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Graphs and Partially Ordered Sets

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1. Introduction

Since graphs are simple and elegant structures, it is not surprising that they have been studied intensively. In contrast, partially ordered sets have considerably more structure and are therefore viewed by some as being less elegant. In recent years, however, there has been a resurgence of interest in partially ordered sets and their combinatorial properties. In something of a reversal of roles, other mathematical structures, such as graphs, groups, and lattices, have been used to study partial orders, rather than the ordered sets being the tools. Results of these investigations appear to have justified this approach, and the theory of partially ordered sets has shed light on a number of combinatorial problems.

In this chapter, we survey some of the theory of partially ordered sets. In keeping with the theme of this book, we concentrate on topics related to graphs. Using the concepts of the dimension and rank of partially ordered sets, we explore topics involving graph colorings, planar graphs, forbidden subgraphs, and extremal digraphs.

The next two sections provide the fundamental definitions and notation for partially ordered sets, and introduce the concepts of dimension and rank.

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In Section 4, we give a construction of a hypergraph from a partially ordered set in such a way that the chromatic number of the former equals the dimension of the latter. (In many cases, the hypergraph is a graph whose chromatic number is easily found.) In Section 5, we discuss the connections between the dimension of a partially ordered set and the planarity of its Hasse diagram. Section 6 is devoted to the interplay between dimension and certain intersection graphs, such as interval graphs.

In Section 7, we turn our attention to the concept of rank, and present an algorithm for its determination as the maximum number of arcs in a digraph of a particular type. For one class of partially ordered sets, this algorithm reduces to an extremal digraph problem whose solution generalizes Turán's theorem. We close the chapter with a compilation of some open problems.

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2. Definitions and Terminology

A **partially ordered set** (X, P), or **poset** for short, consists of a non-empty set X and a binary relation P on X which is reflexive, anti-symmetric and transitive. The elements of X are called **points**, and the relation P is called a **partial ordering** on X. An example is the collection of pairs $P = \{(a, a), (b, b), (c, c), (d, d), (e, e), (a, b), (a, c), (d, c), (e, a), (e, d), (e, c), (e, b)\}$ on the set $X = \{a, b, c, d, e\}$.

For convenience, both xPy and $x \leqslant y$ are used to denote $(x, y) \in P$. In addition, we let $x < y$ denote that $x \leqslant y$ and $x \neq y$, in the customary way. If $x < y$, and if there is no point z such that $x < z < y$, then y is said to **cover** z. For finite posets, it is clear that the entire relation is determined by the covering relation. In our example, this is just the set $\{(a, b), (a, c), (d, c), (e, a), (e, d)\}$.

This covering relation is frequently used in representing a poset diagrammatically. A **Hasse diagram** of a poset (X, P) is a drawing in which the points of X are placed so that if y covers x, then y is placed at a higher level than x and joined to x by a line segment. The corresponding graph is called the **Hasse graph** of the poset. For our example, a Hasse diagram is shown in Fig. 1.

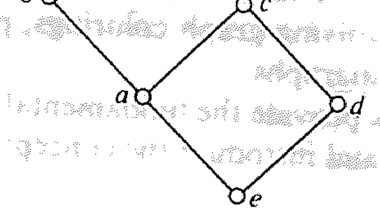


Fig. 1

Two distinct elements x and y in a poset (X, P) are called **comparable** if either $x < y$ or $y < x$, and **incomparable** otherwise. We denote the fact that x and y are incomparable by $x \| y$. The binary relation C_P consisting of all comparable pairs of (X, P) is called the **comparability relation**. It is clearly symmetric, and its graph is called the **comparability graph** of the poset. The **incomparability relation** I_P and **incomparability graph** are defined similarly. Figure 2 shows these graphs for the example given above. Note that the incomparability graph of a poset is always the complement of the comparability graph.

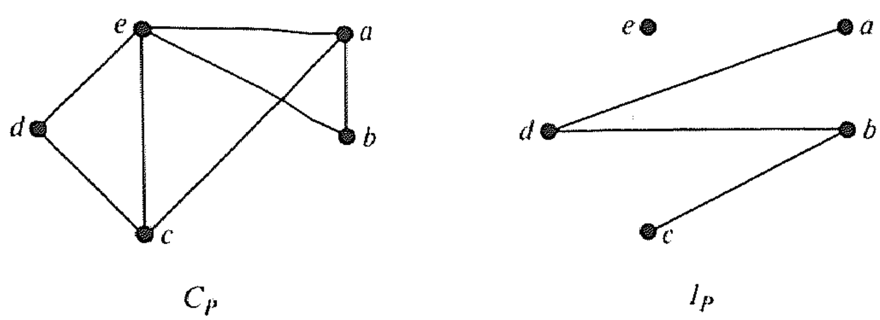


Fig. 2

A poset in which any two elements are comparable is called a **chain** (or **linear order** or **total order**), and one in which no two elements are comparable is called an **antichain** (or **unordered set**). The size of a largest chain in a poset is called the **length** of the poset, and that of a largest antichain is called its **width**. Thus, the length of the poset shown in Fig. 1 is 3, and its width is 2.

At times, it is convenient to use a single symbol to denote a poset, such as **X** for (X, P). In particular, we denote by

R, the real numbers with the usual order;
$\underline{n}$, an n-chain;
and $\bar{n}$, an n-antichain.

The **dual** $\hat{P}$ of a binary relation P is the set of pairs (x, y) for which $(y, x) \in P$. When P is a partial ordering on X, $\hat{P}$ is also a partial ordering, and it is natural to refer to $(X, \hat{P})$ as the **dual** of (X, P). Note that if a Hasse diagram of (X, P) is inverted, then the result is a diagram of its dual.

A **subposet** of a poset (X, P) is a poset (Y, Q) in which $Y \subseteq X$ and Q is the restriction P_Y of P to $Y \times Y$. Note that under this definition a subposet is determined by its set of points.

Two posets (X, P) and (X', P') are called **isomorphic** if there is a one-to-one correspondence $\phi: X \to X'$ such that $x \leqslant y$ in P if and only if $\phi(x) \leqslant \phi(y)$ in P'. In general, we do not distinguish between isomorphic posets, and we frequently use equality to denote isomorphism.

The poset (Y, Q) is said to be **embedded** or **contained** in (X, P), denoted by $(Y, Q) \subseteq (X, P)$, if (Y, Q) is isomorphic to a subposet of (X, P).

Our next concept, which will be the basis of the definitions of dimension and rank in the next section, is in contrast to an embedding. If P and Q are two partial orders on the same set X, we call Q an **extension** of P if $P \subseteq Q$; it is a **linear extension** if (in addition) Q is a chain.

We conclude this section with a theorem on linear extensions, due to Szpilrajn [**37**]:

Theorem 2.1. (*i*) *Every partial ordering P of a set X has a linear extension;* (*ii*) *the intersection of all linear extensions of P is P itself.* ‖

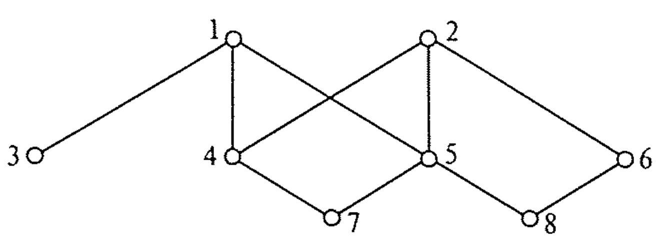
3. Dimension and Rank

In this section we introduce two concepts which are of interest, both in their own right and in connection with more graph-theoretic concepts (as we shall see in later sections).

To begin with, we recall Theorem 2.1(*ii*) which states that every partial ordering is determined as the intersection of its linear extensions. This result can be restated as follows: for any two incomparable elements x and y in a poset (X, P), there is one linear extension of P in which $x < y$, and another in which $y < x$.

In general, however, one does not need all of its linear extensions to determine a partial order P. A **realizer** of P is any collection R of linear extensions whose intersection is P. Alternatively, a collection $R = \{L_1, L_2, \ldots, L_t\}$ of linear extensions of P is a realizer of P when $x < y$ in P if and only if $x < y$ in every L_i. Notationally, it is convenient to let L: $[x_1, x_2, \ldots, x_n]$ denote the linear order on $\{x_1, x_2, \ldots, x_n\}$ in which $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$. For an example of a realizer, consider the poset in Fig. 3, for which the four linear extensions indicated constitute a realizer.

The **dimension** $\dim(X, P)$ of a poset (X, P) is the minimum order of a realizer of P. This definition was first made in the historic paper of Dushnik and Miller [**10**]. We observe that a poset has dimension 1 if and only if it is a chain. An example of a poset of dimension 2 is the n-element antichain $\bar{n}$ (for $n \geqslant 2$) since, for any linear order L, $\{L, \hat{L}\}$ is a realizer.



