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A COMBINATORIAL PROBLEM INVOLVING GRAPHS AND MATRICES

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In this paper we discuss a combinatorial problem involving graphs and matrices. Our problem is a matrix analogue of the classical problem of finding a system of distinct representatives (transversal) of a family of sets and relates closely to an extremal problem involving 1-factors and a long standing conjecture in the dimension theory of partially ordered sets. For an integer $n \geq 1$, let $\mathbf{n}$ denote the n element set $\{1, 2, 3, \ldots, n\}$. Then let A be a $k \times t$ matrix. We say that A satisfies property $P(n, k)$ when the following condition is satisfied: For every k-tuple $(x_1, x_2, \ldots, x_k) \in \mathbf{n}^k$, there exist k distinct integers $j_1, j_2, \ldots, j_k$ so that $x_i = a_{ij_i}$ for $i = 1, 2, \ldots, k$. The minimum value of t for which there exists a $k \times t$ matrix A satisfying property $P(n, k)$ is denoted by $f(n, k)$. For each $k \geq 1$ and n sufficiently large, we give an explicit formula for $f(n, k)$; for each $n \geq 1$ and k sufficiently large, we use probabilistic methods to provide inequalities for $f(n, k)$.

1. Introduction

Let $\mathscr{F} = \{A_i: 1 \leq i \leq k\}$ be an indexed family of sets. A set $S = \{s_1, s_2, \ldots, s_k\}$ of k distinct elements is called a system of distinct representatives (SDR) of $\mathscr{F}$ when $s_i \in A_i$ for $i = 1, 2, \ldots, k$. The following well-known theorem of P. Hall [2] gives a necessary and sufficient condition for the existence of a SDR of a family $\mathscr{F}$.

Theorem 1 (Hall). *A family $\mathscr{F} = \{A_i: 1 \leq i \leq k\}$ has a SDR if and only if $|\bigcup \mathscr{G}| \geq |\mathscr{G}|$ for every subfamily $\mathscr{G} \subseteq \mathscr{F}$.*

In this paper we consider a combinatorial problem involving the determination of systems of distinct representatives for families of sets formed by selecting subsets of the entries in the rows of a matrix. For an integer $n \geq 1$, let $\mathbf{n}$ denote the n element set $\{1, 2, 3, \ldots, n\}$. We refer to the elements of $\mathbf{n}$ as *letters*; consequently, it is natural to refer to a k-tuple $(x_1, x_2, x_3, \ldots, x_k)$ from $\mathbf{n}^k$ as a *word* and use the notations $\mathbf{x}$ and $x_1 x_2 x_3 \cdots x_k$ for this word. When $x_1 x_2 x_3 \cdots x_k$ is a word and $1 \leq i_1 < i_2 < \cdots < i_m \leq k$, we call $x_{i_1} x_{i_2} x_{i_3} \cdots x_{i_m}$ a *subword*. We then say that the $k \times t$ matrix $A = (a_{ij})$ satisfies property $P(n, k)$ when the following condition holds:

For every word $x_1 x_2 x_3 \cdots x_k \in \mathbf{n}^k$, there exist k distinct integers (columns) $j_1, j_2, j_3, \ldots, j_k$ so that $a_{ij_i} = x_i$ for $i = 1, 2, 3, \ldots, k$.

This definition may be rephrased in terms of systems of distinct representatives as follows. Let $\mathbf{x}=x_1x_2x_3\cdots x_k\in\mathbf{n}^k$ and then let $\mathcal{F}_\mathbf{x}(A)=\{A_i: 1\leq i\leq k\}$ be defined by $A_i=\{j: a_{ij}=x_i\}$. It is easy to see that A satisfies property $P(n,k)$ if and only if $\mathcal{F}_\mathbf{x}(A)$ has a SDR for every $\mathbf{x}\in\mathbf{n}^k$.

Example 2. A satisfies $P(3,2)$ and B satisfies $P(4,3)$.

$$A=\begin{pmatrix}1 & 1 & 2 & 3\\ 1 & 2 & 3 & 3\end{pmatrix},\qquad B=\begin{pmatrix}1 & 2 & 4 & 4 & 3 & 3 & 4\\ 3 & 3 & 1 & 2 & 4 & 4 & 4\\ 4 & 4 & 3 & 3 & 1 & 2 & 4\end{pmatrix}\qquad\square$$

Note that a matrix A may satisfy $P(n,k)$ yet contain entries which are not elements of $\mathbf{n}$. We adopt the convention of using an asterisk to denote such entries.

Example 3. A satisfies $P(7,2)$.

$$A=\begin{pmatrix}1 & 1 & 2 & 3 & 4 & 4 & 5 & 6 & 7 & *\\ 1 & 2 & 3 & 3 & 4 & 5 & 6 & 6 & * & 7\end{pmatrix}\qquad\square$$

The minimum value of t for which there exists a $k\times t$ matrix A satisfying property $P(n,k)$ is denoted by $f(n,k)$. The remainder of this paper is devoted to the study of this function and related combinatorial problems. For each $k\geq 1$, we will provide an explicit formula for $f(n,k)$ which holds for n sufficiently large. The determination of the least value of n for which our formula holds leads to an interesting extremal problem involving 1-factors. On the other hand, it appears that a precise determination of $f(n,k)$ is not possible when k is relatively large compared to n. In this case we use probabilistic methods to determine a nontrivial upper bound on $f(n,k)$.

We begin our study of $f(n,k)$ with some elementary inequalities and a complete determination of $f(n,k)$ when $n\leq 3$.

Theorem 4. *$f(n,k)\geq k+n-1$ for each $n\geq 1$, $k\geq 1$.*

Proof. Suppose that $f(n,k)=t$ and let A be a $k\times t$ matrix satisfying property $P(n,k)$. For $i=1,2,3,\ldots,k$, choose a letter $x_i\in\mathbf{n}$ so that $x_i\neq a_{ij}$ for each $j=1,2,3,\ldots,n-1$. Then let $S=\{j_1,j_2,j_3,\ldots,j_k\}$ be a SDR for the family $\mathcal{F}_\mathbf{x}(A)$ where $\mathbf{x}=x_1x_2x_3\cdots x_k$. Since $x_i\neq a_{ij_i}$ for each $i=1,2,3,\ldots,k$, we observe that $S\subseteq\{n,n+1,\ldots,t\}$. Since $|S|=k$, we conclude that $k\leq t-n+1$, and thus $f(n,k)=t\geq k+n-1$. $\square$

