Discrete Mathematics 31 (1980) 297–313

LARGE MINIMAL REALIZERS OF A PARTIAL ORDER II

Stephen B. MAURER*

Department of Mathematics, Swarthmore College, Swarthmore, PA 19081, U.S.A.

I. RABINOVITCH

Department of Mathematics, Concordia University, Montreal, Que. Canada H3G 1M8

William T. TROTTER Jr.

Department of Mathematics, Computer Science and Statistics, University of South Carolina, Columbia, SC 29208, U.S.A.

Received 22 August 1979
Revised 14 February 1980

The rank (resp. dimension) of a poset P is the cardinality of a largest (resp. smallest) set of linear orders such that its intersection is P and no proper subset has intersection P. Dimension has been studied extensively. Rank was introduced recently by Maurer and Rabinovitch in [4], where the rank of antichains was determined. In this paper we develop a general theory of rank. The main result, loosely stated, is that to each poset P there corresponds a class of graphs with easily described properties, and that the rank of P is the maximum number of edges in a graph in this class.

1. Introduction

A *partially ordered set* (X, P) consists of a set of elements X, always finite in this paper, and a binary, transitive, irreflexive relation P on X. For short, we speak of the *poset* P, or just P. When $(a, b) \in P$, we think of a as over b. A *realizer* $\mathbf{L}$ of P is a set $\{L_1, L_2, \ldots, L_n\}$ of linear orders on X whose intersection is P, i.e., $P = \bigcap \mathbf{L} = \{(a, b) \mid (a, b) \in L \text{ for every } L \in \mathbf{L}\}$. An early result of Szpilrajn [9] implies that every poset has a realizer. A realizer $\mathbf{L}$ *of* P is *minimal* if no proper subset of $\mathbf{L}$ is also a realizer. The *dimension* of P, $d(P)$, is the cardinality of a minimum (smallest) a realizer. Dimension has been studied extensively and is generally hard to compute; see [10] for many references. It is relatively easy to find a minimal realizer for a given P, but not all minimal realizers are minimum. How big can a minimal realizer be? This question prompted the first two authors to define $r(P)$, the *rank* of P, to be the cardinality of a maximum (largest) minimal realizer in [4]. There the rank of the antichain of n elements was determined, and the ranks of a few other posets were announced.

*Work supported by the United States National Science Foundation Grant MCS 76-19670, while the author was in the Department of Mathematics at Princeton University.

In this paper we develop a general theory of rank. This theory allows us to determine rank for many more posets. Most of the concepts of the theory are refinements of ideas in [4]. The key refined concept is that of a "critical digraph" on X as vertex set. The key additional idea is that of a "nonforcing" edge. Each critical digraph D of P is constructed from a realizer $\boldsymbol{L}$, using nonforcing edges, and the number of edges of D is always $|\boldsymbol{L}|$. Moreover, D necessarily satisfies certain easily stated graph-theoretical properties which depend on P but not $\boldsymbol{L}$. In important cases, these properties are also sufficient for D to be a critical digraph. Specifically, the Main Theorem is: if D has the largest number of edges of a graph meeting these properties, then D is a critical digraph and the number of edges is the rank of P. The significance of this result is that one can compute $r(P)$ by examining digraphs; the explicit construction of realizers is entirely bypassed.

The purpose of this paper is to prove this Main Theorem and to give a few brief but representative examples of its applications. Other applications and further results in the theory of rank appear elsewhere [6, 7]. Additional material is available in the preliminary manuscript [5], which also gives an earlier version of the material included here.

The proof of the Main Theorem is quite lengthy. Therefore, we organize the material as follows; First, in Section 2, we state the Main Theorem. Actually, we state three different forms. Sections 3–5 provide the proof. Section 3 contains the basic results on the central ideas of forcing and criticality. Section 4 is the proof of a key result we call the Preparation Lemma. This result is stated at the beginning of the section for easy reference. It is then straightforward, in Section 5, to prove the various forms of the Main Theorem. In Section 6 we give the promised applications, with indication of the further results which appear elsewhere. Specifically, we compute the rank of several classes of posets; we subsume the results of Rabinovitch and Rival on rank of distributive lattices [8] under our general theory; we give a counter-example to a conjecture of Golumbic that rank depends only on the comparability graph; and we characterize those posets on n elements which have the largest rank. It is possible to read the applications in Section 6 immediately after Section 2.

Further notation and terminology. Let $B(X)=B$, for binary, denote $\{(a,b)\,|\,a,b\in X, a\neq b\}$. Ordered pairs (a,b) are always in B. We will find it convenient to interpret B as a digraph. Thus $a\in X$ is an element or a vertex interchangeably, and (a,b) is a pair or a directed edge from a to b. A sequence $(a_1,a_2),(a_2,a_3),\ldots,(a_{n-1},a_n)$, where the a_i are distinct and $n\geq 2$, will therefore be called a (directed) *path*. If the a_i are distinct except that $a_n=a_1$, we have a (directed) *cycle*. If the a_i are not known to be distinct, we have a (directed) *walk*. We may abbreviate such sequences as $a_1\cdots a_n$, and we will omit the word "directed", but *in all cases it is understood that all edges are directed uniformly from* a_1 *to* a_n. By a *largest* (di)graph with some property, we always mean one with the most edges. Similarly, for a digraph D, $|D|$ is the number of edges.

For any $S\subset B$, $\mathrm{Tr}(S)$ denotes the transitive closure of S. Let $I(P)=I$ be the set of *incomparable* pairs of P, i.e., all $(a, b) \in B$ with both $(a, b), (b, a) \notin P$. For $(a, b) \in I$, $\mathrm{Tr}(P \cup (a, b))$ is always a poset.

If P, Q are posets and $P \subset Q$, we say Q *extends* P. Unless otherwise indicated, P and Q are defined on X. The poset induced on $A \subset X$ by P is denoted P_A, and is called a *subposet* of P.

Let $\underline{n}$ denote a chain (linear order) on n elements. Let $\bar{n}$ denote an antichain (P empty). For any two posets $(X, P), (Y, Q)$ with $X \cap Y = \emptyset$, the *join* $P \oplus Q$ is the poset on $X \cup Y$ with pairs $P \cup Q \cup (X \times Y)$.

Any set $\boldsymbol{L}$ of linear orders is a realizer of $\bigcap \boldsymbol{L}$. When $\boldsymbol{L}$ is a minimal realizer of $\bigcap \boldsymbol{L}$, we say $\boldsymbol{L}$ is *irredundant.* The distinction between minimal and irredundant is that a minimal $\boldsymbol{L}$ realizes a predetermined poset, usually P, whereas an irredundant $\boldsymbol{L}$ need not. Even so, "minimal realizer of P" and "irredundant realizer of P" mean the same thing, and are used interchangeably in our papers.