L_1: [3, 7, 8, 4, 5, 1, 6, 2]
L_2: [7, 4, 8, 5, 6, 2, 3, 1]
L_3: [8, 7, 5, 3, 4, 6, 1, 2]
L_4: [8, 6, 7, 3, 4, 5, 1, 2]

Fig. 3

For $n \geqslant 3$, we define the poset $\mathbf{S}_n^0$ to consist of n maximal elements $a_1, a_2, \ldots, a_n$, and n minimal elements $b_1, b_2, \ldots, b_n$, with $b_i < a_j$ if $i \neq j$. This poset is indicated in Fig. 4, and we note that $\mathbf{S}_n^0$ is isomorphic to the poset of the 1-element and $(n-1)$-element subsets of an n-element set, ordered by inclusion.

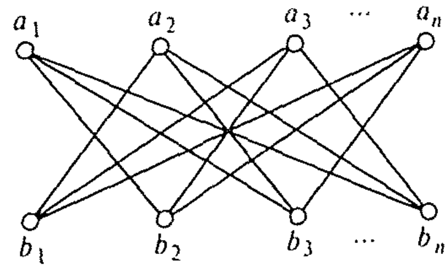


Fig. 4

Theorem 3.1. *The dimension of* $\mathbf{S}_n^0$ *is* n.

Proof. First consider the set $R = \{L_1, L_2, \ldots, L_n\}$ of linear extensions, where

$$L_k\colon [b_1, \ldots, b_{k-1}, b_{k+1}, \ldots, b_n, a_k, b_k, a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n].$$

We observe that, if $i \neq j$, then $b_j < a_i < b_i < a_j$ in L_i, and $b_i < a_j < b_j < a_i$ in L_j. It follows that R is a realizer of $\mathbf{S}_n^0$, and hence that $\dim \mathbf{S}_n^0 \leqslant n$.

On the other hand, if S is any realizer of $\mathbf{S}_n^0$, then for each $k = 1, 2, \ldots, n$. some element of S must have $a_k < b_k$. Furthermore, it is easy to see that no linear extension L of $\mathbf{S}_n^0$ can have $a_i < b_i$ and $a_j < b_j$, for $i \neq j$. It follows that $\dim \mathbf{S}_n^0 \geqslant n$, and the proof is complete. ‖

This poset $\mathbf{S}_n^0$ is known as the **standard *n*-dimensional poset**. As was shown first by Hiraguchi [16], and later by Bogart [2], it has the minimum number of elements among the n-dimensional posets. In this sense, it plays a role in dimension theory analogous to that played by the complete graph in chromatic graph theory. We shall say more about this analogy in the next section.

An alternative definition of dimension, in terms of coordinates, was given by Ore [31]. Let $\mathbf{R}^t$ denote the poset of all t-tuples of real numbers, partially ordered by inequality in each coordinate—that is, $(a_1, \ldots, a_t) \leqslant (b_1, \ldots, b_t)$ if and only if each $a_i \leqslant b_i$. Then $\dim(X, P)$ is the minimum number t such that $(X, P) \subseteq \mathbf{R}^t$. For example, consider the poset (X, P) in Fig. 5. The given coordinates show that its dimension is at most 3. This coordinatization corresponds to the three linear extensions $L_1\colon [e, b, f, d, a, c]$; $L_2\colon [e, f, c, b, d, a]$; $L_3\colon [f, d, e, b, c, a]$. You may like to show that $\dim(X, P) \geqslant 2$ by proving that there is no realizer of order 2; for a more systematic approach, see Section 4.

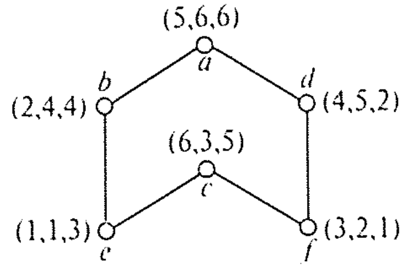


Fig. 5

We now turn our attention to the concept of *rank*, which was first defined by Maurer and Rabinovitch [27]. A realizer R of a poset $\mathbf{X}$ is called **irredundant** if no proper subset of R is a realizer of $\mathbf{X}$. The **rank** of a poset $\mathbf{X}$, denoted by rank $\mathbf{X}$, is the maximum order of an irredundant realizer. Clearly, a realizer of $\mathbf{X}$ with order dim $\mathbf{X}$ is irredundant, and so the dimension of a poset never exceeds its rank. A poset has rank 1 if and only if it is a chain.

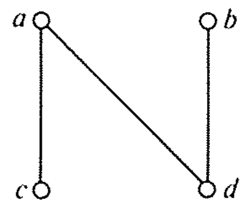


Fig. 6

As a further example, consider the poset $\mathbf{X} = (X, P)$ in Fig. 6. It has only five linear extensions:

$$L_1: [c, d, a, b]; \quad L_2: [d, c, a, b]; \quad L_3: [c, d, b, a];$$
$$L_4: [d, c, b, a]; \quad L_5: [d, b, c, a].$$

Therefore, $2 \leqslant \dim \mathbf{X} \leqslant \operatorname{rank} \mathbf{X} \leqslant 5$. However, b and c are incomparable in P so that, since $c < b$ in each L_i except L_5, any realizer must contain L_5. Furthermore, since $\{L_1, L_5\}$ is a realizer, $\dim \mathbf{X} = 2$, and since the set of all five is redundant, rank $\mathbf{X} \leqslant 4$. It is not difficult to use this to verify that rank $\mathbf{X} = 3$, and that $\{L_2, L_3, L_5\}$ is the only maximum irredundant realizer of $\mathbf{X}$.

Our next example, due to Maurer and Rabinovitch [27], shows that the rank of a 2-dimensional poset can be arbitrarily large. This example is the antichain $\overline{2n}$, in which we take X to be $\{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$. The family R of n^2 linear extensions

$$L_{ij}: [a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n, b_j, a_i, b_1, \ldots, b_{j-1}, b_{j+1}, \ldots, b_n]$$

is easily seen to be a realizer. Its irredundancy follows from the fact that, except for L_{ij}, $a_i < b_j$ in each extension in R. The rank of $\overline{2n}$ is therefore at least n^2.

We now turn to the rank of the standard n-dimensional poset:

Theorem 3.2. *The rank of S_n^0 is n.*

Proof. That n is a lower bound for the rank of S_n^0 follows from Theorem 3.1. To see that it is an upper bound, we observe that a family R of linear extensions of S_n^0 is a realizer if and only if, for $i = 1, 2, \ldots, n$, there exists $L_i \in R$ in which $a_i < b_i$. ‖

We note in passing that Maurer, Rabinovitch and Trotter [30] have determined those posets **X** for which dim **X** = rank **X**.

These examples suggest that it would be useful to have some further techniques for deciding whether or not a family of linear extensions of a partial order P is a realizer of P. To this end, we define an incomparable pair (x, y) of (X, P) to be a **non-forced pair** if $P \cup \{(x, y)\}$ is also a partial order. In other words, an incomparable pair (x, y) is non-forced if and only if $z < x$ implies $z < y$, and $z > y$ implies $z > x$. The set N_P of non-forced pairs can be considered as a digraph with vertex-set X; an example is shown in Fig. 7. Given a family R of linear extensions of a partial order P, and a subset S of the incomparable pairs of P, we say that R **reverses** S if, for each pair (x, y) in S, $(y, x) \in L$ for some L in R. Our interest in non-forced pairs is explained by the following elementary result of Maurer, Rabinovitch and Trotter [28]:

Theorem 3.3. *A family R of linear extensions of a poset (X, P) is a realizer of P if and only if R reverses the set N_P of non-forced pairs.* ‖

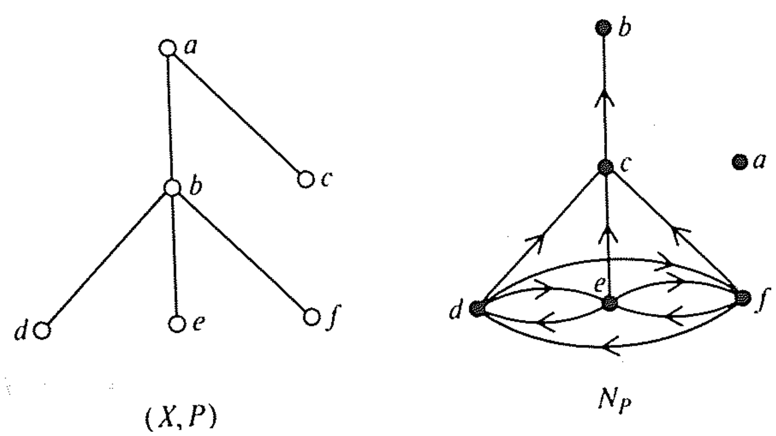


Fig. 7

Another question of some interest is when a given set of incomparable pairs can be included in a linear extension. In order to answer this, we need some further definitions. A sequence of incomparable pairs $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ in I_P is called a ***P*-alternating cycle** if $b_1 \leqslant a_2$, $b_2 \leqslant a_3, \ldots,$ and $b_n \leqslant a_1$, and is called a **strong *P*-alternating cycle** if, in addition, $b_i \nleqslant a_j$ for all other pairs. For example, in the poset of Fig. 8, the five pairs labeled (a_i, b_i) form an alternating cycle, whereas $(a_1, b_1)(a_3, b_3)$ is a strong alternating cycle.