Lemma 5. *$f(n,k)\leq nk$ for each $n\geq 1$, $k\geq 1$.*

Proof. The following $k \times nk$ matrix satisfies $P(n, k)$ and thus $f(n, k) \leqslant nk$

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n & * & * & * & \cdots & * & & * & * & * & \cdots & * \\ * & * & * & \cdots & * & 1 & 2 & 3 & \cdots & n & & * & * & * & \cdots & * \\ * & * & * & \cdots & * & * & * & * & \cdots & * & & * & * & * & \cdots & * \\ & & \vdots & & & & & \vdots & & & \ddots & & & & & \\ * & * & * & \cdots & * & * & * & * & \cdots & * & & 1 & 2 & 3 & \cdots & n \end{pmatrix} \quad \square$$

Corollary 6. *$f(n, 1) = n$ for every $n \geqslant 1$ and $f(1, k) = k$ for every $k \geqslant 1$.*

Lemma 7. *$f(2, k) = k + 1$ for every $k \geqslant 1$.*

Proof. We have $f(2, k) \geqslant k + 1$ by Theorem 4. On the other hand, the following $k \times (k+1)$ matrix shows that $f(2, k) \leqslant k + 1$.

$$\begin{pmatrix} 1 & 2 & * & * & * & \cdots & * & * \\ 1 & 1 & 2 & * & * & \cdots & * & * \\ 1 & 1 & 1 & 2 & * & \cdots & * & * \\ & & \vdots & & & & \vdots & \\ 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 2 \end{pmatrix} \quad \square$$

Lemma 8. *$f(3, k) = k + 2$ for every $k \geqslant 1$.*

Proof. Theorem 4 implies that $f(3, k) \geqslant k + 2$. On the other hand, the following $k \times (k+2)$ matrix shows that $f(3, k) \leqslant k + 2$.

$$\begin{pmatrix} 1 & 2 & 3 & 3 & 3 & 3 & \cdots & 3 & 3 & 3 \\ 1 & 1 & 2 & 3 & 3 & 3 & \cdots & 3 & 3 & 3 \\ 1 & 1 & 1 & 2 & 3 & 3 & \cdots & 3 & 3 & 3 \\ 1 & 1 & 1 & 1 & 2 & 3 & \cdots & 3 & 3 & 3 \\ & & \vdots & & & & & & & \\ 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 2 & 3 \end{pmatrix} \quad \square$$

If A is a $k \times t$ matrix satisfying property $P(n, k)$, then there are three elementary operations which may be performed which produce a $k \times t$ matrix A^* also satisfying $P(n, k)$.

Lemma 9. *Let A be a $k \times t$ matrix satisfying $P(n, k)$. If A^* is obtained from A by any of the following three operations, then A^* also satisfies $P(n, k)$.*

(i) *Permute the rows of A.*

(ii) *Permute the columns of A.*

(iii) *Let σ be a permutation of $\mathbf{n}$ and i an integer with $1 \leq i \leq k$. Then permute the entries in the ith row of A by the rule $a_{ij}^* = \sigma(a_{ij})$.*

Note that there is no analogue for the operation in (iii) of Lemma 9 for columns.

Now let A be a $k \times t$ matrix, $x \in \mathbf{n}$, and i an integer with $1 \leq i \leq k$. We then define the *multiplicity* of x in the ith row of A, denoted $m(x, i, A)$, as the number of times x appears in the ith row of A. Note that if A satisfies $P(n, k)$, then $1 \leq m(x, i, A) \leq t$ for each $x \in \mathbf{n}$ and every integer i with $1 \leq i \leq k$. A letter x is called a *single* in row i when $m(x, i, A) = 1$. Similarly, we will speak of *doubles*, *triples*, ..., and use the generic term *multiple* for a letter whose multiplicity is at least two.

Lemma 10. *Let A be a $k \times t$ matrix satisfying $P(n, k)$. Suppose $a_{i_1j_1}$ and $a_{i_2j_2}$ are distinct entries of A with $a_{i_1j_1}$ a single in row i_1 and $a_{i_2j_2}$ a single in row i_2. Then $j_1 \neq j_2$.*

Proof. The conclusion is immediate when $i_1 = i_2$. Now suppose $i_1 \neq i_2$ and let $\mathbf{x} = x_1x_2x_3, \cdots x_k$ be any word in which $x_{i_1} = a_{i_1j_1}$ and $x_{i_2} = a_{i_2j_2}$. Choose a SDR $S = \{s_1, s_2, \ldots, s_k\}$ for the family $\mathcal{F}_x(A)$. Then it follows that $s_{i_1} = j_1$ and $s_{i_2} = j_2$ and thus $j_1 \neq j_2$. □

We use the terminology "singles do not overlap" to indicate that the condition in Lemma 10 is satisfied.

Theorem 11. *$f(n, k) \geq \lceil 2kn/(k+1) \rceil$ for every $n \geq 1$ and every $k \geq 1$.*

Proof. Suppose $f(n, k) = t$ and let A be a $k \times t$ matrix satisfying $P(n, k)$. For each $i = 1, 2, \ldots, k$, let m_i be the number of letters in $\mathbf{n}$ which are singles in the ith row, and let $m = \min\{m_i : 1 \leq i \leq k\}$. Each row of A contains at least $n - m$ letters in $\mathbf{n}$ which are multiples. It follows that $t \geq m + 2(n - m) = 2n - m$, and thus $m \geq 2n - t$. On the other hand, since singles do not overlap, we know that $t \geq m_1 + m_2 + \cdots + m_k \geq km \geq k(2n - t) = 2kn - kt$. Therefore $(k+1)t \geq 2kn$ and the desired result follows since t is an integer. □

The preceding inequality for $f(n, k)$ was derived solely from the fact that singles do not overlap in matrices which satisfy $P(n, k)$. Since there are more stringent requirements which such matrices must satisfy, one might expect that this inequality is quite weak. However, it will turn out to be suprisingly strong.

2. Graphs, cycles, and 1-factors

We begin this section by developing an elementary characterization of a family $\mathscr{F}$ of sets which does not have a SDR in the special case where $1 \leq |A| \leq 2$ for every $A \in \mathscr{F}$. We say that $\mathscr{F}$ is *critical* when $\mathscr{F}$ is a family of nonempty sets of size at most two, $\mathscr{F}$ does not have a SDR, but every nonempty proper subfamily of $\mathscr{F}$ has a SDR.

Example 12. The following families are critical.

(a) For each $m \geq 0$, let $\mathscr{D}(m) = \{\{1\}, \{1,2\}, \{2,3\}, \{3,4\}, \ldots, \{m, m+1\}, \{m+1\}\}$.