Throughout this paper, a path will always be R rather than P to avoid confusion with P for poset. Families of orders will always be boldface $\boldsymbol{L}$.

2. Main theorems

Definition. Given distinct pairs $(a, b), (c, d) \in I(P)$, (a, b) *forces* (c, d) if $(c, d) \in \mathrm{Tr}(P \cup (a, b))$. Call (a, b) *unforced* if no pair forces it. Call (a, b) *nonforcing* if it forces no pair. Let $N = N(P)$ be the set of nonforcing pairs of $I(P)$.

Thus (a, b) forces (c, d) iff either

(1) $(c, a), (b, d) \in P$;

(2) $c = a$ and $(b, d) \in P$; or

(3) $b = d$ and $(a, c) \in P$.

We sometimes need to use this longer equivalent in proofs. Also, note that $(a, b) \in I$ is nonforcing iff $\mathrm{Tr}(P \cup (a, b)) = P \cup (a, b)$.

It is easy to verify that forcing is a poset with ground set I, and that (a, b) forces (c, d) iff (d, c) forces (b, a). Hence (a, b) is uniforced iff (b, a) is nonforcing.

The forcing relation and unforced pairs were introduced by Rabinovitch and Rival [8]. In light of the duality between unforced and nonforcing pairs, it might seem appropriate to follow their lead and use the former only. However, we will use the latter only. As we show in Section 3, it is $N(P)$ which interacts nicely with P. The unforced pairs do not. Of course, the unforced pairs do interact nicely with $\hat{P}$, the dual of P, but it is not natural to think in terms of $\hat{P}$.

We now introduce the key graph-theoretical property, alluded to earlier, such that graphs with this property are intimately related to realizers of P.

Definition. A digraph $D \subset B$ is *unipathic relative to* P, abbreviated $\mathrm{U_P}$, if whenever there are two edge-disjoint paths from a to b in D, then $(a, b) \in P$.

For instance, if $P=\emptyset$, to be U_P is simply to be unipathic.

Note: we do not consider single vertices to be null paths; in particular, a U_p graph may have cycles.

Main Theorem (Form 1). *Let P be a poset which is not a subposet of any* $\underline{n}\oplus\bar{3}\oplus\underline{m}$. *Let N be the set of nonforcing edges for P. Then the rank of P is the largest number of edges in any acyclic* U_P *subgraph of N.*

We call subposets of $\underline{n}\oplus\bar{3}\oplus\underline{m}$ *rank degenerate.*

Main Theorem (Form 2). *Let P be any nonlinear poset. Then r(P) is the largest number of edges in any subgraph of N which is either a cycle or is acyclic* U_P.

Form 2 is appealing, since there are no exceptional posets, except linear orders, for which $N=\emptyset$ but $r=1$. However, Form 2 requires looking at a larger class of graphs to cover what is after all a very small and well-behaved set of exceptions. Form 1 also has the advantage that it leads to Form 3. Namely, although N can have many cycles, when P is not rank degenerate one can restrict attention to a subset of N so as to destroy all cycles and still compute the rank.

Definition. Let L be any linear order on X. L need not extend P. As usual, let $N=N(P)$. Define $N^*=N^*(P)=\{(a,b)\in N \mid \text{either } (b,a)\notin N \text{ or } (a,b)\in L\}$.

Later we will show easily that N^* is acyclic and independent of L (up to graph isomorphism).

Main Theorem (Form 3). *Let P be any poset which is not rank degenerate. Then r(P) is the largest number of edges in any* U_P *subgraph of* N^*.

Although there will be little use of Form 3 in this paper, Form 3 is nonetheless the motivating force behind the development of still further structure and further applications [6].

3. Criticality and nonforcing

In this section we first give the basic definitions and results concerning critical edges and critical digraphs. Then we give the main result about the interaction of the poset P and the nonforcing edges N. The principal result of this section is that every critical digraph of P is either acyclic U_P or a cycle.

Definition. Edge (a,b) is *critical* for $L\in\mathbf{L}$ if $(a,b)\notin L$ but for every $L'\in\mathbf{L}-L$, $(a,b)\in L'$. That is, L alone excludes (a,b) from $\bigcap\mathbf{L}$.

Lemma 3.1. *Let $\mathbf{L}$ be a minimal realizer of (X, P), where $|X|\geq 2$. Then for each $L\in\mathbf{L}$, there is at least one edge which is critical for L. Moreover, distinct linear orders in $\mathbf{L}$ have disjoint sets of critical edges.*

The following is the dual of a statement in Rabinovitch and Rival [8].

Lemma 3.2. *Let $\mathbf{L}$ be a minimal realizer of P. Assume $|\mathbf{L}|\geq 2$, i.e., P is nonlinear. Then every $L\in\mathbf{L}$ has at least one critical edge in N.*

Proof. By Lemma 3.1, any $L\in\mathbf{L}$ has a critical edge (a, b). Since $|\mathbf{L}|\geq 2$, $(a, b)\in I$. If $(a, b)\in I-N$, choose any $(c, d)\in \mathrm{Tr}(P\cup(a, b))\cap N$. For each $L'\in\mathbf{L}-L$, we have $(c, d)\in L'$. But $(c, d)\notin P=\bigcap\mathbf{L}$. Hence $(c, d)\notin L$. Thus (c, d) is critical for L.

Corollary 3.3. *If P is nonlinear*

$$2\leq r(P)\leq|N|. \tag{1}$$

Definition. Let $\mathbf{L}$ be a minimal realizer for a nonlinear P. A *critical digraph* of $\mathbf{L}$, abbreviated CD, is a subgraph of N with one edge for each $L\in\mathbf{L}$ and such that each edge is critical for its corresponding L. Digraph D is a CD of P if it is a CD of some minimal realizer of P.

Note: for an edge to be in a CD it must be in N. This is a refinement of the definition of CD in [4]. However, for the edge simply to be critical for some $\mathbf{L}$ it need *not* be in N. The reason for allowing this is that we will often take a subset $N'\subset N(P)$ and create an irredundant set $\mathbf{L}$ for which each edge of N' is critical as defined. We will want to refer to the edge in N' which is critical for a given $L\in\mathbf{L}$ and vice versa. Yet at that stage we will not know, and may never know, if $\bigcap\mathbf{L}=P$. If not, then it may be that $N'\not\subset N(\bigcap\mathbf{L})$ so that N' is not a CD for $\mathbf{L}$.