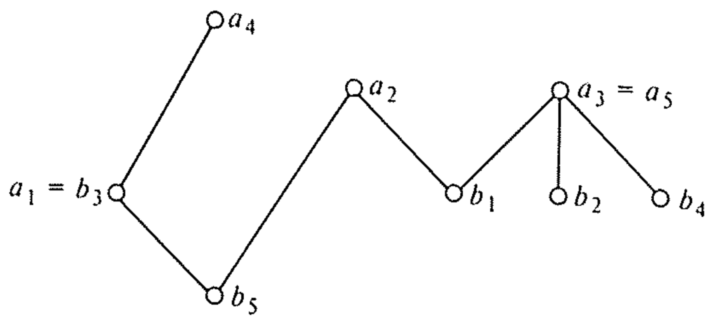


Fig. 8

The following result was proved by Trotter and Moore [**50**]:

Theorem 3.4. *Let (X, P) be a poset, and let S be a set of incomparable pairs. Then the following statements are equivalent:*

(*1*) *there exists a linear extension L of P such that $S \subseteq L$;*

(*2*) *no subset of S forms a P-alternating cycle;*

(*3*) *no subset of S forms a strong P-alternating cycle.* ‖

Alternating cycles may be used to provide an alternative definition of dimension:

Theorem 3.5. *If (X, P) is not a chain, then* dim (X, P) *is the least number t for which there exists a partition of I_P into t subsets, none of which contains a subset which forms a P-alternating cycle.* ‖

4. Dimension and Chromatic Number

In this section we limit our attention to those posets which are not chains; thus $I_P \neq \varnothing$. We define the **associated hypergraph** $H_{\mathbf{X}}$ of such a poset $\mathbf{X} = (X, P)$ as follows: the vertex-set of $H_{\mathbf{X}}$ is the set N_P of non-forced pairs, and a subset S of N_P is an edge if and only if its dual $\hat{S}$ is a strong P-alternating cycle. We define the **chromatic number $\chi(H)$ of a hypergraph H** to be the minimum number of colors required to color the vertices of H so that no edge of H has all of its vertices colored the same. The following result is actually a corollary of Theorems 3.4 and 3.5:

Theorem 4.1. *Let $\mathbf{X} = (X, P)$ be a poset and let $H_{\mathbf{X}}$ be its associated hypergraph. Then* dim $\mathbf{X} = \chi(H_{\mathbf{X}})$. ‖

We now present several partial orders whose dimensions can be readily computed using the preceding theorem, and we discuss their role in some of the theorems in dimension theory. Our first example clarifies our previous comment on the analogy between the standard example of an n-dimensional poset and a complete graph on n vertices. Let $n \geqslant 3$, and let $\mathbf{X}$ be $\mathbf{S}_n^0$, the standard n-dimensional poset. Then $N_P = \{(b_i, a_i): 1 \leqslant i \leqslant n\}$, and the associated hypergraph is a simple graph—the complete graph K_n. Thus $\dim \mathbf{S}_n^0 = \chi(K_n) = n$.

The posets $\mathbf{S}_n^0$ figure in several theorems in dimension theory. Hiraguchi [**16**] proved that, if $|X| \geqslant 4$, then $\dim(X, P) \leqslant \frac{1}{2}|X|$. Bogart and Trotter [**4**] and Kimble [**23**] gave a forbidden subposet characterization of this inequality, which can be summarized by saying that, if $|X| \leqslant 2n + 1$ and $n \geqslant 4$, then $\dim(X, P) < n$, unless (X, P) contains $\mathbf{S}_n^0$. Hiraguchi also proved that the dimension of a poset does not exceeds its width, and the posets $\mathbf{S}_n^0$ show that this inequality is best possible. Finding a forbidden subposet characterization of this inequality appears to be a difficult problem.

Kimble [**23**] and Trotter [**41**] proved a dual result by showing that, if A is an antichain in a poset (X, P), and if $|X - A| \geqslant 2$, then $\dim(X, P) \leqslant |X - A|$. Trotter [**43**] gave a forbidden subposet characterization of this inequality, involving a family of posets whose regular structure can be explicitly described. When $|X - A| = n$, this family includes $\mathbf{S}_n^0$. In [**39**], Trotter constructed a family of posets called *crowns*, and computed their dimension; this family also contains $\mathbf{S}_n^0$.

We next consider the poset $\mathbf{X}$, previously discussed in Section 3, whose Hasse diagram is given in Fig. 9. The associated hypergraph is again a simple graph—the circuit graph C_5.

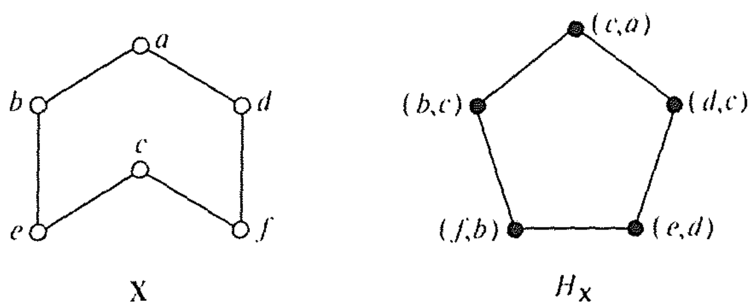


Fig. 9

For $n \geqslant 3$, let $\mathbf{X}$ be the $(n + 1)$-dimensional poset indicated in Fig. 10, whose associated hypergraph is the simple graph $H_\mathbf{X}$. To see that $\chi(H_\mathbf{X}) = n + 1$, note that $\chi(H_\mathbf{X}) \geqslant n$, since the subgraph $\{(a_i, c_i): 1 \leqslant i \leqslant n\}$ is complete. Now suppose that $\chi(H_\mathbf{X}) = n$, and that f is an n-coloring. Then we may assume, without loss of generality, that $f((a_i, c_i)) = i$, for $i = 1, 2, \ldots, n$. Since (q, b_i) is adjacent to (a_j, c_j) when $i \neq j$, we must have $f(q, b_i) = i$, for $i = 1, 2, \ldots, n$. However, it is then impossible for f to assign a color to the vertex (p, q). Thus $\chi(H_\mathbf{X}) > n$. On the other hand, assigning to (p, q) the color $n + 1$ shows that $\chi(H_\mathbf{X}) = n + 1$.

In [**42**], Trotter proved that if A is the set of maximal (or minimal) elements in a poset (X, P), and if $X - A \neq \varnothing$, then the dimension of (X, P)

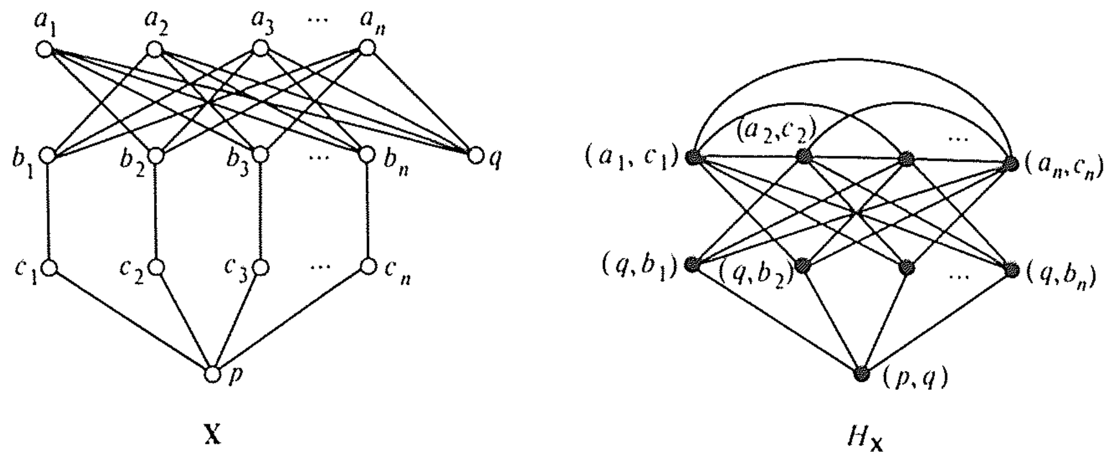


Fig. 10

does not exceed one more than the width of the subposet $(X - A, P_{X-A})$. The posets in Figs 9 and 10 show that this inequality is best possible. Trotter **[42]** also proved that if A is an arbitrary antichain in a poset (X, P), and if $X - A \neq \varnothing$, then the dimension of (X, P) does not exceed one more than twice the width of $(X - A, P_{X-A})$. The posets constructed in **[41]** show that this inequality is also best possible.