(b) For each $m \geq 1$, $p \geq 1$, with $m \geq p$, let $E(m,p) = \{\{1\}, \{1,2\}, \{2,3\}, \ldots, \{m, m+1\}, \{p, m+1\}\}$.

(c) For each $m \geq 1$, $p \geq 1$, $q \geq 1$, with $m \geq p$ and $m \geq q$, let $\mathscr{F}(m, p, q) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \ldots, \{m, m+1\}, \{1, m+1-q\}, \{p, m+1\}\}$. □

We now show that the three critical families constructed in the preceding example are essentially the only critical families (up to relabeling the elements in the sets). The following elementary result, which follows immediately from Hall's theorem will prove useful in our argument.

Lemma 13. *Let $\mathscr{F}$ be a critical family and let $a \in \bigcup \mathscr{F}$. Then there exist distinct sets $A_1, A_2 \in \mathscr{F}$ so that $a \in A_1$ and $a \in A_2$.*

We will also find it convenient to use the concept of paths and cycles. We call $\mathscr{P}(m) = \{\{1,2\}, \{2,3\}, \{3,4\}, \ldots, \{m, m+1\}\}$ a path of length m and $\mathscr{C}(m) = \{\{1,2\}, \{2,3\}, \{3,4\}, \ldots, \{m, m+1\}, \{1, m+1\}$ a cycle of length m. Note that paths and cycles have SDR's.

Theorem 14. *Let $\mathscr{F}$ be a critical family.*

(i) If $\mathscr{F}$ contains two singletons, then (after relabeling) $\mathscr{F} = \mathscr{D}(0)$.

(ii) If $\mathscr{F}$ contains one singleton, then there exist unique integers $m, p \geq 1$, with $m \geq p$, so that (after relabeling) $\mathscr{F} = \mathscr{E}(m, p)$.

(iii) If $\mathscr{F}$ contains no singletons, then there exist unique integers $m, p, q \geq 1$, with $m \geq p$ and $m \geq q$, so that (after relabeling) $\mathscr{F} = \mathscr{F}(m, p, q)$.

Proof. If every set in $\mathscr{F}$ is a singleton, then it is clear that $\mathscr{F} = \mathscr{D}(0)$, so we may assume that $\mathscr{F}$ contains at least one doubleton. Now let $\mathscr{P}(m) = \{\{1,2\}, \{2,3\}, \{3,4\}, \ldots, \{m, m+1\}\}$ be a path of maximum length contained in $\mathscr{F}$. By Lemma 13 we may choose sets $A_1, A_2 \in \mathscr{F} - \mathscr{P}(m)$ so that $1 \in A_1$ and $m+1 \in A_2$. Note that the maximality of m requires that $A_1 \cup A_2 \subseteq \{1, 2, \ldots, m+1\}$.

Suppose first that $A_1 \neq A_2$. If A_1 and A_2 are both singletons, then $\mathscr{F} = \mathscr{D}(m)$. If one of A_1 and A_2 is a singleton and the other is a doubleton, say $A_1 = \{1\}$ and

$A_2=\{p, m+1\}$ where $1\leqslant p\leqslant m$, then $\mathscr{F}-\mathscr{E}(m,p)$. If A_1 and A_2 are both doubletons and $A_1=\{1, m+1-q\}$ and $A_2=\{p, m+1\}$, then $\mathscr{F}=\mathscr{F}(m,p,q)$.

Now suppose that $A_1=A_2$. Then $A_1=\{1, m+1\}$. Let $\mathscr{G}=\mathscr{P}(m)\cup\{A_1\}=\mathscr{C}(m)$. Since $\mathscr{G}$ has a SDR, $\mathscr{G}$ is a proper non-empty subfamily of $\mathscr{F}$, and it follows that $(\bigcup\mathscr{G})\cap(\bigcup(\mathscr{F}-\mathscr{G}))\neq\emptyset$. We may therefore assume without loss of generality that there exists a set $A_1'\in\mathscr{F}-\mathscr{G}$ so that $1\in A_1'$. Since $A_1'\neq A_2'$, we are back in the preceding case and the proof is complete. □

We now turn our attention to the problem of determining $f(n,k)$ when $n\equiv 0 \bmod k+1$. Let $n=s(k+1)$; then by Theorem 11, we know that $f(n,k)\geqslant 2ks$. We suppose that $f(n,k)=2ks$ and investigate the implications of this equation.

Lemma 15. *For an integer $k\geqslant 1$, let A be a $k\times 2ks$ matrix satisfying $P(s(k+1),k)$. Then each row of A contains s letters of n which are singles while the remaining $n-s$ letters are doubles.*

Proof. Following the notation used in the proof of Theorem 11, let m_1 denote the number of singles in the ith row of A, and let $m=\min\{m_i: 1\leqslant i\leqslant k\}$. Then we note that $m\geqslant 2s(k+1)-2ks=2s$ and $2ks\geqslant m_1+m_2+\cdots+m_k\geqslant km$. Therefore, $2ks\geqslant km$, $2s\leqslant m$, and thus $m=2s$. Since $2ks\geqslant m_1+m_2+\cdots+m_k\geqslant 2ks$ and $m_i\geqslant m$ for each i, we conclude that $m_i=m=2s$ for each $i=1,2,3,\ldots,k$.

Now consider an arbitrary row of A. We know that $2s$ of the entries in this row are singles. Without loss of generality, we may assume that the letters $y_1, y_2, \ldots, y_{n-2s}$ are multiples in this row with multiplicities (respectively) $d_1, d_2, \ldots, d_{n-s}$. It follows that $2ks\geqslant 2s+d_1+d_2+\cdots+d_{n-2s}\geqslant 2s+2(n-2s)=2s(k+1)-2s=2sk$, and thus $d_i=2$ for $i=1,2,\ldots,n-2s$. □

Now suppose that A is a $k\times 2ks$ matrix satisfying $P(s(k+1),k)$. We may use Lemmas 9, 10, and 15, to permute the columns of A so that for each $i=1,2,\ldots,k$, the entries a_{ij} are singles in row i and column $j=(i-1)2s+r$ where $1\leqslant r\leqslant 2s$. The matrix A is then partitioned into k blocks $B_1, B_2, \ldots, B_k$, with each block consisting of $2s$ consecutive columns of A. These blocks are called *single blocks*.

$$A=\left(\begin{array}{c|c|c|c}
1\ 2\ 3\ \cdots\ 2s & & & \\
 & 1\ 2\ 3\ \cdots\ 2s & \cdots & B_k \\
B_1 & B_2 & & \\
 & & & 1\ 2\ 3\ \cdots\ 2s
\end{array}\right)$$

When A is a $k\times 2ks$ matrix satisfying $P(s(k+1),k)$ and A has been transformed into this form, we say that A is in *canonical* form.