The key point about CDs is that the cardinality of $\mathbf{L}$ is the same as the number of edges in any of its CDs. Thus, whenever P is not linear, $r(P)$ is the cardinality of any largest CD of P; $d(P)$ is the cardinality of any smallest CD. The ideal situation would be to characterize CD subgraphs of N graph-theoretically. Although we have not done this, we come close enough to handle rank. We give necessary conditions to be a CD which, for maximal graphs with these properties, are also sufficient.

Theorem 3.4. *Suppose D is a* CD *for a minimal realizer $\mathbf{L}$ of P. Then D is either acyclic* U_P *or a cycle.*

Proof. For $(a, b)\in D$, let $L(a, b)$ be the linear order in $\mathbf{L}$ not containing (a, b). Since $D-(a, b)\subset L(a, b)$ for each (a, b), D can contain a cycle C only if $D=C$. Now suppose R_1, R_2 are edge-disjoint paths from a to b in D. For any $(c, d)\in R_1$,

$$(a, b)\in \mathrm{Tr}(P\cup R_2)\subset L(c, d).$$

For any $(c', d') \in D - R_1$,

$$(a, b) \in \mathrm{Tr}(P \cup R_1) \subset L(c', d').$$

Thus $(a, b) \in \bigcap \boldsymbol{L} = P$. So D is U_P.

When P is an antichain, it is easy to find cyclic CDs, acyclic U_P CDs, and an $\boldsymbol{L}$ with many different CDs. Also, the converse of Theorem 3.4 is false. If D consists of k vertex-disjoint edges, D is acyclic U_P whatever P is. But if $\dim(P) > k$, D cannot be a CD. Posets with high dimension and many vertex disjoint edges in N are easy to construct.

The proof of Theorem 3.4 does not use the fact that $D \subset N$. Also, by essentially the same argument, it is easy to give necessary conditions on a CD D which seem much stronger. Specifically,

(i) if $P \cup D$ contains a cycle, all edges of D are on it.

(U'_P) if $P \cup D$ contains two paths from a to b with no D edges in common, then $(a, b) \in P$.

Indeed, we will use (i) and (U'_P) implicitly in constructing realizers from subsets of N in Section 4 (see Lemma 4.2). Fortunately, for $D \subset N$, these seemingly stronger conditions are *not* stronger. This is implicit in Lemmas 3.6 and 3.7 below.

Lemma 3.5. *Suppose $R = a_1 a_2 \cdots a_n$ is a walk in $P \cup N$. Suppose at least one edge is in P. Then $(a_1, a_n) \in P$.*

Proof. We induct on the number k of edges of N in R. For $k = 0$, the result holds since P is transitive. For $k = 1$, recall that $(a, b) \in N$ means $\mathrm{Tr}(P \cup (a, b)) = P \cup (a, b)$. Since $P \cap R \neq \emptyset$ and P contains no cycles, $(a_1, a_n) \in P$. Finally, assume the lemma for $k < m$. If R has m edges from N, pick a subwalk $a_i a_{i+1} \cdots a_j$ which has at least one edge from P and exactly one from N. Then $(a_i a_j) \in P$. Thus the walk $a_1 a_2 \cdots a_i a_j \cdots a_n$ satisfies the hypotheses for $k = m - 1$.

Lemma 3.6. *For any $D \subset N$, if C is a cycle in $P \cup D$, then in fact $C \subset D$.*

Proof. If not, by the previous lemma P would contain pairs of the form (a, a).

Lemma 3.7. *For any $D \subset N$, the condition U'_P (above Lemma 3.5) is equivalent to U_P.*

Proof. Clearly U'_P implies U_P. As for the converse, suppose R_1, R_2 are paths of $P \cup D$ satisfying the hypotheses of U'_P. If either path contains an edge of P, $(a, b) \in P$ by Lemma 3.5 and we are done. So suppose $R_1, R_2 \subset D$. The hypothesis of U'_P then says that R_1, R_2 are edge-disjoint. Thus by U_P, $(a, b) \in P$.

Note: if we weakened U_P by replacing the conclusion $(a, b) \in P$ with $(a, b) \notin N$, then for subsets D of N the weakened U_P would still be equivalent to the original U_P and hence to U'_P. The proof appears as part of the first paragraph of the proof of the next lemma. Since acyclicity and U_P are essentially the only properties we will look for, and since we will only look for them in subsets of N, we could switch to the weakened form of U_P. It has the advantage that, once N is found, we could completely ignore P. However, we choose to continue with the original definition of U_P.

Lemma 3.8. $\mathrm{Tr}(P \cup N) = P \cup N$, *except that the left-hand side will also include those pairs* (a, a) *such that* N *contains cycles through* a. *Moreover, if there is a path* $a_1 a_2 \cdots a_n$ *in* N, *and* $(a_1, a_n) \notin P$, *then for all* i, j *with* $1 \leq i < j \leq n$, *we have* $(a_i, a_j) \in N$.

Proof. By Lemma 3.5, $\mathrm{Tr}(P \cup N) = P \cup \mathrm{Tr}(N)$. So to prove the first sentence of the lemma, consider any path R from a to b in N. If $(a, b) \notin P$, then either $(b, a) \in P$, or $(a, b) \in I - N$, or we are done. In the first case, the cycle $R \cup (b, a)$ violates Lemma 3.6. In the second case, $\mathrm{Tr}(P \cup (a, b))$ includes some $(c, d) \in N$. Consequently, there is a walk from c to d in $P \cup R$ using at least one edge from P. By Lemma 3.5, $(c, d) \in P$, a contradiction.

As for the second sentence of the lemma, we have just shown that all $(a_i, a_j) \in P \cup N$. If some (a_i, a_j) were in P, then applying Lemma 3.5 to the path $a_1 a_2 \cdots a_i a_j \cdots a_n$, we get $(a_1, a_n) \in P$, contradicting the hypothesis.

The subset $A \subset X$ is said to have *duplicate holdings* if for all $a, a' \in A$ and $x \in X$, $(a, x) \in P \Leftrightarrow (a', x) \in P$, and $(x, a) \in P \Leftrightarrow (x, a') \in P$. It follows that $P|_A$ is an antichain and that, for all $x \in X - A$, $(a, x) \in I \Leftrightarrow (a', x) \in I$.