Next let $n \geqslant 1$, and let $\mathbf{X}_n$ be the $(2n + 5)$-element poset given in Fig. 11.

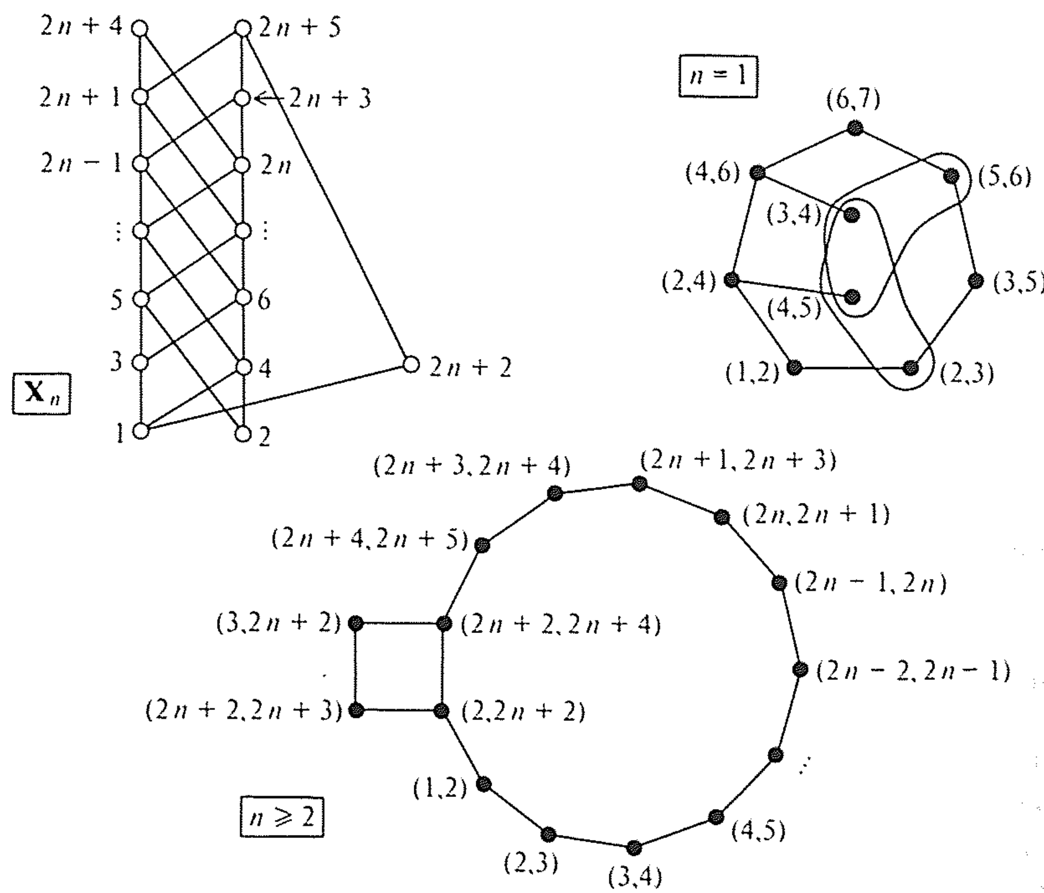


Fig. 11

When $n = 1$, the hypergraph associated with this poset is a 3-chromatic hypergraph. This hypergraph has two edges, each containing three vertices. The remaining nine edges form a simple graph which contains a circuit of length seven. When $n \geqslant 2$, the hypergraph associated with $\mathbf{X}_n$ is a 3-chromatic simple graph containing an odd circuit of length $2n + 5$.

For any integer $t \geqslant 2$, a poset $\mathbf{X}$ is said to be ***t*-irreducible** if the dimension of $\mathbf{X}$ is t and if the dimension of every proper non-empty subposet of $\mathbf{X}$ is less than t. Hiraguchi [16] proved that the removal of a point from a poset can decrease its dimension by at most 1, so a poset $\mathbf{X} = (X, P)$ is t-irreducible if $\dim(X, P) = t$ and if $\dim(X - \{x\}, P_{X-\{x\}}) = t - 1$ for every $x \in X$. The only 2-irreducible poset is a 2-element antichain. There are infinitely many 3-irreducible posets, and they can be conveniently grouped into nine infinite families with eighteen odd examples left over. The complete determination of these posets was made independently by Kelly [20] and by Trotter and Moore [49]; Kelly's approach was lattice-theoretic whereas Trotter and Moore's was graph-theoretic. We discuss this subject in greater detail in Sections 5 and 6.

Each of the posets in the last four examples is irreducible, and other examples of irreducible posets were given in [21], [42], [52] and [53]. Using a construction motivated by Toft's construction [38] of color-critical graphs with a large number of edges relative to the number of vertices, Trotter and Ross [52] proved that every t-irreducible poset can be embedded in a $(t + 1)$-irreducible poset. Using the family of irreducible posets in Fig. 11 and Kelly's dimension product [21], Trotter and Ross [53] subsequently proved that, for $t \geqslant 3$, every t-dimensional poset is a subposet of a $(t + 1)$-irreducible poset. Note that this result is false when $t = 2$, since no 2-dimensional poset whose length and width both exceed 5 can be a subposet of a 3-irreducible poset.

A review of the examples presented thus far may mislead one into believing that the hypergraph $H_{\mathbf{X}}$ always contains a subgraph which is a simple graph with the same chromatic number. Cogis [5] and Doignon [8], who have investigated many topics related to dimension theory, conjectured that if we let $G_{\mathbf{X}}$ denote the graph whose vertex-set is N_P and whose edge-set contains only those edges in $H_{\mathbf{X}}$ which contain exactly two vertices, then $G_{\mathbf{X}}$ and $H_{\mathbf{X}}$ have the same chromatic number. The next example shows that this conjecture is false, and explains why we must respect edges of all sizes in coloring the hypergraph $H_{\mathbf{X}}$.

We let $n \geqslant 3$, and construct a poset $\mathbf{X} = \mathbf{X}_n$ which contains three disjoint subposets $\mathbf{Y}_1$, $\mathbf{Y}_2$ and $\mathbf{Y}_3$, each of which is a copy of $\mathbf{S}_n^0$ and in which the minimal elements of $\mathbf{Y}_i$ are less than the maximal elements of $\mathbf{Y}_{i+1}$ in a cyclic fashion. It is easy to see that, if (x, y) is a non-forced pair in $\mathbf{X}$, then x is a minimal element and y is a maximal element. Let $N_P = N_1 \cup N_2$, where N_1 contains those non-forced pairs (x, y) such that x and y come from the same copy of $\mathbf{S}_n^0$, and N_2 contains those pairs (x, y) such that x and y come from different copies of $\mathbf{S}_n^0$. Note that $|N_1| = 3n$, and $|N_2| = 3n^2$. We observe that the hypergraph H_X contains n^3 edges, each containing exactly three vertices from N_1, and none of which is present in the simple graph G_X. At this point, note that the chromatic number of the hypergraph H_X is at least $\frac{3}{2}n$, since no three vertices in N_1 can be assigned the same color. (Of course, it is also true that certain pairs of vertices in N_1 cannot be assigned the same color.) Now consider the problem of coloring the graph G_X. It is easy to see that n colors suffice to color N_1. Furthermore, one additional color may be used to color all of the vertices in N_2, since no two of these vertices are adjacent in G_X. Thus $\chi(G_X) \leqslant n + 1$. Furthermore, when $n \geqslant 4$,

$$\chi(G_X) \leqslant n + 1 < \tfrac{3}{2}n \leqslant \chi(H_X) = \dim \mathbf{X}.$$

The analogies between graph coloring and dimension theory suggest many problems for future investigation. Central among them are developing analogues of Brooks' theorem and Vizing's theorem for posets, and studying irreducible posets by classifying their associated hypergraphs.

5. Dimension and Planarity

Perhaps no topic in graph theory has attracted more attention than the subject of planar graphs and their chromatic numbers. In this section, we discuss the relationship between the dimension of posets and the planarity of their Hasse diagrams. In addition to some interesting mathematical consequences, this relationship yields some tantalizing unsolved problems of intrinsic graph-theoretic concern.