Lemma 16. *Let A be a canonical $k\times 2ks$ matrix satisfying $P(s(k+1), k)$ and let i_1, i_2 be integers with $1\leqslant i_1, i_2 \leqslant k$ and $i_1\neq i_2$. Then let x be a letter in row i_2 of block B_{i_1}. Then x is a double and the other occurrence of x in row i_2 of A is also in block B_{i_1}.*

Proof. The fact that x is a double is immediate. Now suppose that the other occurrence of x is in block B_{i_3} where $i_3\neq i_1$. Since A is in canonical form and x is a double, we note that $i_3\neq i_2$. Choose the columns j_1, j_2 in A so that $a_{i_2j_1}=x=a_{i_2j_2}$. Then let $\mathbf{y}=y_1y_2y_3\cdots y_k$ be any word in which $y_{i_1}=a_{i_1}$, $y_{i_2}=x$, and $y_{i_3}=a_{i_3j_2}$. We observe that $\mathcal{F}_{\mathbf{y}}(A)$ contains $\{\{j_1\},\{j_1,j_2\},\{j_2\}\}$ as a subfamily, but this subfamily (after relabeling) is the critical family $\mathcal{D}(1)$. This is a contradiction and completes the proof. □

When a letter x is a double in some row of a matrix A, it is natural to refer to the two occurrences of x in this row as *mates*. In view of Lemma 16, we may also refer to the actual positions where x appears as *mates* since the symbol which occupies these positions is arbitrary.

We next show how the preceding development will allow us to characterize those integers s for which $f(s(k+1), k)=2ks$. First we want to extend the concept of canonical form to arbitrary matrices. We say that a $k\times 2ks$ matrix A (which may or may not satisfy $P(s(k+1), k)$) is in canonical form when it satisfies the following properties.

(i) For each $i=1,2,\ldots,k$ and each $r=1,2,\ldots,2s$, the letter r is a single in row i and column $(i-1)2s+r$. Furthermore the letters $2s+1, 2s+2,\ldots,n$ are doubles in each row of A.

(ii) For each i_1, i_2 with $1\leqslant i_1, i_2\leqslant k$ and $i_1\neq i_2$, and for each letter x which appears in row i_2 of block B_{i_1}, the other occurrence of x in row i_2 is also in B_{i_1}, i.e., mates occur in the same single block.

Theorem 17. *Let A be a cononical $k\times 2ks$ matrix. Then A satisfies $P(n, k)$ if and only if there is no word $\mathbf{x}=x_1x_2\cdots x_k$ in $\mathbf{n}^k$ for which $\mathcal{F}_{\mathbf{x}}(A)$ contains a cycle.*

Proof. Suppose first that there is some $m\geqslant 1$ and a word $\mathbf{x}=x_1x_2\cdots x_k$ so that (after relabeling) $\mathcal{F}_{\mathbf{x}}(A)$ contains the cycle $\mathcal{C}(m)=\{\{j_1,j_2\},\{j_2,j_3\},\ldots,\{j_m,j_{m+1}\},\{j_1,j_{m+1}\}\}$. Then there exist distinct integers $i_1, i_2,\ldots,i_{m+1}$ so that $x_{i_\alpha}=a_{i_\alpha j_\alpha}=a_{i_\alpha j_{\alpha+1}}$ for $\alpha=1,2,\ldots,m$ and $x_{i_{m+1}}=a_{i_{m+1}j_{m+1}}=a_{i_{m+1}j_1}$. It follows that there is a single block B_i so that the columns $j_1, j_2,\ldots,j_{m+1}$ of A occur in block B_i. Furthermore $i\neq i_\alpha$ for $\alpha=1,2,\ldots,m+1$. Now let $\mathbf{y}=y_1y_2\cdots y_k$ be any word for which $y_{i_\alpha}=x_{i_\alpha}$ for $\alpha=1,2,\ldots,m+1$ and $y_i=a_{ij_1}$. Then $\mathcal{F}_{\mathbf{y}}(A)$ contains the critical family $\mathcal{E}(m,1)=\mathcal{C}(m)\cup\{\{j_1\}\}$ and we conclude that A does not satisfy $P(n,k)$.

Conversely, suppose that A does not satisfy $P(n,k)$. Then there exists a word $\mathbf{x}=x_1x_2\cdots x_k$ so that $\mathcal{F}_{\mathbf{x}}(A)$ contains a critical subfamily $\mathcal{G}$. Since each critical family $\mathscr{E}(m, p)$ and $\mathscr{F}(m, p, q)$ contains a cycle, we may assume without loss of generality that $\mathscr{G} = \mathscr{D}(m) = \{\{j_1\}, \{j_1, j_2\}, \{j_2 j_3\}, \ldots, \{j_m, j_{m+1}\}, \{j_{m+1}\}\}$. Then choose distinct integers $i_0, i_1, i_2, \ldots, i_{m+1}$ so that $x_{i_0} = a_{i_0 j_1}$, $x_{i_{m+1}} = a_{i_{m+1} j_{m+1}}$, and $x_{i_\alpha} = a_{i_\alpha j_\alpha} = a_{i_\alpha j_{\alpha+1}}$ for $\alpha = 1, 2, \ldots, m$. Then we conclude that columns j_α and $j_{\alpha+1}$, for $\alpha = 1, 2, \ldots, m$, belong to the distinct single blocks B_{i_0} and $B_{i_{m+1}}$ which is impossible. The contradiction completes the proof. □