Lemma 3.9. *For any* $A \subset X$ *with* $|A| \geq 2$, *the following are equivalent*:

(1) *A has duplicate holdings.*
(2) *$N|_A$ is a complete two-way digraph.*
(3) *N contains a cycle whose vertex set is A.*

In particular, the maximal two-way complete subgraphs of N are vertex-disjoint.

Proof. (1)$\Leftrightarrow$(2) is a direct application of the definitions. (2)$\Rightarrow$(3) is immediate. As for (3)$\Rightarrow$(2), let $a_1 a_2 \cdots a_n a_1$ be the cycle. No (a_i, a_j) is in P by Lemma 3.6. Thus they are all in N by Lemma 3.8. The last statement in the lemma follows because duplicate holdings provide an equivalence relation on X.

4. The preparation lemma

Theorem 4.1 (The Preparation Lemma). *Suppose $N = N(P)$ is neither empty nor a 2-cycle. Let N' be any maximal acyclic U_P subgraph of N, that is, any proper* *supergraph of N' in N either contains a cycle or violates U_P. Then N' is a critical diagraph of P.*

Given this theorem and Theorem 3.4, it should be clear that the Main Theorems are not far away. Indeed, if the Preparation Lemma also asserted that every cycle in N is a CD, or even that just some largest cycle in N is also a CD, then Form 2 of the Main Theorem would be immediate. Unfortunately, these strengthenings can be false, as we show in the next section.

The proof of the Preparation Lemma is the rest of this section. The proof involves two steps:

(1) Show that the assumptions imply the existence of some irredundant $\mathbf{L}$ containing $|N'|$ linear orders, all extending P, such that each edge of N' is critical for one $L \in \mathbf{L}$. This step is easy; in fact, the use of N instead of I is only a simplifying convenience here. Necessary and sufficient conditions for a set $N' \subset N$ (or $D \subset I$) to allow this step to work are implicit in the proof of Lemma 4.3 (and the remarks before Lemma 3.5).

(2) Show that the assumptions imply that one such $\mathbf{L}$ realizes P. This step is harder. Also, the use of N here seems to be crucial.

Definition. For $N' \subset N$ and $(a, b) \in N'$, let $N'(a, b) = P \cup N' \cup (b, a) - (a, b)$.

Lemma 4.2. *If N' is a CD of a minimal realizer $\mathbf{L}$ of P, then each $N'(a, b)$ is acyclic. Conversely, if each $N'(a, b)$ is acyclic, then there is an irredundant $\mathbf{L}$ with $|N'|$ linear orders, all extending P, such that each edge of N' is critical for one $L \in \mathbf{L}$.*

Proof. First assertion: if (a, b) is critical for $L \in \mathbf{L}$, then $N'(a, b) \subset L$ by definition of a CD. Thus $N'(a, b)$ is acyclic. Converse: any acyclic set is contained in a linear order. For now, let $L(a, b)$ be any linear order extending $N'(a, b)$. Since $P \subset N'(a, b)$, $L(a, b)$ extends P. Let $\mathbf{L} = \{L(a, b) | (a, b) \in N'\}$. $L(a, b)$ is the only order in $\mathbf{L}$ which excludes (a, b). Thus $\mathbf{L}$ is irredundant and (a, b) is critical for $L(a, b)$ in $\mathbf{L}$.

Lemma 4.3. *If $N' \subset N$ is either acyclic U_P or a cycle, then for each $(a, b) \in N'$, $N'(a, b)$* is acyclic.

Proof. Suppose C were a cycle in $N'(a, b)$. If $(b, a) \notin C$, then $C \subset P \cup N'$, so $C \subset N'$ by Lemma 3.6. Hence $C \cup (a, b) \subset N'$, so N' is neither acyclic nor a cycle. If $(b, a) \in C$, then $P \cup N'$ contains two edge-disjoint paths from a to b, namely (a, b) *and* $C - (b, a)$. Since N' is U_P, This conclusion about $P \cup N'$ violates lemma 3.7.

We now treat Step (2).

Lemma 4.4. *Suppose* $\boldsymbol{L}$ *is a set of linear extensions of* P, *and each edge of* $N' \subset N$ *is critical for some* $L \in \boldsymbol{L}$. *Then* $\bigcap \boldsymbol{L} = P$ *iff* $(\bigcap \boldsymbol{L}) \cap (N - N') = \emptyset$.

Proof. "Only if" is obvious. For "if" we need to show that $(\bigcap \boldsymbol{L}) \cap (\hat{P} \cup I) = \emptyset$, where $\hat{P}$ is the dual of P. $(\bigcap \boldsymbol{L}) \cap \hat{P} = \emptyset$ because $P \subset \bigcap \boldsymbol{L}$ and $\bigcap \boldsymbol{L}$ is a poset. Suppose $(a, b) \in (\bigcap \boldsymbol{L}) \cap (I - N)$. We know that some $(c, d) \in N$ is forced by (a, b). Hence, for each $L \in \boldsymbol{L}$, $(c, d) \in \mathrm{Tr}(P \cup (a, b)) \subset L$, that is, $(c, d) \in \bigcap \boldsymbol{L}$. Since we assume $(\bigcap \boldsymbol{L}) \cap (N - N') = \emptyset$, $(c, d) \in N'$. But (c, d) is not in the L for which (c, d) is critical, a contradiction. Thus $(\bigcap \boldsymbol{L}) \cap I = \emptyset$.

The following result is a special case of a lemma of Bogart [1]. We omit the straightforward proof.

Lemma 4.5. *Let* Q *be a poset on* X. *For* $x \in X$, *define* $Q(x) = \{(y, x) \mid (y, x) \in I(Q)\}$. *Then for any* $(b, a) \in Q$, *there exists a linear order* $L(Q, a, b)$ *which extends* $Q \cup Q(a) \cup Q(b)$.

Definition. Let $N'(a, b)$ be as before and suppose it is acyclic. Let $Q = \mathrm{Tr}(N'(a, b)) \supset P$. Then $L_0(a, b)$ shall denote any linear order $L(Q, a, b)$ as in the previous lemma.

Lemma 4.6. *Suppose* N' *is an acyclic* U_P *subgraph of* N. *Suppose* N' *contains a path* $a_1 a_2 \cdots a_n$ *with* $n > 2$. *Suppose further that* $(a_1, a_n) \notin P$. *Then* $(a_n, a_1) \in L_0(a_1, a_2)$ *and* $(a_1, a_n) \in L_0(a_{n-1}, a_n)$.