A poset (X, P) is said to be **planar** if it is possible to draw its Hasse diagram in the plane without edge-crossings. If a poset is planar, then clearly so is its Hasse graph. On the other hand, it is possible for a non-planar poset to have a planar Hasse graph, as in the poset of Fig. 12. No plane drawing of the Hasse graph is a Hasse diagram of the poset.

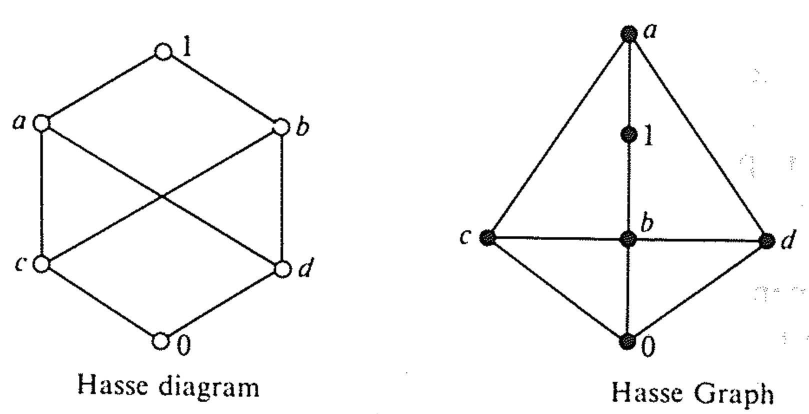


Fig. 12

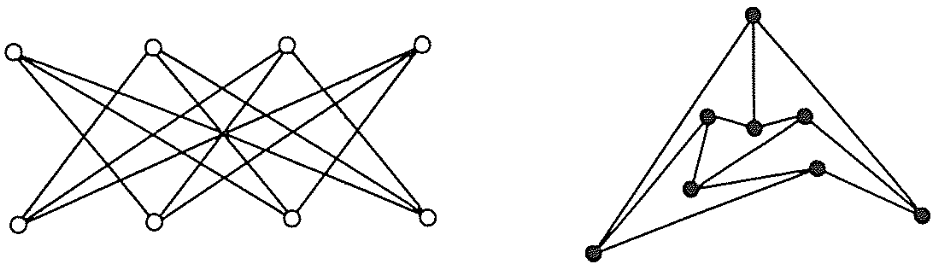


Fig. 13

The planarity of a particular poset may not be obvious, however; for example, S_4^0 is planar, as shown in Fig. 13.

In order to discuss planarity for posets in a more general setting, we call a simple graph in which each edge is assigned a direction an ***AO*-graph** (short for acyclic oriented graph) when it contains no directed circuits. Diagrams for *AO*-graphs can be presented without arrowheads by using the same convention as for a Hasse diagram of a poset—we require that y be higher in the figure than x whenever the *AO*-graph contains an arc from x to y; these diagrams are called **order diagrams**. Figure 14 shows an *AO*-graph and its order diagram.

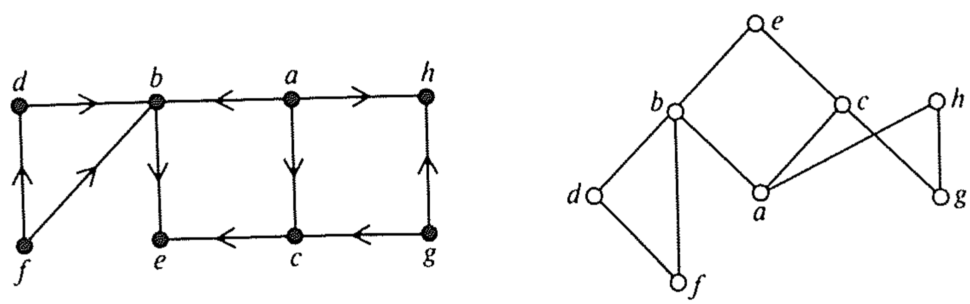


Fig. 14

Among the many unsolved problems involving *AO*-graphs is the characterization of planar *AO*-graphs; these graphs should admit a Kuratowski-type characterization. Figure 15 contains the order diagrams of some of the forbidden subgraphs. Each order diagram in the figure is non-planar, but the deletion of any edge leaves a planar diagram. Note that only the first diagram can be judged to be non-planar by Kuratowski's theorem (see Chapter 1); the others require special arguments.

The last three posets in Fig. 15 illustrate a concept for *AO*-graphs similar to outerplanarity for simple graphs. An *AO*-graph G is said to be **zero-join planar** if the *AO*-graph G_0 formed by adding a new vertex 0 and edges from 0 to all vertices in G is planar. These graphs should also admit a forbidden subgraph characterization; removing the lowest point from the last three diagrams in Fig. 15 leaves three of the *AO*-graphs which must appear in this characterization.

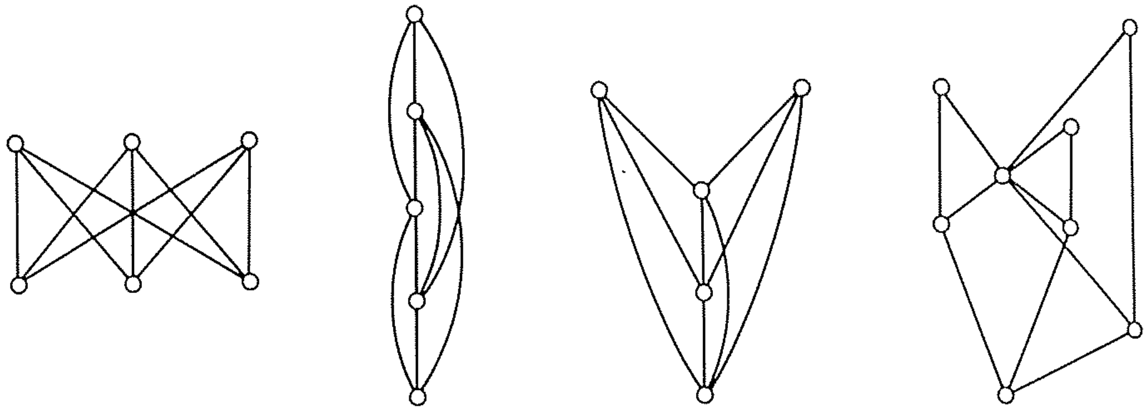


Fig. 15

The interplay between dimension and planarity begins with planar lattices. A finite poset **X** is a **lattice** if, for each pair of points x and y, there are unique points z and w such that, if $a \geqslant x$ and $a \geqslant y$ then $a \geqslant z$, and if $b \leqslant x$ and $b \leqslant y$ then $b \leqslant w$. The points z and w are called the **join** and **meet** of x and y, and are denoted by $x \vee y$ and $x \wedge y$, respectively. (Figure 16 contains two posets which are lattices.) The algebraic, geometric and topological properties of lattices have been studied extensively (see [**1**], for example); here we discuss briefly some of their combinatorial properties.

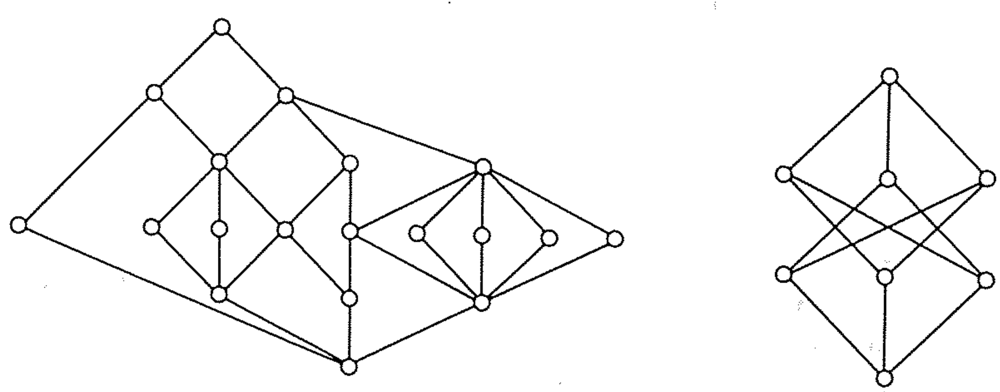


Fig. 16

The following elementary results are a combination of a theorem of Zilber (see [**1**, Exercise 7c on page 32]) and a theorem of Dushnik and Miller [**10**]:

Theorem 5.1. *A poset* **X** *has dimension* 2 *if and only if its comparability graph is an incomparability graph—that is, if and only if there exists a poset* **Y** *such that* x *is comparable to* y *in* **X** *if and only if* $x \parallel y$ *in* **Y**. ‖

Figure 17 contains two complementary 2-dimensional posets.