We will find it convenient to provide a graph theoretic interpretation of the preceding result. Recall that a 1-factor F of a graph $G = (V, E)$ is a partitioning of the vertex set V into 2-element subsets so that each subset in the partition is an edge in E. Now suppose that A is a cononical $k \times 2ks$ matrix satisfying $P(s(k+1), k)$ and let i_0 be an integer with $1 \leq i_0 \leq k$. Then let G be a complete graph with vertex set $V = (v_1, v_2, \ldots, v_{2s}\}$. For each i with $1 \leq i \leq k$ and $i \neq i_0$, we define a 1-factor F_i of G by $F_i = \{\{j_1, j_2\}$: there exist integers j_3, j_4 and a letter $x \in \mathbf{n}$ so that $a_{i_0 j_3} = j_1$, $a_{i_0 j_4} = j_2$, and $a_{ij_3} = a_{ij_4} = x.\}$ Then let $G^* = (V, E^*)$ be the subgraph of G consisting of those edges which belong to at least one 1-factor in the collection $\{F_i: 1 \leq i \leq k, i \neq i_0\}$. We note that G^* has $s(k-1)$ edges since it follows that if distinct 1-factors have a common edge, then there exists a word $\mathbf{x}$ so that $\mathscr{F}_{\mathbf{x}}$ contains the cycle $\mathscr{C}(2)$. Furthermore, we conclude from Theorem 18 that G^* does not contain a cycle in which each edge comes from a distinct 1-factor. Conversely, if $G = (V, E)$ is a complete graph and $F_1, F_2, \ldots, F_{k-1}$ are edge disjoint 1-factors of G so that G does not contain a cycle in which each edge comes from a distinct 1-factor, then we may employ these 1-factors to construct each of the single blocks of a canonical $k \times 2ks$ matrix satisfying $P(s(k+1), k)$. We have therefore established the following result.

Theorem 18. *$f(s(k+1), k) = 2ks$ if and only if there exist $k-1$ edge disjoint 1-factors of a complete graph G on $2s$ vertices so that G does not contain a cycle in which each edge comes from a distinct 1-factor.*

Example 19. Let $s = 3$ and $k = 4$ and consider the following 4×24 matrix which satisfies $P(15, 4)$.

$$\left(\begin{array}{cccccc|cccccc|cccccc|cccccc}
1 & 2 & 3 & 4 & 5 & 6 & 10 & 11 & 12 & 10 & 11 & 12 & 15 & 13 & 13 & 14 & 14 & 15 & 7 & 7 & 8 & 8 & 9 & 9 \\
7 & 7 & 8 & 8 & 9 & 9 & 1 & 2 & 3 & 4 & 5 & 6 & 13 & 14 & 15 & 13 & 14 & 15 & 2 & 10 & 10 & 11 & 11 & 12 \\
9 & 7 & 7 & 8 & 8 & 9 & 10 & 10 & 11 & 11 & 12 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 13 & 14 & 15 & 13 & 14 & 15 \\
7 & 8 & 9 & 7 & 8 & 9 & 12 & 10 & 10 & 11 & 11 & 12 & 13 & 13 & 14 & 14 & 15 & 15 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}\right)$$

The first single block of this matrix produces three 1-factors of a complete graph on 6 vertices (see Fig. 1). □

In view of these results, it is natural to define the combinatorial function $g(s)$ as the largest integer p for which there exist p edge disjoint 1-factors of a complete

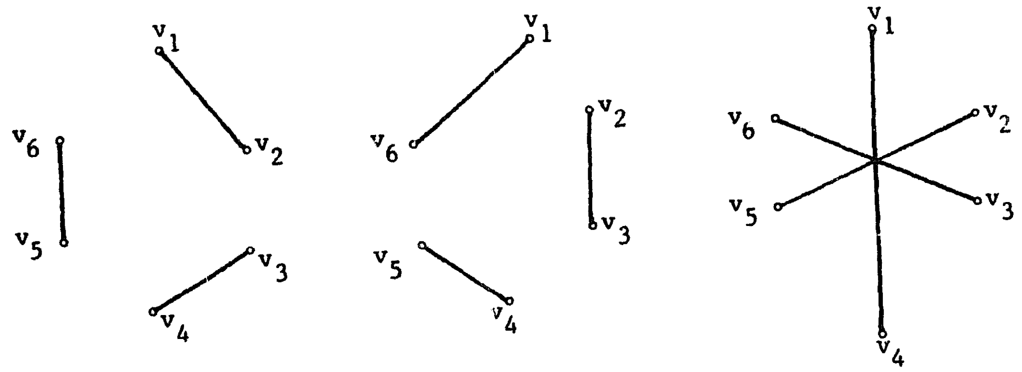

Fig. 1.

graph on $2s$ vertices so that G does not contain a cycle in which each edge comes from a distinct 1-factor.

Example 20. It follows immediately from Turan's theorem that $g(s) \leq s$ for if $g(s) > s$, then there are $sg(s) > s^2 = (2s)^2/4$ edges of G which belong to the 1-factors. Therefore G contains a triangle each of whose edges comes from a 1-factor, but these 1-factors are necessarily distinct. Note that from Example 3, we have $g(3) = 3$, but it is easy to show that $g(4) = 3$ so that this inequality is not best possible. □

Lemma 21. *Let $F_1, F_2, \ldots, F_p$ be 1-factors in a complete graph G on $2s$ vertices so that G does not contain a cycle in which each edge comes from a distinct 1-factor. Then let G^* be the subgraph of G consisting of those edges which come from these 1-factors. Then G^* does not contain a $K_{2,4}$.*

Proof. Suppose G^* contains a $K_{2,4}$ labeled as in Fig. 2. Without loss of generality we may assume that $\{a, x_i\} \in F_i$ for $i = 1, 2, 3, 4$. Now consider the 4-cycle $\{a, x_1, b, x_2\}$. We must either have $\{b, x_2\} \in F_1$ or $\{b, x_1\} \in F_2$. If $\{b, x_2\} \in F_1$, consider the 4-cycle $\{a, x_2, b, x_3\}$. We must have $\{b, x_3\} \in F_2$, but this implies that the 4-cycle $\{a, x_2, b, x_4\}$ has edges from distinct 1-factors. A similar argument holds when $\{b, x_1\} \in F_2$. □

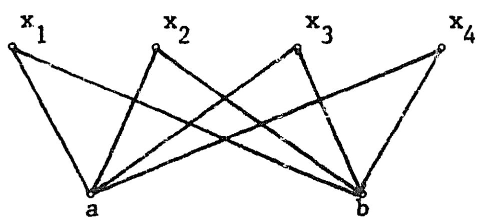

Fig. 2.

Theorem 22. $g(s) \leq \lfloor -2+\sqrt{1+8s} \rfloor$.

Proof. Let $g(s)=p$ and let $F_1, F_2, \ldots, F_p$ be edge disjoint 1-factors of a complete graph on $2s$ vertices so that G does not contain a cycle in which each edge comes from a distinct 1-factor. Then let G^* be the subgraph of G consisting of those edges contained in the 1-factors. Note that G^* is a regular graph of degree p.