Proof. To show that $(a_n, a_1) \in L_0(a_1, a_2)$, we must show that $(a_1, a_n) \notin Q = \mathrm{Tr}(N'(a_1, a_2))$; for then (a_n, a_1) is in Q or $I(Q)$, and in either case, for different reasons, $(a_n, a_1) \in L_0(a_1, a_2)$, (In fact, the latter case holds.) So suppose there were a path R from a_1 to a_n in $N'(a_1, a_2)$. Clearly, $(a_2, a_1) \notin R$, and it is given that $(a_1, a_n) \notin P$. Hence, by Lemma 3.5, $R \subset N' - (a_1, a_2)$. Let a_k be the first vertex among $a_2, a_3, \ldots, a_n$ which R reaches after leaving a_1. Let R' be the segment of R from a_1 to a_k. Then R' and $a_1 a_2 \cdots a_k$ are disjoint paths in N' from a_1 to a_k. Since N' is U_P, $(a_1, a_k) \in P$. By Lemma 3.5, $(a_1, a_n) \in P$, a contradiction.

Similarly, to prove that $(a_1, a_n) \in L_0(a_{n-1}, a_n)$, suppose there were a path R from a_n to a_1 in $N'(a_{n-1}, a_n)$. If $(a_n, a_{n-1}) \notin R$, then $a_1 \cdots a_n$ followed by R is a cycle in $P \cup N'$, or contains one. If $(a_n, a_{n-1}) \in R$, then $a_1 \cdots a_{n-1}$ followed by $R - (a_n, a_{n-1})$ is a cycle in $P \cup N'$, or contains one (here we use $n > 2$). Either way, N' contains a cycle by Lemma 3.6, contradicting the first sentence of this lemma.

Proof of the Preparation Lemma. We show that $\{L_0(a, b) \mid (a, b) \in N'\}$ is a minimal realizer of P with N' as a CD. Given the previous lemmas, it suffices to show that for each $(c, d) \in N - N'$, there is some $(a, b) \in N'$ such that $(c, d) \notin L_0(a, b)$.

$N' \cup (c, d)$ either violates U_P or it contains a cycle. Suppose it contains cycle C. Then $(c, d) \in C$. If there exists an $(a, b) \in N' - C$, then it is easy to see that (c, d) is not contained in *any* linear extension of $N'(a, b)$, so $(c, d) \notin L_0(a, b)$. If $N' \subset C$, we argue that $|N'| \geq 2$. Otherwise N' would be a single edge, but since $|N| \geq 2$ by inequality (1), and N is not a 2-cycle by hypothesis, a single edge cannot be maximal acyclic U_P. Consequently, $N' \subset C$ implies that N' is a path $a_1 \cdots a_n$ with $n > 2$ and $(c, d) = (a_n, a_1)$. Hence $(c, d) \notin L_0(a_{n-1}, a_n)$ by Lemma 4.6.

Suppose $N' \cup (c, d)$ violates U_P. Thus $N' \cup (c, d)$ contains two edge-disjoint paths between some x, y, where $(x, y) \notin P$. One of these paths, call it R, is entirely in N'. Write $R = a_1 \cdots a_n$, where $x = a_1$, $y = a_n$. If $n = 2$, that is, if $(x, y) \in N'$, then we consider the other path to conclude that (c, d) is not in any linear extension of $N'(x, y)$. if $n > 2$, $(x, y) \notin L_0(a_1, a_2)$ by Lemma 4.6. Therefore $(c, d) \notin L_0(a_1, a_2)$ either.

5. Proof of the main theorem

Form 1. Define a poset to be *rank degenerate* if it is linear or if every largest CD is a cycle. Recall that if D is a CD of L, then $|D| = |L|$. Thus Theorem 3.4 and the Preparation Lemma tell us immediately that the conclusion of Form 1 holds for every poset which is not rank degenerate. (Note that the one nonlinear exception to the Preparation Lemma, N a 2-cycle, can only arise from a rank degenerate P.) It happens that rank degenerate posets, as defined here, are precisely the subposets of $\underline{n} \oplus \bar{3} \oplus \underline{m}$ excluded in Form 1 and already called rank degenerate there. That the subposets of $\underline{n} \oplus \bar{3} \oplus \underline{m}$ are all rank degenerate as defined here is straightforward to check directly. As for the converse, let k be the length of a largest CD cycle C of a rank degenerate poset (X, P). Let A be the vertex set of C. By Lemma 3.9, all edges between vertices of A are in N. Suppose $k \geq 4$. Partition A into A_1, A_2 with $\lceil \frac{1}{2}k \rceil$ and $\lfloor \frac{1}{2}k \rfloor$ elements, and let $N' = \{(b, c) \mid b \in A_1, c \in A_2\}$. note $|N'| = \lfloor \frac{1}{4}k^2 \rfloor \geq k$. Moreover, N' is acyclic U_P, since N' contains no consecutive edges. Thus some acyclic U_P supergraph of N' is a CD by the Preparation Lemma, contradicting rank degeneracy. Thus $k \leq 3$. Moreover, there cannot be any other edge $e \in N$ except those between elements of A. For otherwise pick any $e' \in C$ and consider $C + e - e'$. This is acyclic U_P, so again the Preparation Lemma says that some supergraph violates rank degeneracy. Thus, except for A, P is linear. That is, P is a subposet of $\underline{n} \oplus \bar{3} \oplus \underline{m}$.

It is equally easy to characterize those posets such that *some* largest CD is a cycle [5].

Form 2. It suffices to show that, for any P, no cycle in N has more edges than the largest CD. If every cycle in N were a CD, this would be immediate. However, this is false. Suppose A is the largest set with duplicate holdings in (X, P) and suppose $|A| < d(P)$. Then by Lemma 3.9, every set of 2 or more points in A forms a cycle, but no cycle in N is a CD. Moreover, it is easy to construct P with $|A|$ large but less than $d(P)$.

Fortunately, we need only that every cycle in N as large as some CD is itself a CD.