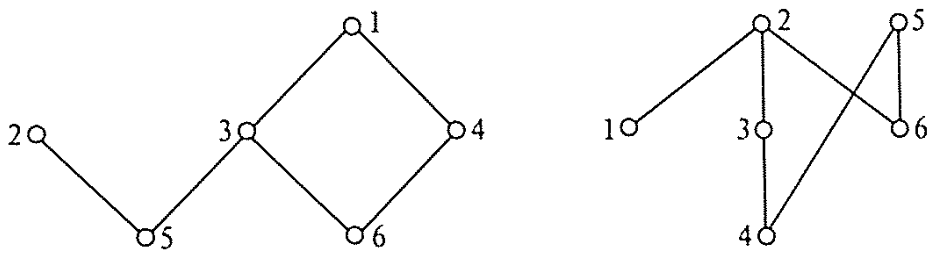


Fig. 17

Theorem 5.2. *Any planar poset with a greatest and a least element is a 2-dimensional lattice.* ‖

The second poset in Fig. 16 is non-planar since it is 3-dimensional (it contains S_3^0).

The poset **X** in Fig. 18, which is a 2-dimensional non-planar poset with greatest and least elements, demonstrates that the converse of Theorem 5.2 is invalid. One way to settle the question of planarity for this poset using only dimension theory is to insert points on two of the edges in this diagram to form a new poset **Y**. Clearly this does not affect planarity, but **Y** is 3-dimensional (see Fig. 9) and is thus non-planar.

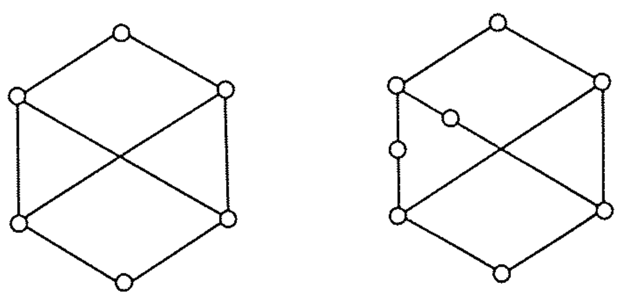


Fig. 18

Kelly and Rival [**22**] gave a forbidden subposet characterization of planar lattices by determining the minimum collection $\mathscr{L}$ of non-planar (3-dimensional) lattices so that a lattice **L** is non-planar if and only if it contains a lattice from $\mathscr{L}$ as a subposet. Baker (unpublished) proved that the completion of a poset **X** is a lattice of the same dimension. These two results were used by Kelly [**20**] to determine the collection of all 3-irreducible posets. Kelly's argument is quite complex and requires clever organization to handle the nine infinite families and eighteen odd examples present in the final list of all 3-irreducible posets. Trotter and Moore [**50**] investigated the dimension of planar posets in general, and extended Theorem 5.2 to the case where only one bound is present:

Theorem 5.3. *If* **X** *is a planar poset with either a greatest or a least element, then* dim **X** ≤ 3. ‖

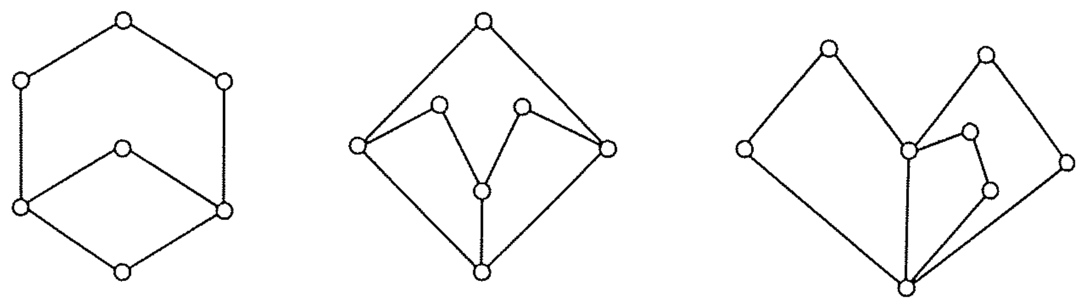


Fig. 19

Each of the posets in Fig. 19 is 3-dimensional. Removing the least element from the last one leaves a poset whose Hasse graph is a tree. In [**50**], Trotter and Moore proved that the dimension of a poset whose Hasse graph is a tree is at most 3; this was accomplished by showing that its Hasse diagram is zero-join planar. They also constructed an infinite family of 4-dimensional posets. (Recall from Fig. 13 that the 4-dimensional poset $\mathbf{S}_4^0$ is planar.)

For some time, we believed that there might be a theory relating the maximum dimension of a poset to the minimum genus of a surface on which the Hasse diagram can be embedded. This dream began to fade with the discovery that embedding *AO*-graphs on the plane is different from embedding them on a sphere, and the realization that there exist posets of arbitrarily large dimension whose Hasse diagrams can be embedded on the sphere (see [**42**]). Kelly [**21**] removed any lingering doubt about the viability of such a theory with the following result:

Theorem 5.4. *There exist planar posets of arbitrary dimension.*

Proof. It suffices to show that, for each $n \geqslant 3$, there exists a planar poset $\mathbf{X}$ containing $\mathbf{S}_n^0$. Figure 20 gives the Hasse diagram of such a poset. ||

It is not known whether this result can be extended to irreducible planar posets.

It also remains to investigate which properties of a poset are determined by its Hasse graph. The characterization of Hasse graphs themselves remains unsolved. (A Hasse graph is triangle-free, but Nešetřil (personal communication) has shown that there exist graphs of arbitrarily large girth which are not Hasse graphs.)

6. Dimension and Forbidden Subgraphs

Any subgraph of a planar graph is also planar, so it is possible to characterize planar graphs by providing a minimum list of forbidden subgraphs (as Kuratowski did in his famous theorem). In this section, we discuss several other properties of graphs which admit forbidden subgraph characterizations. These characterization problems involve families of intersection

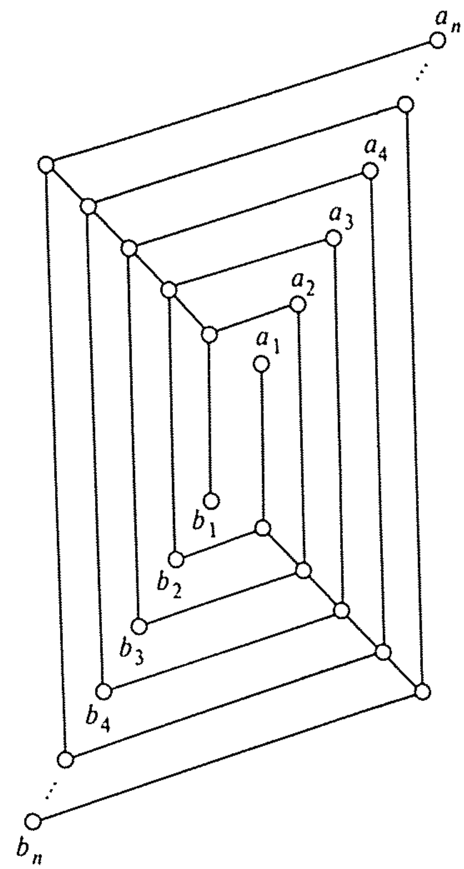


Fig. 20

graphs, and are related to the problem of determining the collection of all 3-irreducible posets.

Recall from Chapter 3 that a graph G is called an **interval graph** if it is the intersection graph of a family of intervals on a line. For example, the graph G in Fig. 21 is an interval graph, as indicated; for clarity, the intervals in the representation have been displaced vertically. Interval graphs belong to the family of **rigid-circuit graphs** (also called *triangulated graphs*) in which every circuit of length at least 4 has a chord [**25**]. Thus the circuit graph C_n is not an interval graph for $n > 3$.