Choose an arbitrary vertex $v_0 \in G$ and let $v_1, v_2, \ldots, v_p$ be the neighbors of v_0. Note that no two vertices in the set $\{v_1, v_2, \ldots, v_p\}$ are adjacent since G^* does not contain a triangle. Now let $v_{p+1}, v_{p+2}, \ldots, v_{2p-1}$ be the neighbors of v_1 other than v_0. Since G^* has no $K_{2,4}$, v_2 has at least $p-3$ neighbors which do not come from the set $\{v_0, v_1, \ldots, v_{2p-1}\}$. Label these vertices $v_{2p}, v_{2p+1}, \ldots, v_{3p-4}$. Similarly, v_3 has at least $p-5$ neighbors which do not come from the set $\{v_0, v_1, \ldots, v_{3p-4}\}$. Label these vertices $v_{3p-3}, v_{3p-2}, \ldots, v_{4p-9}$. Continuing in this fashion, we conclude that G^* must contain at least $1+p+(p-1)+(p-3)+(p-5)+\cdots$ vertices, and thus $p \leq \lfloor -2+\sqrt{1+8s} \rfloor$. □

Corollary 23. $g(s) \geq \lfloor 1+\log_2 s \rfloor$.

Proof. Let $p=\lfloor 1+\log_2 s \rfloor$. For each integer $i=1,2,\ldots,p$, let F_i be the 1-factor on $\{1,2,\ldots,2s\}$ defined by $F_i=\{\{j, j+2^i-1\}: 1 \leq j \leq 2s-1,\ j \text{ odd, with } j+2^i-1$ interpreted cyclically$\}$. Suppose that G^* contains a cycle $v_1, v_2, \ldots, v_m$ of length m so that each of the edges $\{v_i, v_{i+1}\}$, for $i=1,2,\ldots,m$, and $\{v_1, v_m\}$ come from distinct 1- factors. Since each edge consists of an odd integer and an even integer, we know that m is even, say $m=2q$. Now suppose that $\{v_i, v_{i+1}\} \in F_{\alpha_i}$, for $i=1,2,\ldots,m-1$, and $\{v_1, v_m\} \in F_{\alpha_m}$. Then it follows $(2^{\alpha_1}-1)-(2^{\alpha_2}-1)+(2^{\alpha_3}-1)-(2^{\alpha_4}-1)+\cdots-(2^{\alpha_m}-1)=0$ which is impossible. □

Although we have not been able to obtain better inequalities for $g(s)$, we note that we have at least proved the following result.

Corollary 24. *For each $k \geq 1$, there exists a constant s_k so that if $s \geq s_k$ and $n=s(k+1)$, then $f(n,k)=\lceil 2kn/(k+1) \rceil = 2ks$.*

Hereafter, we will use the short phrase "the cycle condition is satisfied" to mean that a particular collection $\{F_1, F_2, \ldots, F_p\}$ of 1- factors of a graph (equivalently, locations for mates in rows of a matrix) satisfies the hypothesis given in Lemma 21 and Theorem 22, specifically that the graph determined by the edges in these 1-factors does not contain a cycle in which the edges come from distinct 1-factors.

We now turn our attention to the general problem of determining $f(n,k)$ when n is large compared to k. Surprisingly enough, most of the work has already been done.

Theorem 25. *For each $k \geq 1$, there exists a constant n_k so that if $n \geq n_k$ and $n=s(k+1)+r$ where $0 \leq r < \frac{1}{2}(k+1)$, then $f(n,k)=\lceil 2kn/(k+1) \rceil = 2ks+2r$.*

Proof. The case $r=0$ was treated in Corollary 24, so we may assume that $0<r<\frac{1}{2}(k+1)$. Also note that the conclusion holds for all values of n when $k=1$ or $k=2$, so we may assume that $k\geq 3$. We observe that $\lceil 2kn/(k+1)\rceil = 2ks+2r$ so that $f(n,k)\geq 2ks+2r$. To show that $f(n,k)\leq 2ks+2r$ we simply construct a $k\times(2ks+2r)$ matrix A satisfying $P(n,k)$. We assume that $k-1\leq g(s)$ and that $k-1\leq g(s+r)$.

We consider the matrix A as being partitioned into $k+1$ blocks $B_1, B_2, \ldots, B_{k+1}$. Each of the blocks $B_1, B_2, \ldots, B_k$ is called a single block and contains $2s$ columns of A. Block B_1 contains the first $2s$ columns of A, B_2 contains the next $2s$ columns, etc. Block B_{k+1} contains the last $2r$ columns of A and is called the "dump".

For each $i=1,2,\ldots,k$, the symbols $1,2,\ldots,2s$ are singles in rwo i of A and occur consecutively in single block B_i. Each of the remaining $n-2s$ symbols of $\mathbf{n}$ will be a double in every row of the matrix A so that in order to complete the construction of A, it suffices to describe the location of mates. Blocks $B_1, B_2, \ldots, B_{k-2}$ are constructed as in the proof of Theorem 18, i.e., the location of mates in one of the rows in these blocks (except the row of singles) is viewed as a 1-factor chosen so that the cycle condition is satisfied. Note that we have assumed that $k-1\leq g(s)$ so that this construction is possible. Also note that for each $i=1,2,\ldots,k-2$, the symbols $\{(i-1s+2s+j\colon 1\leq j\leq s\}$ are doubles in each row of block B_i other than row i.

We need only construct the blocks B_{k-1}, B_k, and B_{k+1}. We begin by placing the letters $n-2r+1, n-2r+2, \ldots, n$ in the first $2r$ position in the last row of B_{k-1}. Similarly, we place these same letters in the last row of B_{k+1}.

To complete the construction of B_{k-1}, we choose locations for mates in the first $k-2$ rows of B_{k-1} and the last $2s-2r$ positions in row k of B_{k-1} so that the cycle condition is satisfied. Note that we permit the mate of a letter in block B_k to be in block B_{k+1}.