Lemma 5.1. *Let $C = a_1 \cdots a_n$ be any cycle in N with $n \geq d(P)$. Then C is a* CD *of P.*

Proof. Let $A = \{a_1, \ldots, a_n\}$. Let $X' = (X - A) \cup \{a_1\}$. For any P and any $X' \subset X$, it is an easy standard result that $d(P|_{X'}) \leq d(P)$. So let $m = d(P|_{X'})$. (In fact, in the case at hand, $m = d(P)$ except when $m = 1$, in which case $d(P) = 2$.) Let $\boldsymbol{L} = \{L_1, \ldots, L_m\}$ be a set of linear orders on X' which realize $P|_{X'}$. For $i = 1, 2, \ldots, m-1$ we will extend L_i to a linear order L_i' on X, and we will extend L_m to linear orders $L_m', L_{m+1}', \ldots, L_n'$ on X, so that $\boldsymbol{L}' = \{L_1', \ldots, L_n'\}$ realizes P with C as CD. For $i = 1, \ldots, n$, let L_i'' be the unique linear order on A which extends $C \cup (a_{i+1}, a_i) - (a_i, a_{i+1})$. L_i' is constructed by inserting A with order L_i'' in place of a_1 in L_i (or in L_m if i>m). It is easily seen that $\bigcap \boldsymbol{L}' = P$, and that (a_i, a_{i+1}) is critical for L_i'. Since $C \subset N$ by hypothesis, C is a CD for $\boldsymbol{L}'$.

Form 3. First we show that for any P and any acyclic U_P subgraph N' of N, there is a U_P subgraph of N^* isomorphic to N'. Then, by Form 1, for posets which are not rank degenerate, we need only find the largest acyclic U_P subgraph of N^*. Second we show that N^* is acyclic. Thus we need only find the largest U_P subgraph of N^*.

Let L be the arbitrary linear ordering of X used to define N^*. Let L' be any linear extension of $P \cup N'$; such exists by Lemma 3.8, since N' is acyclic. For each equivalence class $A \subset X$ of points with duplicate holdings under P, L and L' impose linear orders on A. Define $f: X \to X$ as the unique map such that for each such A, and $i = 1, 2, \ldots, f$ maps the ith element from the top of A under L' to the ith from the top under L. Clearly f is an automorphism of P, so the induced map on N is also an automorphism, one which maps U_P subgraphs to U_P subgraphs. Finally, note that $f(N') \subset N^*$. (In fact, if N^{**} is the "version" of N^* obtained by using L' in the definition instead of L, f induces an isomorphism of N^{**} and N^*.)

For the second claim, set $N_1^* = \{(a, b) \in N^* \mid (b, a) \in N\}$. By definition of N^*, $N_1^* \subset L$, so N_1^* is acyclic. But by Lemma 3.9, any cycle in N^* is entirely in N_1^*, so N^* is acyclic.

6. Sample applications

(1) *Computation of some ranks.* We preface our examples with some useful facts.

Corollary 6.1. *If P is nonlinear, $r(P) = |N|$ iff N is either a cycle or acyclic* U_P.

Proof. Immediate from the Main Theorem, Form 2.

It might seem an unlikely event that $r(P)=|N|$, but in fact it happens in a number of interesting posets.

For the next definition and theorem, recall that we have chosen $(a,b)\in P$ to mean that a is over b.

Definition. Let $I_m \subset I(P)$ be defined as

$$\{(a,b)\in I(P)\mid b \text{ minimal in } P,\ a \text{ maximal and not minimal in } P\}.$$

Unless P contains isolated elements, incomparable to everything else, we need merely require of a that it be maximal in P.

Theorem 6.2. *I_m is always an acyclic U_P subgraph of N. Thus $r(P)\geq |I_m|$.*

Proof. No $(a,b)\in I_m$ is on any path whose edges are in P, by the definition of maximal and minimal. So $I_m\subset N$. There are no paths with more than one edge in I_m by definition. So I_m is acyclic and U_P. Now use the main Theorem, Form 2 (if $N=\emptyset$, our present result is trivial).

Corollary 6.3. *$I_m=N$ iff*

$$\text{for all } (a,b)\in I,\quad \mathrm{Tr}(P\cup(a,b))\cap I_m\neq\emptyset. \tag{2}$$

Thus, if (2) holds, $r(P)=|I_m|$.

Proof. For any $(a,b)\in I$, $\mathrm{Tr}(P\cup(a,b))\cap N\neq\emptyset$. Thus $I_m=N$ implies (2). If $I_m\neq N$, pick $(a,b)\in N-I_m$. Then $\mathrm{Tr}(P\cup(a,b))=P\cup(a,b)$, which is disjoint from I_m, contradicting (2). If $I_m=N$, *then* $r(P)=|I_m|$ follows from the previous theorem.

There is much more that can be said about necessary conditions and sufficient conditions for $r(P)$ to equal $|I_m|$ (see [5]).

Boolean Posets. Let X be an n-set. Let S_k and S represent the collections of all k-subsets and all subsets of X. Let $S_{k,j}=S_k\cup S_j$. Denote by B_n, $B_{n,k}$, and $B_{n,k,j}$ the partial orders induced by set inclusion on S, S_k, and $S_{k,j}$ respectively.

Theorem 6.4. *Suppose $0<j<k<n$. Let S' be any subset of S such that $S_{k,j}\subset S'$ and $S_i\cap S'=\emptyset$ for all $i<j$ and all $i>k$. Let B' be the poset of set inclusion on S'. Then*

$$r(B')=\binom{n}{j}\left[\binom{n}{k}-\binom{n-j}{k-j}\right]. \tag{3}$$

In particular, if $S'=\bigcup_{i=1}^{m}S_{k(i)}$, where $0<j=k(1)<k(2)<\cdots<k(m)=k<n$, then B' satisfies (3).

Proof. Clearly S_j is the set of all minimal elements in B' and S_k is the set of all maximal elements. Since every j-set is included in $\binom{n-j}{k-j}$ k-sets, the quantity on the right of (3) is just $|I_m|$. It is enough now to verify condition (2) of Corollary 6.3. Suppose $Y, Z \in S'$ and $(Y, Z) \in I(B')$. Then there is an element $z \in Z - Y$. Let Z' be any j-subset of Z containing z. Let Y' be any k-superset of Y not containing z. Since $0<j$ and $k<n$, Y' and Z' exist. Moreover there are in S'. Thus $(Y', Z') \in \mathrm{Tr}(B' \cup (Y, Z)) \cap I_m$.

When $S' = S_{k,j}$, this result was announced in [4]. For every B', $d(B') \leq n$ since $d(B_n) = n$ (see [3]). Thus (3) once again indicates how much greater rank can be than dimension. However, when $j = 1$ and $k = n-1$, (3) gives $r(B_{n,n-1,1}) = n$. It is also well-known that $d(B_{n,n-1,1}) = n$ (see [3]). Indeed, we have now characterized all posets for which rank equals dimension [7]. They are essentially the subposets of B_n containing $B_{n,n-1,1}$, except that elements may be stretched into chains.

Rank of joins. Recall the definition of the join from Section 1.

