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MAXIMAL DIMENSIONAL PARTIALLY ORDERED SETS II. CHARACTERIZATION OF 2n-ELEMENT POSETS WITH DIMENSION n

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Abstract. In this paper, we show that if a partially ordered set has $2n$ elements and has dimension n, then it is isomorphic to the set of $n-1$ element subsets and 1element subsets of a set, ordered by inclusion, or else it has six elements and is isomorphic to a partially ordered set we call the *chevron* or to its dual.

1. Introduction

In 1941, Dushnik and Miller [4] introduced the concept of the *dimension* of a partial ordering – the smallest number of linear orderings whose intersection is the original partial ordering. They gave examples of partial orderings of dimension n on a $2n$ element set. Their examples consist of the $n-1$ element subsets and 1 element subsets of an n element set, ordered by inclusion. Their methods were combined with Szpilrajn's theorem [10] by Komm [8] to show that the set of all subsets of an n element set also has dimension n. Dushnik and Miller also showed that a partial ordering has dimension 2 if and only if there is a partial ordering whose comparability relation is the incomparability relation of the original partial ordering. Ghouila-Houri [5] and Gilmore and Hoffman [6] proved theorems about the representability of graphs as comparability graphs of partial orderings and Baker, Fishburn and Roberts [1] related these theorems to Dushnik and Miller's result to obtain additional characterizations of partial orderings of dimension 2.

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The Baker–Fishburn–Roberts paper contains an excellent literature survey, many results about general dimension theory of partial orderings (for instance the theory of partial orderings of dimension n is not finitely axiomatizable in first order logic), many good examples, additional results on partial orderings of dimension 2 and an explanation of the relation of dimension theory to measurement theory and other mathematical theories of interest to social scientists. The result that the theory of partially ordered sets of dimension n is not finitely axiomatizable is especially interesting in light of Dilworth's finite axiomatization of distributive lattices of dimension n in his paper on decomposition of partially ordered sets into chains [3]. This characterization is in Theorem 1.2 of Dilworth's paper; to interpret this theorem in the terminology used here, we need Ore's remark [9] that the dimension of a partially ordered set is the smallest number of chains such that the partially ordered set is isomorphic to a subposet of a product of that number of chains.

The subject of this paper is slightly different. In 1951, Hiraguchi [7] proved that the dimension of a partial ordering of an n element set is at most $[\frac{1}{2}n]$. The value of this result is indicated by some of the computations made in [1]. Recently, Bogart gave a considerably simpler proof of this theorem [2] and suggested that it would be of interest to find those partial orderings on n element sets which have dimension $[\frac{1}{2}n]$.

In this paper, we solve this problem for even n – we show that the only "maximum dimensional" partial orderings are the well known ones, the Dushnik-Miller example mentioned above and the six element *chevron* shown in Fig. 1 (and, of course, its dual). This example is discussed in [1]. This simple characterization, though not a finite axiomatization, is striking where contrasted with the complexity of the general theory of dimension described in [1].

Fig. 1. The maximal dimensional partial orderings on six elements.

2. Basic concepts

We shall use the symbol S_{2n} for the standard example of a maximal dimensional partially ordered set – the set of all subsets of an n element set with either 1 or $n-1$ elements, ordered by set inclusion. S_6 is the first example shown in Fig. 1. We shall use the symbol C_6 to denote the six element *chevron* ordering which is the second ordering shown in Fig. 1.

Our proofs involve detailed examinations of covering pairs. We say (a, b) is a *cover* or *a covers b* in a partial ordering P of a set X if a is above b (i.e., $(a, b) \in P$) and no element of X is between them (i.e., (a, c) and (c, b) in P imply $a = c$ or $b = c$). Following Hiraguchi, we say that a cover (a, b) has *rank* 0 if each element above b is above each element below a. Equivalently, each element covering b is greater than each element covered by a.

By the *height* of a partially ordered set we mean the number equal to one less than the maximum number of points in any chain of the partially ordered set. By the *width* of a partially ordered set we mean the cardinality of a largest antichain – i.e., the largest possible number of elements in a set of pairwise incomparable elements. The symbol X denotes a set partially ordered by an ordering P.

We shall use each of the following lemmas – proofs of them may be found in [7] or [2].