In order to conclude that the matrix we have constructed satisfies $P(n,k)$, we must show that no $\mathscr{F}_{\mathbf{x}}$ contains a critical subfamily. To the contrary, suppose that $\mathbf{x}\in\mathbf{n}^k$ and that $\mathscr{F}_{\mathbf{x}}$ contains a critical subfamily $\mathscr{F}$. Since singles do not overlap in A, $\mathscr{F}\neq\mathscr{D}(0)$. Since the mate of a double in a row in B_i is also in B_i for $i=1,2,\ldots,k-1$, and the mate of a double in a row in $B_k\cup B_{k+1}$ is also in $B_k\cup B_{k+1}$, it follows that $\mathscr{F}\neq\mathscr{D}(m)$ for every $m\geq 1$. On the other hand, since the cycle condition is satisfied, $\mathscr{F}$ cannot be one of the critical families $\mathscr{E}(m,p)$ or $\mathscr{F}(m,p,q)$; and with this observation we have completed the argument that A satisfies $P(n,k)$. Thus $f(n,k)\leq 2ks+2r=\lceil 2kn/(k+1)\rceil$ and the proof of our theorem is complete. □

Theorem 26. *For each $k\geq 1$, there exists a constant n_k so that if $n\geq n_k$ and $n=s(k+1)+r$, where $\frac{1}{2}(k+1)\leq r\leq k-1$, then*

$$f(n,k)=1+\lceil 2kn/(k+1)\rceil = 2ks+2r.$$

Proof. Note that when $n=s(k+1)+r$ and $\frac{1}{2}(k+1)\leq r\leq k-1$, we have

$\lceil 2kn/(k+1)\rceil = 2ks+2r-1$, so that $f(n,k) \geq 2ks+2r-1$. Suppose first that $f(n,k) = 2ks+2r-1$, and let A be a $k \times (2ks+2r-1)$ matrix satisfying property $P(n,k)$.

Then it follows that each row of A contains at least $2n-(2ks+2r-1) = 2s+1$ singles. Since $k(2s+2) = 2ks+2k > 2ks+2r-1$, we may assume that the first α rows (where $\alpha > 0$) of A contain exactly $2s+1$ singles, and the last $k-\alpha$ rows of A contain at least $2s+2$ singles. Note that $2s+1+(k-1)(2s+2) = 2ks+2k-1 > 2ks+2r-1$ so that $\alpha \geq 2$. We may then assume that A has been partitioned into blocks $B_1, B_2, \ldots, B_k, B_{k+1}$ with the blocks $B_1, B_2, \ldots, B_k$ being single blocks, having single letters in row i of block B_i for $i = 1, 2, \ldots, k$, and B_{k+1} be the dump. We also assume that $B_1, B_2, \ldots, B_\alpha$ each contain $2s+1$ columns and $B_{\alpha+1}, B_{\alpha+2}, \ldots, B_k$ each contain at least $2s+2$ columns.

Since $2s+1$ is odd and no $\mathcal{F}_x$ can contain the critical subfamily $\mathcal{D}(1)$, it follows that for each $i = 2, 3, \ldots, \alpha$, there is at least one double in the first row of A with one appearance in B_i and the other in the dump. This requires that the dump contain at least $\alpha - 1$ columns, and thus A must contain at least $\alpha(2s+1) + (k-\alpha)(2s+2) + \alpha - 1 = 2ks+2k-1$ columns. This is a contradiction since $k > r$.

On the other hand it is strightforward to show that $f(n,k) \leq 2ks+2r$ by constructing a $k \times (2ks+2r)$ matrix A satisfying $P(n,k)$. We begin by partitioning A into blocks $B_1, B_2, \ldots, B_{k+1}$ with $B_1, B_2, \ldots, B_k$ being single blocks each containing $2s$ columns and B_{k+1} being designated as the dump. For each $i = 1, 2, \ldots, k-1$, we construct the block B_i as in the proof of Theorem 26, i.e., we treat the $i-1$ rows of B_i (other than row i where the letters in B_i are singles) as 1-factors of a graph on $2s$ vertices chosen so that the desired cycle condition is satisfied.

To construct B_k and B_{k+1} we choose k 1-factors of a graph on $2s+2r$ vertices so that the cycle condition is satisfied. Clearly we may assume that the last of these 1-factors satisfies the additional requirement that the mate of any vertex in the last $2r$ vertices is also in the last $2r$ vertices, i.e., the restriction of this 1-factor to the last $2r$ vertices is also a 1-factor. We then use the first $k-1$ of these 1-factors to determine the location of mates in the first $k-1$ rows of $B_k \cup B_{k+1}$. Finally, we use the last 1-factor to determine the location of mates in the last row of B_{k+1}. It is easy to see that A satisfies $P(n,k)$, and therefore $f(n,k) \leq 2ks+2r = 1 + \lceil 2kn/(k+1)\rceil$. □

Note that the first part of the argument in the proof of Theorem 26 fails when $r = k$. Although we do not include the details of the construction, we note that if n is sufficiently large and $n = s(k+1)+k$, then $f(n,k) = \lceil 2kn/(k+1)\rceil = 2ks+2k-1$. In this case, a $k \times (2ks+2k-1)$ matrix A satisfying $P(n,k)$ can be constructed by partitioning A into single blocks $B_1, B_2, \ldots, B_k$ where B_1 contains $2s+1$ columns and $B_2, B_3, \ldots, B_k$ each contain $2s+2$ columns. The odd entry in each row of B_1 other than row 1 is an asterisk.

We summarize the preceding results as follows.

Theorem 27. *For each $k \geq 1$, there exists a constant n_k so that if $n \geq n_k$ and $n = s(k+1)+r$, where $0 \leq r \leq k$, then*

$$f(n,k) = \begin{cases} \left\lceil \frac{2kn}{k+1} \right\rceil & \text{when } 0 \leq r < \frac{1}{2}(k+1) \text{ and when } r = k, \\ 1 + \left\lceil \frac{2kn}{k+1} \right\rceil & \text{when } \frac{1}{2}(k+1) \leq r < k. \end{cases}$$

It is interesting to comment that the second part of Theorem 27 never applies when $k = 2$. In fact it is easy to establish the first few cases in order to derive the following result.

Corollary 28. *For each $n \geq 1$, $f(n, 2) = \lceil \frac{4}{3}n \rceil$.*

Although we do not include the details here, the reader may verify that $f(1,3) = 3$, $f(2,3) = 4$, $f(3,3) = 5$, $f(4,3) = 7$, and that the formula in Theorem 27 holds for $f(n, 3)$ when $n \geq 5$.

3. Probabilistic methods

In the preceding section, a precise formula for $f(n, k)$ which holds when n is sufficiently large compared to k was given. The situation is more complicated when k is large compared to n. First note that the inequality $f(n,k) \geq \lceil 2kn/(k+1) \rceil$ is weaker than the inequality $f(n,k) \geq k+n-1$ when k is large. In fact, $2kn/(k+1) \leq k+n-1$ when $k \geq n-1$. In this section we use probabilistic methods to determine an upper bound on $f(n, k)$ when k is large compared to n. Precise determination of $f(n, k)$ appears to be extremely difficult.