Theorem 6.5. *Suppose $P = P_1 \oplus \cdots \oplus P_t$, where P is not rank degenerate and k of the P_i are rank degenerate. Then*

$$r(P) = \sum_{i=1}^{t} r(P_i) - k.$$

Proof. Clearly, $N(P) = \bigcup N(P_i)$ and the union is disjoint. Since P is not rank degenerate, we seek the largest acyclic U_P subgraph of $N(P)$. Clearly, $N' \subset N(P)$ is acyclic U_P iff each piece $N' \cap N(P_i)$ is. Thus if P_i is not rank degenerate, we may pick $r(P_i)$ edges from $N(P_i)$ for N'. If P_i is rank degenerate, it is easily seen that we may pick only a path of $r(P_i) - 1$ edges from $N(P_i)$ for N'.

This result contrasts with the fact $d(P) = \max\{d(P_i)\}$.

A *weak order*, a concept found in the literature [2], may be defined as any poset of the form $\bar{m}_1 \oplus \cdots \oplus \bar{m}_t$. In [4] it was proved that $r(\bar{m}) = \lfloor \frac{1}{4} m^2 \rfloor$ if $m \geq 4$, and $r(\bar{m}) = m$ if $m = 1, 2, 3$. The cases $m = 1, 2, 3$ are also the cases where $\bar{m}$ is rank degenerate. Thus Theorem 6.5 immediately gives.

Corollary 6.6. *If the weak order $\bar{m}_1 \oplus \cdots \oplus \bar{m}_t$ is not rank degenerate, then its rank is $\Sigma_i f(m_i)$, where $f(m) = \lfloor \frac{1}{4} m^2 \rfloor$ if $m \geq 4$ and $f(m) = m - 1$ if $m = 1, 2, 3$.*

The join is a special case of poset composition. For partial results on composition and rank, see [5].

Rank of distributive lattices. In [8], Rabinovitch and Rival derive a several part formula for the rank of an arbitrary distributive lattice D. They use (indeed introduce) the idea of forcing, but then their argument involves decomposing the lattice into linearly nondecomposable parts, and a lattice characterization of when linear extensions of D may contain exactly one unforced edge. By using the general results of our theory, we can shorten both the derivation and the statement of their result.

For $x \in D$, let $f(x) = \bigvee \{y \mid x \not\leq y\}$. It is well-known that, for a distributive lattice, if f is restricted to the set $J(D)$ of join irreducible elements, then f is an order isomorphism between $J(D)$ and the meet irreducible elements $M(D)$. Also, for $x \in J(D)$, $f(x)$ is the unique maximal element of D not equal or above x. From this, it is not hard to show that

$$N = N(D) = \{(f(x), x) \mid x \in J(D) \text{ and } f(x) \not< x\}.$$

So far, all this is explicit or implicit in [8].

Suppose $x \neq y$ in D have the property that every other $z \in D$ is either above both or below both. If x and y are comparable, call $\{x, y\}$ *a dividing pair*. If they are incomparable, call $\{x, y\}$ a *complementary pair*. Then it is also not hard to show the following.

(1) The pairs $\{x, f(x)\}$ with $x \in J(D)$ and $f(x) < x$ are precisely the dividing pairs of D.

(2) Every cycle in N is a 2-cycle, and the pairs of vertices in such 2-cycles are precisely the complementary pairs.

(3) Since each vertex x has at most one edge of N exiting from it, N is U_D.

Thus, if D is rank degenerate, D is a subposet of some $\underline{m} \oplus \bar{2} \oplus \underline{n}$, with rank 1 if D is linear, 2 otherwise. If D is not rank degenerate, then the largest acyclic U_D subgraph of N is obtained by removing one edge from each cycle, i.e., N^* is U_D. Summarizing, we have

Theorem 6.7. *Let D be a finite distributive lattice which is not rank degenerate. Let m, n, p respectively be the number of join irreducible elements, dividing pairs, and complementary pairs. Then $r(D) = m - n - p$.*

This is equivalent to the main Theorem 2 in [8].

Rank and Turan type problems. Consider the poset $(X(n, m), P(n, m))$ with $X(n, m) = \{1, 2, \ldots, n\}$ and $(i, j) \in P(n, m)$ iff $i \geq j + m$. We were led to this class through the subclass $P(2n, n)$, which arose naturally as the class of height one posets which are as far as possible from satisfying a certain condition (omitted here) for $r(P)$ to equal $|I_m|$. Not surprisingly, $r(P(n, m))$ is fairly difficult to compute, even with the machinery developed here. What *is* surprising is that both the problem and the form of the solution can be viewed as a generalization of Turan's theorem on largest k-clique free subgraphs of K_n (see [12]). The evaluation of $r(P(n, m))$ will appear elsewhere [6], but we now briefly suggest the connection to the Turan problem.

Consider $P = P(n, n) = \emptyset$. if P is not rank degenerate $(n \geq 4)$, we may pass immediately to N^*, which is a linearly directed K_n. A small part of the U_P condition for $N' \subset N^*$ is that N' does not contain triangles. Nonetheless, the largest graphs N' satisfying U_P and the largest with no triangles are the same: complete bipartite subgraphs with $\lfloor\frac{1}{2}n\rfloor$ and $\lceil\frac{1}{2}n\rceil$ vertices on the two sides. (For the U_P problem, where direction counts, we must choose the bipartition so all arrows point to the same side.) This observation for $P=\emptyset$ goes back to [4] and will be used again later.

It is possible to use the material developed in Sections 2–5 to formulate the problem of computing the rank of $P(n, m)$ as an extremal problem which is clearly a generalization of the Turan problem. In [6], the authors solve this extremal problem and characterize the extremal graphs. As is the case with Turan's problem, the extremal graphs are complete multipartite graphs. However, in our problem, the part sizes need not be uniform.

(2) *Disproof of a conjecture on rank and the comparability graph.* The *comparability graph* of (X, P) is the set of undirected edges ab such that either (a, b) or (b, a) is in P. Different posets on X can have the same comparability graph, but surprisingly, they all have the same dimension [11]. Indeed, all have the same number of linear extensions.