Any induced subgraph of an interval graph is also an interval graph, and so this family of graphs admits a forbidden subgraph characterization. The following result is due to Lekkerkerker and Boland [**25**]:

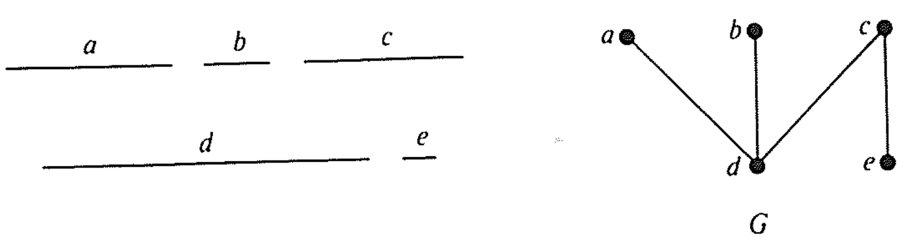
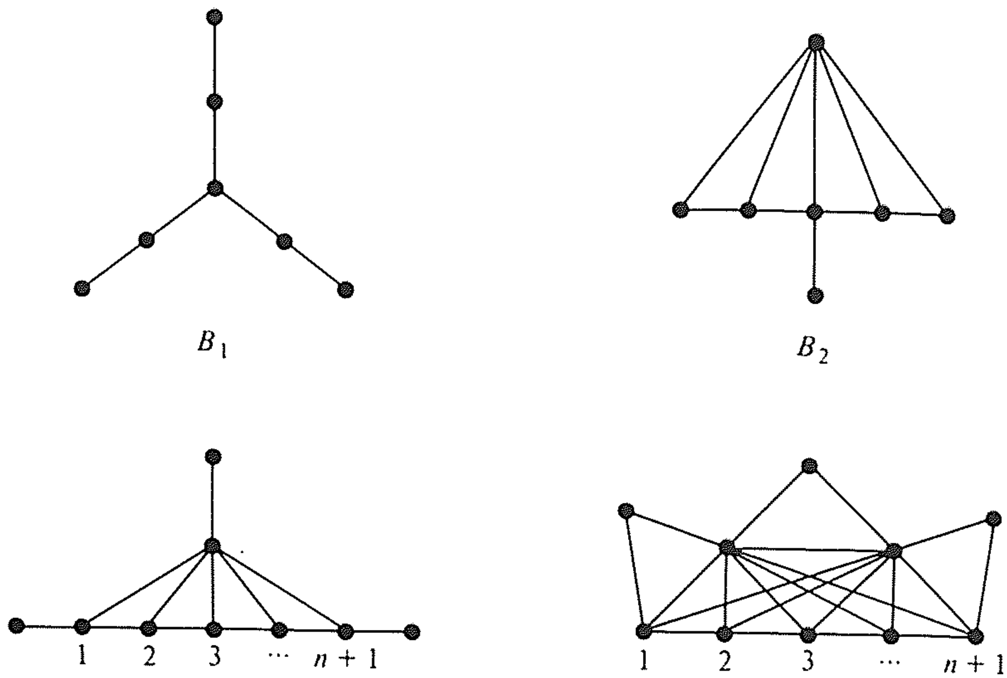


Fig. 21

Theorem 6.1. *A rigid-circuit graph is an interval graph if and only if it does not contain any of the graphs in Fig. 22 as an induced subgraph.* ‖

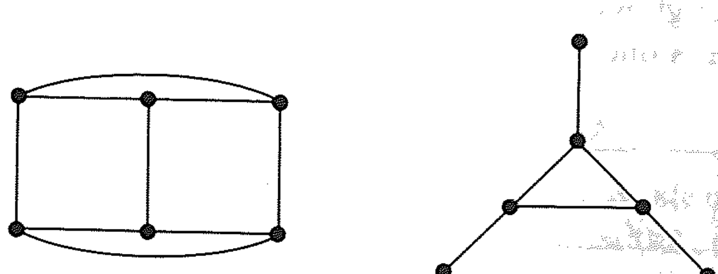


G_n; $n \geqslant 1$ H_n; $n \geqslant 1$

Fig. 22

Interval graphs can also be used to define a class of partial orders and a variant of the concept of dimension, which constitute a connecting link between dimension theory and several forbidden subgraph problems. The discussion of this link begins with comparability graphs. Any induced subgraph of a comparability graph is also a comparability graph. Gallai [13] determined the list $\mathscr{C}$ of forbidden subgraphs for comparability graphs. This list is quite long and the argument is necessarily quite complicated, and so in Fig. 23 we present only two of the forbidden subgraphs.

From Theorem 5.1, we know that a poset has dimension at most 2 if and only if its incomparability graph is a comparability graph. It follows that if **X**



G_1 G_2

Fig. 23

is a 3-irreducible poset, and if G is the comparability graph of $\mathbf{X}$, then the complement of G is one of the graphs in Gallai's list. Notice, for example, that the graph G_1 in Fig. 23 is the incomparability graph of $\mathbf{S}_3^0$. The list of all 3-irreducible posets can be determined by systematically examining each graph G in $\mathscr{C}$. If the complement of G is a comparability graph, then any transitive orientation of $\bar{G}$ is a 3-irreducible poset. Furthermore, every 3-irreducible poset arises in this fashion. This process is simplified considerably by the fact that the comparability graph of an irreducible poset admits a unique transitive orientation up to duality (see [**51**]).

If $\mathscr{I}$ is a collection of intervals of the real line, then a partial order of $\mathscr{I}$ is obtained by setting $[a_1, b_1] < [a_2, b_2]$ if and only if $b_1 < a_2$. A poset which arises in this fashion is called an **interval order**. A poset is an interval order if and only if its incomparability graph is an interval graph. The following forbidden subposet characterization is due to Fishburn [**12**]:

Theorem 6.2. *A poset is an interval order if and only if it does not contain the poset in Fig. 24 as a subposet.* ‖

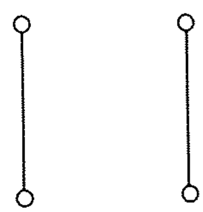


Fig. 24

As we suggested earlier, the concept of interval order yields a natural generalization of dimension, first defined by Trotter and Bogart. An **interval realizer** of a poset (X, P) is a family $\{P_1, P_2, \ldots, P_t\}$ of extensions such that each (X, P_i) is an interval order and $P = P_1 \cap P_2 \cap \ldots \cap P_t$. The **interval dimension** of (X, P), denoted by $\dim_I(X, P)$, is the minimum number of extensions in any interval realizer. Since every linear order is an interval order, it follows that $\dim_I \mathbf{X} \leqslant \dim \mathbf{X}$ for every poset $\mathbf{X}$. There exist interval orders of arbitrarily large dimension [**3**]; however, the dimension of a poset of length 2 never exceeds the interval dimension by more than 1.

For $t \geqslant 2$, a poset $\mathbf{X}$ is called ***t*-interval irreducible** if $\dim_I \mathbf{X} = t$, but $\dim_I \mathbf{Y} < t$ for every proper subposet $\mathbf{Y}$ of $\mathbf{X}$. The only 2-interval irreducible poset is the 4-point example in Fig. 24, so we turn to the case $t = 3$.

Let $\mathbf{X}$ be a 3-interval irreducible poset of length 2, and let $G(\mathbf{X})$ and $\bar{G}(\mathbf{X})$ denote its comparability and incomparability graphs, respectively. Let A denote the set of maximal elements and B the set of minimal elements of $\mathbf{X}$, and denote by $G'(\mathbf{X})$ the graph obtained by adding to the comparability graph $G(\mathbf{X})$ all edges between any two vertices in A and all edges between any two vertices in B. Note that a vertex $a \in A$ is adjacent to a vertex $b \in B$ in

$G'(\mathbf{X})$ if and only if a is not adjacent to b in $\bar{G}(\mathbf{X})$; but in both graphs, the subgraphs induced by A and B are complete.

Figure 25 shows $\bar{G}(\mathbf{X})$ and $G'(\mathbf{X})$ for a particular poset $\mathbf{X}$. Since $\mathbf{X}$ has length 2, we consider a Hasse diagram for $\mathbf{X}$ as a diagram for the comparability graph $G(\mathbf{X})$ as well. You are encouraged to verify that $\mathbf{X}$ is 3-interval irreducible in order to be convinced that such determinations can be extremely difficult without the assistance of some general theorems.

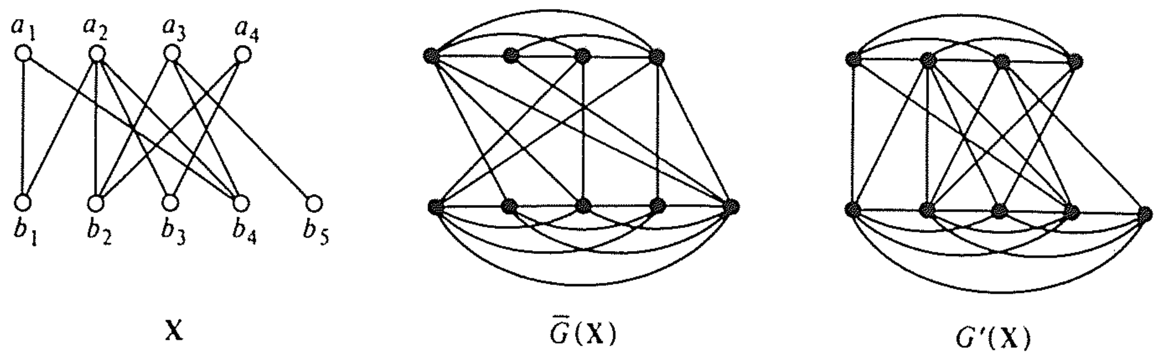


Fig. 25

We are now ready to discuss two important forbidden subgraph problems closely related to dimension theory. A graph G is a **circular-arc graph** if it is the intersection graph of a family of arcs on a circle. The problem of characterizing these graphs was posed by Klee [**24**], and partial solutions have been provided by Tucker [**53**], [**54**] and Hopkins [**17**]. Although the general problem of providing a forbidden subgraph characterization of circular-arc graphs remains unsolved, the dimension theory of posets contributes some significant partial results [**49**]:

Theorem 6.3. *The incomparability graph of every 3-interval irreducible poset* $\mathbf{X}$ *of length 2 is a forbidden subgraph in the characterization of circular-arc graphs.* ‖

The converse of Theorem 6.3 also holds, in the sense that every forbidden subgraph with clique covering number 2 in the characterization of circular-arc graphs arises in this fashion. In [**46**], Trotter used combinatorial techniques for posets to determine completely all 3-interval irreducible posets of length 2.