Theorem 29. *If $f(n, k) \geq t$, then*

$$\sum_{\alpha=1}^{k} n^{\alpha} \binom{k}{\alpha} \binom{t}{\alpha-1} (n-1)^{\alpha(t-\alpha+1)} n^{kt-\alpha(t-\alpha+1)} \geq n^{kt}.$$

Proof. Let $\mathcal{M}$ be the collection of all $k \times t$ matrices with entries from $\mathbf{n}$. If $f(n,k) > t$, then every matrix in $\mathcal{M}$ fails to satisfy property $P(n, k)$, i.e., for every $A \in \mathcal{M}$, there exists a word $\mathbf{x} \in \mathbf{n}^k$ for which the family $\mathcal{F}_{\mathbf{x}}(A)$ does not have a SDR. If $\mathcal{F}_{\mathbf{x}}(A) = \{A_i : 1 \leq i \leq k\}$, then there is a subfamily $\mathcal{F}'_{\mathbf{x}}(A) = \{A_{i_1}, A_{i_2}, \ldots, A_{i_\alpha}\}$ with $1 \leq i_1 < i_2 < \cdots < i_\alpha \leq k$ so that $\mathcal{F}'_{\mathbf{x}}(A)$ does not have a SDR, but every proper nonempty subfamily of $\mathcal{F}'_{\mathbf{x}}(A)$ has a SDR. The subfamily $\mathcal{F}'_{\mathbf{x}}(A)$ and the subword $\mathbf{x}' = x_{i_1}, x_{i_2} \cdots x_{i_\alpha}$ are said to be minimal for A. Not that it follows immediately from Hall's theorem that $|\bigcup \mathcal{F}'_{\mathbf{x}}(A)| = \alpha - 1$.

Now let $\mathbf{x}$ be any word from $\mathbf{n}^k$ and let $\mathbf{x}' = x_{i_1} x_{i_2} \cdots x_{i_\alpha}$ be a subword of $\mathbf{x}$. Then let $\mathcal{M}(\mathbf{x}') = \{A \in \mathcal{M}: \mathcal{F}_{\mathbf{x}}(A)$ does not have a SDR and $\mathbf{x}'$ is minimal for $A\}$. We next obtain an upper bound on $|\mathcal{M}(\mathbf{x}')|$ in terms of n, k and α.

Let $R = \{i_1, i_2, \ldots, i_\alpha\}$. Then let $C = \{j$: there exists $i \in R$ so that $x_i = a_{ij}\}$. Note that $|R| = \alpha$ and $|C| = \alpha - 1$. To obtain an upper bound on $|\mathcal{M}(\mathbf{x}')|$, we then note that if $i \in R$ and $j \notin C$, then a_{ij} can be any letter except x_i. On the other hand, it is a generous estimate to allow all other entries in A to be any letter in $\mathbf{n}$. Since C is an $\alpha - 1$ element subset of $\{1, 2, \ldots, t\}$ and there are $\alpha(t - \alpha + 1)$ entries a_{ij} where $i \in R$ and $j \notin C$, it follows that

$$|\mathcal{M}(\mathbf{x}')| \leq \binom{t}{\alpha - 1}(n-1)^{\alpha(t-\alpha+1)} n^{kt-\alpha(t-\alpha+1)}.$$

Since R is an α element subset of $\{1, 2, \ldots, k\}$ and there are n choices for each of the letters in $\mathbf{x}'$, it follows that

$$n^{kt} = |\mathcal{M}| \leq \sum_{\alpha=1}^{k} n^{\alpha}\binom{k}{\alpha}\binom{t}{\alpha - 1}(n-1)^{\alpha(t-\alpha+1)} n^{kt-\alpha(t-\alpha+1)}. \qquad \square$$

In order to obtain an upper bound on $f(n, k)$ it suffices to show that if k is sufficiently large compared to n and if t is suitably large, then the inequality in the preceding theorem fails. Although we do not include the details here, the following bound can be establised by this method.

Theorem 30. *For each $n \geq 1$, there exists a constant k_n so that if $k \geq k_n$, then*

$$f(n, k) \leq k + \log n + n + o(n).$$

It would be interesting to provide a constructive upper bound for $f(n, k)$ when k is large as well as to provide some information on $f(n, k)$ when n and k are of comparable size. However, such results are not likely to be easy.

4. Some comments on the origin of the problem

The problem of computing $f(n, k)$ surfaced originally in attempts to settle a tantalizing combinatorial problem involving partially ordered sets. In the interests of brevity we provide only the basic definitions necessary to discuss this problem and refer the reader to [3] and [6] for additional material. Recall that a partially ordered set is a pair (X, P) where X is a finite set and P is a reflexive, antisymmetric, and transitive relation on X. The dimension of (X, P), denoted $\text{Dim}(X, P)$, is the least positive integer t for which there exists a function f which assigns to each point $x \in X$ a sequence $f(x)(1), f(x)(2), \ldots, f(x)(t)$ of real numbers so that $(x, y) \in P$ if and only if $f(x)(i) \leq f(y)(i)$ for $i = 1, 2, \ldots, t$. One of the best known inequalities for the dimension of posets is Hiraguchi's inequality: $\text{Dim}(X, P) \leq \frac{1}{2}|X|$ when $|X| \geq 4$ (see [3] and [4]). In view of this inequality it seems reasonable to conjecture that every poset (of at least 3 points) contains a pair of points whose removal decreases the dimension at most one, and in fact, there are numerous conditions under which this is true (see [1] and [5]).

On the other hand, if one attempts to construct a poset for which the conjecture fails, it is natural to consider the following family of posets. For an integer $n \geq 1$, let (X_n, P_n) be the poset with n maximal points $a_1, a_2, \ldots, a_n$, n minimal points $b_1, b_2, \ldots, b_n$, and n^2 other points $\{x_{ij}: 1 \leq i \leq n, 1 \leq j \leq n\}$ so that $x_{ij} < a_\alpha$ if and only if $i \neq \alpha$ and $b_\beta < x_{ij}$ if and only if $j \neq \beta$. It is then straightforward to establish the following result which relates the computation of dimension to the determination of $f(n, k)$.

Theorem 31. *For* $n \geq 1$, $\text{Dim}(X_n, P_n) = f(n, 2) = \lceil \frac{4}{3} n \rceil$.

While this family of posets does not settle the conjecture, it comes close as one can establish the following result.

Theorem 32. *If* $n \geq 2$ *and* $n \equiv 1 \pmod 3$, *then the removal of any two maximal points, any two minimal points, or a maximal point and a minimal point decreases the dimension of* (X_n, P_n) *by two.*

We leave it to the reader to find an appropriate pair of points in (X_n, P_n) whose removal decreases the dimension by one (such a pair exists). However, the original conjecture remains unsettled.

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