Lemma 2.1. *If (X, P) is a partially ordered set with a maximum element x (or minimum element x), then*

$$\dim P = \dim P|_{X - \{x\}}.$$

Lemma 2.2. *If $X = X_1 \cup X_2$ and $P = P|_{X_1} \cup P|_{X_2}$, then*

$$\dim P = \max(\dim P|_{X_1}, \dim P|_{X_2}).$$

Lemma 2.3. *The dimension of a partially ordered set is less than or equal to its width.*

Lemma 2.4. *If x is a maximal element of (X, P) and y is a minimal element of (X, P), and x and y are incomparable, then*

$$\dim P \leq 1 + \dim P|_{X-\{x,y\}} \ .$$

Lemma 2.5. *If (x, y) is a cover of rank zero in (X, P), then*

$$\dim P \leq 1 + \dim P|_{X-\{x,y\}} \ .$$

Lemma 2.6. *If the restrictions of P to Y and to Z are chains and the elements of Y are incomparable with the elements of Z, then*

$$\dim P \leq 2 + \dim P|_{X-Y} \ ,$$
$$\dim P \leq 2 + \dim P|_{X-(Y\cup Z)} \ .$$

This lemma is crucial in both the Hiraguchi proof and the Bogart proof of the theorem that the dimension of a partial ordering on an n element set is at most $[\frac{1}{2}n]$. We note that the 2 in the lemma cannot be replaced by 1 – the removal of any two element chain from S_8 (which has dimension 4) gives a six element partially ordered set which is not S_6 or the chevron C_6 and thus has dimension 2.

The following lemma was used by Bogart to avoid some of the complicated reconstructions of chains needed by Hiraguchi.

Lemma 2.7. *Let $Y \subseteq X$ and let Q be an extension of the restriction of P to Y. Then the transitive closure of $P \cup Q$ is a partial ordering.*

This lemma is in essence a version of the basic lemma used in proving Szpilrajn's theorem. Bogart's proof of Hiraguchi's theorem made essential use of the next lemma.

Lemma 2.8. *A partially ordered set contains either a cover of rank 0 or a pair of covers such that the elements of one cover are incomparable with the elements of the other. Further, a partially ordered set of height 2 or more has 9 or more elements if it has no covers of rank 0 and we may assume that both covers contain a maximal element.*

In order to present the results of this paper in the simplest form possible, we examine a circumstance in which removal of a chain reduces the dimension of a partially ordered set by at most one.

Theorem 2.9. *If C is a chain (X, P) such that each element of X is incomparable with at most one element of C, then*

$$\dim(P) \le \dim(P|_{X-C}) + 1 .$$

Proof. Write $P|_{X-C} = L_1 \cap \ldots \cap L_k$ as an intersection of linear orderings. For $i > 1$, let L_i' be a linear extension of the transitive closure of $P \cup L_i$ (such a linear extension exists by Lemma 2.7 and Szpilrajn's theorem). From the ordering L_1 we wish to construct two other linear orderings L_1' and L_1'' such that the intersection of these orderings with the remaining L_i' is P. Suppose $C = \{c_1, \ldots, c_n\}$ and $(c_i, c_{i+1}) \in P$ for $i = 1, 2, \ldots, n-1$. Note that two elements of C cannot both be incomparable with the same element of X by the hypothesis of the theorem. Thus the intersection of the set of elements incomparable with c_i and the set of elements incomparable with c_j is empty if $i \neq j$.

We sketch the remainder of the proof informally to avoid a notational mess. We visualize a linear ordering of a set as a vertical list with larger elements at the top. We construct the list L_1' as follows. At the top of the list we put all elements incomparable with c_1 and all elements larger than c_1 and order them as they were ordered by L_1, then place c_1 immediately below this group. We place elements smaller than c_1 and larger than c_2 between c_1 and c_2 and order them by L_1. We next place elements incomparable with c_2 and c_3, and those below c_2 but above c_3 between c_2 and c_3 and order them by L_1. We continue in this fashion, placing elements incomparable to c_i above c_i if i is odd but below it if i is even. We construct L_1'' by placing elements incomparable to c_i below c_i if i is odd and above c_i if i is even, and then ordering each interval between c_i and c_{i+1} as L_1 orders that set.

Note first that L_1' and L_1'' are extensions of P. If they were not, an ordered pair (x, y) would be in P with (y, x) in (say) L_1'. The only way this could happen is if we placed y above c_i and x below c_i in our construction. Thus y is incomparable with c_i or above c_i and x is incomparable with c_i or below c_i in P. (Here we are using the fact that each element of X is incomparable with at most one element of P.) If y and x are both incomparable with c_i, they are ordered in L_1' in the same way they are ordered in P, so that (y, x) is not in L_1'. If y and x are both comparable with c_i, then (y, x) is in P, so this is impossible. If y is incomparable with c_i and x is below c_i, then x is not above y in P, which is impossible. Finally, if x is incomparable with c_i and y is above

c_i, then x is not above y in P which is again impossible.

