $w \leqslant a_i$ in Y_{x_i}. Hence $b_{i-1} \in Y_{x_i}$ and thus $x_i = y_{i-1}$. However, it is clear that this cannot hold for each i since it would imply cyclically $y_{i-1} > y_i$ for $i = 2, 3, \ldots, m$ as well as $y_m > y_1$.

Now suppose that S_2 contains a TM-cycle

$$\{ (a_i, b_i) : 1 \leqslant i \leqslant m \}.$$

Choose the points x_i and y_i as in the preceding paragraph. Then $b_i \neq y_i$ and $b_i \leqslant a_{i+1}$ in Z imply that $a_{i+1} \in Y_{y_i}$ i.e. $x_{i+1} = y_i$ and again a contradiction has been obtained.

Therefore there exist linear extensions K_{t+1} and K_{t+2} of Z so that $S_1 \subset K_{t+1}$ and $S_2 \subset K_{t+2}$. It is straightforward to verify that $K_1 \cap K_2 \cap \ldots \cap K_{t+2}$ is the partial order on Z. Thus $\mathrm{Dim}(Z) \leqslant t + 2$ and our proof is complete.

We note that if $X \subset Y$, then $S(X) \subset S(Y)$. Now consider an arbitrary poset X to which we repeatedly apply the amalgamation process described earlier where we replace each point by a 3-element chain rooted at the middle point. The net effect of a finite number of such operations is the amalgamation of a family $\{ Y_x : x \in X \}$ of rooted trees. We therefore have the following theorem.

Theorem 8. — *If X is a poset of dimension three or more, then* $\mathrm{Dim}\ S^n(X) \leqslant 2 + \mathrm{Dim}(X)$ *for all* $n \geqslant 2$.

We note that theorem 8 also holds for all posets although for reasons of brevity we do not include the details here.

5. **Splits and cartesian products.** — One of the best known inequalities in dimension theory is

Theorem 9. —

$$\max \{ \mathrm{Dim}(X), \mathrm{Dim}(Y) \} \leqslant$$
$$\leqslant \mathrm{Dim}(X \times Y) \leqslant \mathrm{Dim}(X) + \mathrm{Dim}(Y).$$

The problem is to determine just how accurate the lower bound on $\mathrm{Dim}(X \times Y)$ really is. In particular we may ask the following question.

Problem 5. — *For each $n \geqslant 1$, does there exist an n-dimensional poset X for which $X \times X$ is also n-dimensional?*

If in general we seek posets for which $\mathrm{Dim}(X \times X)$ is substantially less than $2\ \mathrm{Dim}(X)$, we may confine our attention to posets of length 2 for if X is such a poset but the length of X is large, then we may consider the split of X instead. The author and J. I. Moore did prove as a starting point that for the standard example of an n-dimensional poset, the crown S_n^0 (see [11]), we have

$$\mathrm{Dim}(S_n^0 \times S_n^0) = 2n - 2.$$

References

[1] K. Bogart, Maximal Dimensional Partially Ordered Sets I, *Discrete Math.* **5** (1973) 21-32.

[2] K. Bogart and W. T. Trotter, Maximal Dimensional Partially Ordered Sets II, *Discrete Math.* **5** (1973) 33-45.

[3] B. Dushnik and E. Miller, Partially Ordered Sets, *Amer. J. Math.* **63** (1941) 600-610.

[4] T. Hiraguchi, On the Dimension of Orders, *Sci. Rep. Kanazawa Univ.* **4** (1955) 1-20.

[5] D. Kelly and I. Rival, Certain Partially Ordered Sets of Dimension Three, *J. Comb. Theory A* **18** (1975) 239-242.

[6] R. Kimble, Extremal Problems for Partially Ordered Sets, Ph. D. Thesis, *M.I.T.* (1973).

[7] R. Kimble, Personal Communication.

[8] O. Ore, Theory of Graphs, *Amer. Math. Soc. Colloq. Publ.*, Vol. 38 (1962).

[9] I. Rabinovitch, The Dimension Theory of Interval Orders and Semiorders, Ph. D. Thesis, Dartmouth College (1973).

[10] E. Szpilrajn, Sur l'extension der order partiel, *Fund. Math.* **16** (1930) 386-389.

[11] W. T. Trotter, Dimension of the Crown S_n^k, *Discrete Math.* **8** (1974) 85-103.

[12] W. T. Trotter, Inequalities in Dimension Theory for Posets, *Proceedings Amer. Math. Soc.* **47** (1975) 311-316.

[13] W. T. Trotter, Irreducible Posets with Large Height Exist, *J. Comb. Theory A* (1974) 337-344.

[14] W. T. Trotter, A Forbidden Subposet Characterization of an Order Dimension Inequality, *Math. Systems Theory* **10** (1976) 91-96.

[15] W. T. Trotter and J. I. Moore, The Dimension of Planar Posets, *J. Comb. Theory B* **22** (1977) 54-67.

[16] W. T. Trotter and J. I. Moore, Characterization Problems for Graphs, Partially Ordered Sets, Lattices, and Families of Sets, *Discrete Math.* **16** (1976) 361-381.

[17] W. T. Trotter and J. I. Moore, Some Theorems on Graphs and Posets, *Discrete Math.* **15** (1976) 79-84.

[18] W. T. Trotter and K. P. Bogart, On the Complexity of Posets, *Discrete Math.* **16** (1976) 71-82.