M. Golumbic conjectured to us orally that all posets with the same comparability graph also have the same rank. However, this is false. Let P_n be obtained by putting n incomparable elements under a single additional element. For convenience below, take n odd ≥ 3. For any posets (X, P), (Y, Q) with $X\cap Y=\emptyset$, let $P\cup Q$ denote the poset $(X\cup Y, P\cup Q)$. Let $\hat{P}$ be the dual of P. Consider $P_n\cup P_n$ and $P_n\cup\hat{P}_n$. Clearly they have the same comparability graph and the same dimension, 2. Since neither is rank degenerate, we need only consider $N^*(P_n\cup P_n)$, which is the vertex-disjoint union of two complete linearly directed graphs on $n+1$ vertices each, and $N^*(P_n\cup\hat{P}_n)$, which is the vertex-disjoint union of an edge and a linearly directed K_{2n}. Using the comments about Turan's theorem in the previous application, one gets that $r(P_n\cup P_n)=\frac{1}{2}(n+1)^2$ and $r(P_n\cup\hat{P}_n)=n^2+1$. What *is* true, by the Main Theorems and the remarks after Lemma 3.7, is that $r(P)$ depends only on N.

(3) *Posets of Maximum rank.* Let $n=|X|\geq 2$. Define $M(n)=\max\{\lfloor\frac{1}{4}n^2\rfloor, n\}$. Implicit in [4] is the fact that for any poset (X, P), $r(P)\leq M(n)$. The proof is that n is a bound on the number of edges in a cycle on X, and $\lfloor\frac{1}{4}n^2\rfloor$ is a bound on the number of edges in an acyclic U_P subgraph on X by Turan's theorem. Note that $M(n)=n$ for $n\leq 4$, and $M(n)=\lfloor\frac{1}{4}n^2\rfloor$ *for* $n\geq 4$.

Let us say P is of *maximum rank* if $r(P)=M(n)$. All antichains are of maximum rank [4].

Theorem 6.8. *For $n=2, 3$, the antichain $\bar{n}$ is the only poset of maximum rank. For $i\geq 4$, (X, P) is of maximum rank iff P is of the following form: $(a, b)\in P$ iff $a\in W$, $b\in Z$, where Y, Z, W is a partition of X, and $|Y|=\lfloor\frac{1}{2}n\rfloor$ or $\lceil\frac{1}{2}n\rceil$. $W=\emptyset$ is allowed.*

This means that a maximum rank n-element poset can have up to $\frac{1}{16}n^2$ edges, more than $\frac{1}{8}$ the edges of any n-element poset. However, note that all maximum rank posets have dimension 2.

Proof. All posets described in the theorem have maximum rank: if $n<4$ or $W=\emptyset$, P is an antichain. Otherwise $N(P)=N_1\cup N_2$, where N_1 is a 2-way complete digraph on Y and N_2 is a bipartite graph with bipartition $Z\cup W$, Y and with all edges directed from W or to Z. Clearly N_2 is acyclic U_P with $M(n)$ edges, so $r(P)=M(n)$.

Conversely, suppose (X, P) has maximum rank. A cyclic CD on the n-set X can have $M(n)$ edges iff it is a Hamilton cycle and $n=2,3,4$. By Lemma 3.9, P must be an antichain. So suppose every CD N' with $M(n)$ edges is acyclic U_P. N' must be triangle free, so by Turan's theorem, N' is a complete bipartite graph with $\lfloor\frac{1}{2}n\rfloor$ and $\lceil\frac{1}{2}n\rceil$ vertices in the two parts, and $n\geq 4$.

We must determine how the edges of N' are directed. If they all face from the same side to the other, call the receiving side Z, the sending side Y, and set $W=\emptyset$. If the edges are not all uniformly directed, there must exist vertices a, b, c with $(a, b), (c, a)\in N'$. Let the side of N' containing a be called Y and define

$$Z=\{x\in X\,|\,(a,x)\in N'\},\quad W=\{x\in X\,|\,(x,a)\in N'\}.$$

Henceforth, y, z, w are always in Y, Z, W respectively. Clearly Z, W partition $X-Y$. We claim that *every* edge between W and Y goes from W and *every* edge between Z and Y goes to Z. Suppose not. Then either some (z, y) or some (y, w) is in N'. We disprove the first possibility; the second is similar. Clearly $y\neq a$. Thus there is in N' a path $R=cazy$. Since N' is complete bipartite, either (y, c) or (c, y) is in N'. In the former case $R\cup(y,c)$ would form a cycle, but N' is assumed acyclic. In the latter case, R and (c, y) are edge-disjoint paths from c to y, violating U_P whatever P is. This proves the claim.

We can now show that the conditions in the theorem on P when $n\geq 4$ are necessary. First, $P|_Y=P|_Z=P|_W=\emptyset$. For instance, if some (y, y') were in P, the path $yy'z$ would violate Lemma 3.5 for any z. Likewise, if (w, w') or (z, z') were in P, then the path $ww'y$, or yzz', would violate Lemma 3.5 for any y. (Note: since $n\geq 4$, both Y and Z are non-empty.) Next, every edge of the form (w, z) is in P. This is because $|Y|\geq 2$, so between each pair w, z there are two edge-disjoint paths wyz and $wy'z$ in N'. Since N' is U_P, $(w, z)\in P$. The only pairs not yet considered for P are of the form (z, w), which cannot be in P since P must be acyclic.

References

[1] K.P. Bogart, Maximal dimensional partially ordered sets I. Hiraguchi's theorem, Discrete Math. 5 (1973) 21–31.

[2] P.C. Fishburn, Intransitive indifference with unequal indifference intervals, J. Math. Psychol. 7 (1970) 144–149.

[3] H. Komm, On the dimension of partially ordered sets, Am. J. Math. 70 (1948) 507–526.
[4] S.B. Maurer and I. Rabinovitch, Large minimal realizers of a partial order, Proc. AMS 66 (1978) 211–216.
[5] S.B. Maurer and I. Rabinovitch, Large minimal realizers of a partial order II, manuscript (1977).
[6] S.B. Maurer, I. Rabinovitch and W.T. Trotter, A generalization of Turan's theorem, to appear in Discrete Math.
[7] S.B. Maurer, Rabinovitch and W.T. Trotter, Partially ordered sets with equal rank and dimension, submitted.
[8] I. Rabinovitch and I. Rival, The rank of a distributive lattice, Discrete Math. 25 (1979) 275–279.
[9] E. Szpilrajn, Sur l'extension de l'ordre partial, Fund. Math. 16 (1930) 386–389.
[10] W.T. Trotter and J.I. Moore, Characterization problems for graphs, partially ordered sets, lattices, and families of sets, Discrete Math. 16 (1976) 361–381.
[11] W.T. Trotter, J.I. Moore and D.P. Sumner, The dimension of a comparability graph, Proc. AMS 60 (1976) 35–38.
[12] P. Turan, An extremal problem in graph theory, Math. Fiz. Lapok, 48 (1941) 536–552.