Now if (x, y) is in L_1, it is in either L'_1 or L''_1. To see this, we need only check the case in which x and y are incomparable in P. Suppose first that they are both comparable with each element of C. Then one cannot be above any element of C that the other is below, so they both lie above c_1, they both lie below c_n or they both lie between c_i and c_{i+1} for some i. In this situation (x, y) is in both L'_1 and L''_1. Suppose now that only x is comparable with every element of C and y is incomparable with c_i. Then y is below c_{i-1} (unless $i = 1$) and above c_{i+1} (unless $i = n$). Thus (except in the special cases $i = 1, n$) x is between c_{i-1} and c_i or between c_i and c_{i+1}. Thus (x, y) is in L'_1 or L''_1. (If $i = 1$ or n, a similar argument works.) Finally, if x is incomparable with c_i and y is incomparable with c_j, i and j differ by at most 1, since x is below c_{i-1} and above c_{i+1} and y is below c_{j-1} and above c_{j+1}. Thus (x, y) is in L'_1 or L''_1.

It follows that

$$P = L'_1 \cap L''_1 \cap L'_2 \cap \ldots \cap L'_k ,$$

and the theorem is proved.

3. Main result

Theorem 3.1. *If $n \geq 3$, the only partially ordered sets of dimension n with $2n$ elements are the chevron C_6, its dual and the standard maximal dimensional partially ordered sets S_{2n}.*

Proof. The proof is inductive. It is known that the only six element partially ordered sets of dimension 3 are the chevron C_6, its dual, and the standard example S_6 (shown in Fig. 1). We outline a method of proving this fact. Since the width of a poset of dimension 3 must be at least 3 and since a six element poset with dimension 3 has at least two maximal elements and two minimal elements (by Lemmas 2.1 and 2.3 and Hiraguchi's theorem), we need consider only posets of height 1 and 2. By Lemma 2.2 and Hiraguchi's theorem we need only consider those partially ordered sets whose Hasse diagram is connected.

A case by case argument shows that if the poset has height 1 and dimension 3, then it is isomorphic to S_6. If the poset has height 2 and has 3 maximal elements, 2 minimal elements and one element neither maximal or minimal it is easy to verify that it has dimension 2. Otherwise, a maximum antichain must contain two elements not maximal or minimal and one maximal or minimal element. Of the several partially ordered sets of this type, it is easy (but tedious) to verify that only C_6 and its dual have dimension 3.

Now suppose (X, P) is a $2n$ element partially ordered set of dimension n and $n > 3$. We show first that if (X, P) has height 1, then it is isomorphic to S_{2n}. To see this, note that if every maximal element is above every minimal element in a poset of height 1, then the poset has dimension 2. Thus there is a maximal element x of (X, P) incomparable with a minimal element y of (X, P). By Lemma 2.3 and Hiraguchi's theorem,

$$\dim (P|_{X-\{x,y\}}) = n-1.$$

We may assume inductively that $(X-\{x, y\}, P|_{X-\{x,y\}})$ is isomorphic to S_{2n-2}. Thus if x is above each minimal element other than y and y is below each maximal element other than x, then P is isomorphic to S_{2n}. However, each other maximal element of (X, P) is paired with precisely one minimal element with which it is incomparable. By removing one of these pairs from (X, P), we obtain a partially ordered set isomorphic to S_{2n-2} as above, so that x is above each minimal element but the one in the pair removed and y is below each maximal element but the one in the pair removed. Removal of one other pair shows that x is above every minimal element except y and y is below every maximal element except x and thus (X, P) is isomorphic to S_{2n}. Thus we may assume that $n > 3$ and that the height of P is greater than 1.

Now, either (X, P) has a cover (x, y) of rank 0 or (X, P) has more than 9 elements and has two covers (x, y) and (z, w) with both z and w incomparable with both x and y (by Lemma 2.8).

Suppose (X, P) has a cover (x, y) of rank 0. Then

$$P' = P|_{X-\{x,y\}} = P|_{X'}$$

has dimension $n-1$ by Lemma 2.5 and Hiraguchi's theorem. Thus by the inductive hypothesis (X', P') is either C_6, its dual, or S_{2n-2}. If (X', P') is S_{2n-2}, we may assume that y is not minimal in (X, P). To see this, suppose y is minimal. If x is not maximal, we may consider the dual of (X, P) rather than (X, P) itself; in this partially ordered set, (y, x) has rank 0 and x is not minimal. If x is maximal, then (X, P) has height 1 unless x is above some maximal element z of S_{2n-2}. Since no other element of (X, P) can be above z, (x, z) is a cover of rank zero and z is not minimal. Thus we may replace y by z. Now, by Lemma 2.1 and Hiraguchi's theorem, x cannot be above each maximal element of (X', P'), so that there is a maximal element u incomparable with a minimal element v of (X, P) and neither u or v is in $\{x, y\}$. Thus by Lemma 2.4 and Hiraguchi's theorem

$$P^* = P|_{X - \{u,v\}} = P|_{X^*}$$

has dimension $n-1$. Since P^* has a chain of length 3, (X^*, P^*) is isomorphic to C_6 or its dual by induction. Since S_6 is self dual, we may assume (X^*, P^*) is isomorphic to C_6. Thus (X^*, P^*) has a maximal element t of height 1. Since x must be the element of (X^*, P^*) of height 2 and x covers y, t has height 1 in (X, P) and thus must also be a maximal element of (X', P'). However, each maximal element of C_6 is comparable with each minimal element – and thus t is above every element that u is above in (X', P'). This means that (X', P') is not isomorphic to S_6, a contradiction. This series of constructions is illustrated in Fig. 2.

Now assume that removal of any cover of rank 0 from (X, P) gives a poset isomorphic to C_6. (By considering the dual of P if necessary we may assume that we obtain C_6 and not its dual.) Suppose we remove a cover (x, y) of rank 0 from (X, P) to obtain a poset (X', P') = C_6 as labelled in Fig. 1. By Lemma 2.3, (X, P) must have an antichain with four elements; thus either $\{x, b, e, c\}$ or $\{y, b, e, c\}$ is an antichain.

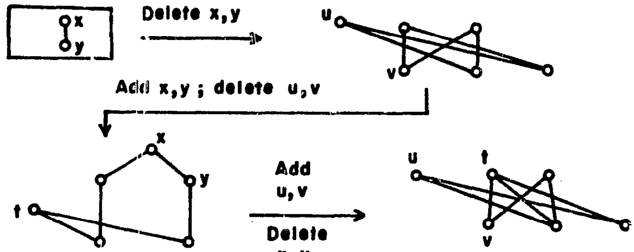

Fig. 2.

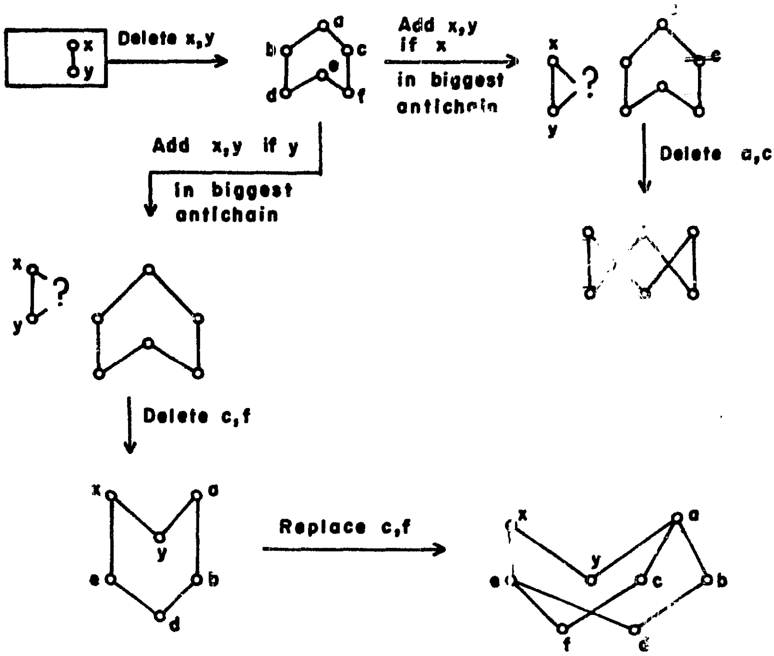

Fig. 3.

If $\{x, b, e, c\}$ is an antichain of (X, P), then no element of X besides a is above c, so that (a, c) is a cover of rank zero. However, then

$$P' = P|_{X - \{a,c\}} = P|_{X'}$$

has dimension $n-1 = 3$. Since (X', P') has three maximal elements, x, b and e, it is isomorphic to S_6, contrary to assumption. Thus $\{y, b, e, c\}$ is the four element antichain of (X, P). Thus no element of X except f is below c in P, so that (c, f) has rank zero. Removal of (c, f) gives a 6 element poset of dimension 3, (X^*, P^*). By our inductive hypothesis, this poset must be isomorphic to C_6 or its dual; since (X^*, P^*) has only one minimal element d not in the three element antichain of (X^*, P^*), this poset is isomorphic to the dual of C_6. Thus y must be below a; e must be below x since it is not below a (and b is below a) and y and d must be incomparable.

Note that a is comparable with y, c, f and b and that d is comparable with e and x, so that each element of (X, P) is incomparable with at most one of the elements of the chain $\{a, d\}$. Thus removal of this chain gives a 6-element poset of dimension 3 by Theorem 2.9. However, this poset has an antichain $\{y, e, c, b\}$ and no six element poset of dimension 3 has width 4. This contradiction completes the case in which (X, P) has a cover of rank zero. The constructions given here are illustrated in Fig. 3. (To avoid use of Theorem 2.9 we could remove other covers of rank 0 to find out all relations between c, f, x and y; however this avoidance makes the proof far more complex.)

Now suppose that (X, P) has no covers of rank 0. Then, by Lemma 2.8, we may assume (X, P) has two covers (x, y) and (w, z) with x and w maximal such that x and y are incomparable with both w and z. Remove these two covers to obtain a partially ordered set (X', P') on $2n-4$ elements with dimension $n-2$ (we use Lemma 2.6 and Hiraguchi's theorem here). This poset is isomorphic to either C_6 or S_{2n-4} by Hiraguchi's theorem (and by dualizing P if necessary).

Suppose that (X', P') is isomorphic to C_6, labelled as in Fig. 1. Then since neither (a, c) nor (c, d) have rank 0, one element of $\{x, y, w, z\}$ must be above b and one element of $\{x, y, w, z\}$ must be below b. Thus two elements of $\{x, y, w, z\}$ are related and neither one covers the other. This is impossible. Thus we may assume (X', P') is isomorphic to S_{2n-4}.

Suppose y is minimal in (X, P). Then, by Lemma 2.4,

$$P^* = P|_{X - \{y,w\}} = P|_{X^*}$$

has dimension $n-1$ and by induction (X^*, P^*) is isomorphic to S_{2n-2} (since X has 10 or more elements). Thus, x is above each minimal element of (X', P') and z is below each maximal element of (X', P'). Also z incomparable with each minimal element of (X', P') and x is incomparable with each maximal element of (X', P'). Then z is also minimal in (X, P). We may repeat the argument just given replacing x and z with w and y to show that (X, P) must be isomorphic to S_{2n}. We may therefore assume that neither y nor z is minimal in (X, P). If a maximal element t of (X', P') is maximal in (X, P) and t' is the minimal element of (X', P') incomparable with t, then $P|_{X-\{t,t'\}}$ has dimension $n-1$ and by induction is isomorphic to S_{2n-2} (since X has 10 or more elements). In this case, an argument similar to the one given above shows that (X, P) is isomorphic to S_{2n}.

We are now left with the case in which each maximal element of (X', P') is below x or w, and neither y nor z is minimal. Since (x, y) is not a cover of rank zero, there is an element u above y in (X, P) which is not below x. (In fact, u is incomparable with something below

x.) However, u must be a maximal element of (X', P'), so that u is below w. However, in this case y is below w and that is impossible. Thus if (X, P) has no covers of rank zero (and has dimension n with $2n$ elements), (X, P) is isomorphic to S_{2n-2}. This completes the proof.

This result does not completely solve the problem of determining all maximal dimensional partially ordered sets, for we have not considered partially ordered sets of size $2n + 1$ having dimension n. It is not true for instance that a partially ordered set on $2n + 1$ elements has dimension n if and only if it contains an n-dimensional subposet on $2n$ elements. To see this, note that on the seven element set $\{x_1, x_2, x_3, y_1, y_2, y_3, 0\}$ the partial ordering P given by $x_i P y_i$ and $x_i P 0$ for all i has dimension 3. Another example is given in [1]. It would certainly be interesting to characterize the maximal dimensional partially ordered sets on an odd number of elements.

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