The determination of all 3-interval irreducible posets of length 2 yields a bonus. A graph G is called a **rectangle graph** if it is the intersection graph of a family of rectangles in the plane, with the sides of the rectangles parallel to the coordinate axes. The graph G in Fig. 26 is an example of a rectangle graph.

The graph $G'(\mathbf{X})$ shown in Fig. 25 is one of the forbidden subgraphs in the characterization of rectangle graphs. In fact, this example is an illustration of the following theorem (see [**46**]):

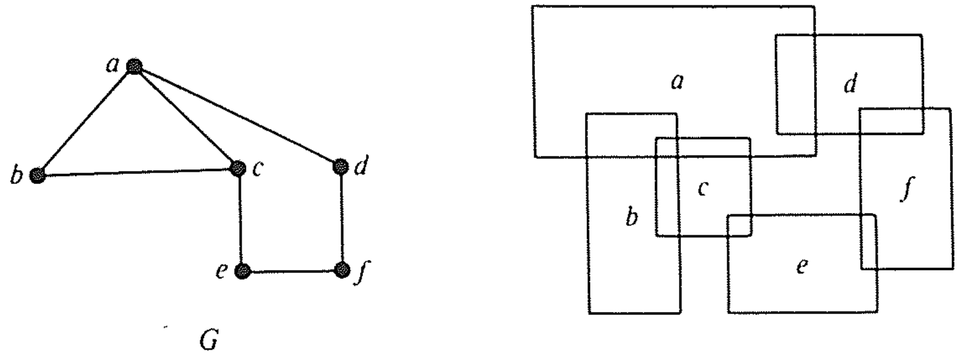


Fig. 26

Theorem 6.4. *For any 3-interval irreducible poset* **X** *of length 2, the graph* $G'(\mathbf{X})$ *is a forbidden subgraph in the characterization of rectangle graphs.* ‖

As with Theorem 6.3, the converse of this theorem is also valid, in the sense that every forbidden subgraph with clique covering number 2 in the characterization of rectangle graphs has the form $G'(\mathbf{X})$ for some 3-interval irreducible poset **X** of length 2.

The general problem of representing graphs and posets by intervals, rectangles and boxes has attracted considerable attention in recent years. Roberts [34] defined the **boxicity** of a graph G as the minimum number t for which G is the intersection graph of boxes in R^t. He showed that, for each $n \geqslant 1$, the boxicity of a graph G with $2n + 1$ vertices does not exceed n. Trotter [45] gave a forbidden subgraph characterization of this inequality which is very similar to the results obtained previously by Bogart and Trotter [4] and Kimble [23] for Hiraguchi's inequality (the dimension of a poset **X** on $2n + 1$ points does not exceed n, for $n \geqslant 2$). Witsenhausen [57] obtained additional results, and Feinberg [11] considered a generalization of boxicity involving circular-arc graphs.

Two other areas of research involving interval graphs and interval orders should be mentioned. The first of these involves restricting the number of different lengths which may be used for intervals in the representation. For an interval graph G, or an interval order **X**, we define the **interval count** to be the least number t for which the graph has a representation using intervals of t different lengths. An interval order with interval count 1 is also called a **semi-order**. These posets admit a simple forbidden subposet characterization, due to Scott and Suppes [35]:

Theorem 6.5. *An interval order is a semi-order if and only if it does not contain the poset in Fig. 27 as a subposet.* ‖

The class of semi-orders is a class of posets which can be enumerated by a relatively simple formula. Dean and Keller [7] showed that the number of semi-orders on n points is the Catalan number $\binom{2n}{n}/(n + 1)$. Roberts [34]

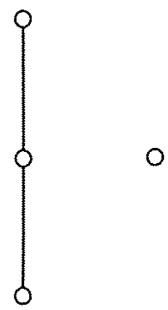


Fig. 27

characterized interval graphs with interval count 1. Liebowitz [**26**] and Couzzens [**6**] obtained several interesting results for the interval count, and Fishburn (personal communication) has conjectured that the interval count of an interval graph with $3n$ vertices does not exceed n. If it is true, this result is best possible, as the example in Fig. 28 (due to Fishburn) shows; it is an interval order with $3n + 1$ points and interval count $n + 1$.

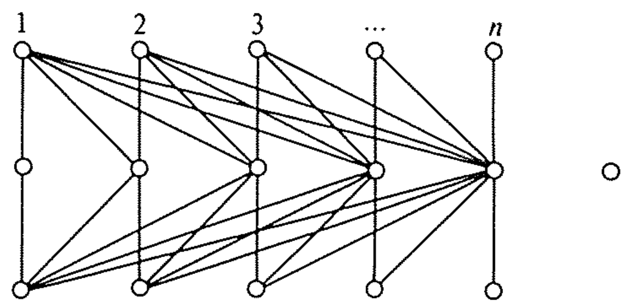


Fig. 28

Another area of interest involves multiple interval graphs. Trotter and Harary [**48**] defined the **interval number** of a graph G to be the least number t for which G is the intersection graph of a family of sets each of which is the union of t intervals on the real line. They derived the following result:

Theorem 6.6. *The interval number of the complete bipartite graph* $K_{m,n}$ *is* $\lceil (mn + 1)/(m + n) \rceil$. ‖

Griggs and West [**15**] established the following upper bound on the interval number of a graph in terms of its maximum valency $\rho_{\max}$:

Theorem 6.7. *The interval number of a graph G is at most* $\lceil \frac{1}{2}(\rho_{\max} + 1) \rceil$ ‖

Griggs and West also proved that if G is regular and triangle-free, then equality holds in the preceding theorem. As a consequence, they showed that the interval number of the n-cube is $\lceil \frac{1}{2}(n + 1) \rceil$.

Griggs [**14**] proved the following result giving the maximum value of the interval number of a graph, and settling a conjecture made by several researchers:

Theorem 6.8. *If $n \geqslant 1$, and if G is an interval graph with $4n - 1$ vertices, then the interval number of G is at most n.* ‖

The complete bipartite graph $K_{2n,2n}$ shows that this result is best possible.

The determination of the interval numbers of complete multipartite graphs is quite complicated and has led to some interesting problems involving Eulerian trails in directed graphs. We refer to [18] and [19] for these results.

7. Rank and Digraphs

In this section, we present a summary of the general theory of poset rank, as developed by Maurer, Rabinovitch and Trotter [28]. The central idea in this theory is the conversion of the problem of determing the rank of a poset into an extremal problem for digraphs. Graph-theoretical concepts make an essential contribution to the development of this theory; in return, we get the solution to a problem of independent interest in graph theory.

Recall that the rank of a poset is the maximum order of an irredundant realizer. Since a poset has rank 1 if and only if it is a chain, we shall restrict our attention to posets which are not chains. Thus, the sets I_P and N_P of incomparable and non-forced pairs are non-empty.

Let $\mathbf{X} = (X, P)$ be a poset, let $R = \{L_1, L_2, \ldots, L_t\}$ be a realizer of $\mathbf{X}$, and let S_i denote the set of non-forced pairs reversed by L_i. Then $N_P = S_1 \cup S_2 \cup \ldots \cup S_t$, and R is irredundant if and only if each S_i has a pair (x_i, y_i) in no other S_j. Just as we consider N_P to be a digraph, so we use graph-theoretical terminology for such a collection of pairs, calling it a **critical digraph** for R, or (more loosely) for $\mathbf{X}$. Note that an irredundant realizer may have more than one critical digraph, but that all must have the same number of arcs—the number of linear extensions in the realizer.

For example, let (X, P) be the poset shown in Fig. 29, and let R be the realizer consisting of

$$L_1: [f, b, c, d, e, a], \qquad L_2: [f, d, e, c, a, b], \qquad L_3: [c, e, d, f, b, a].$$

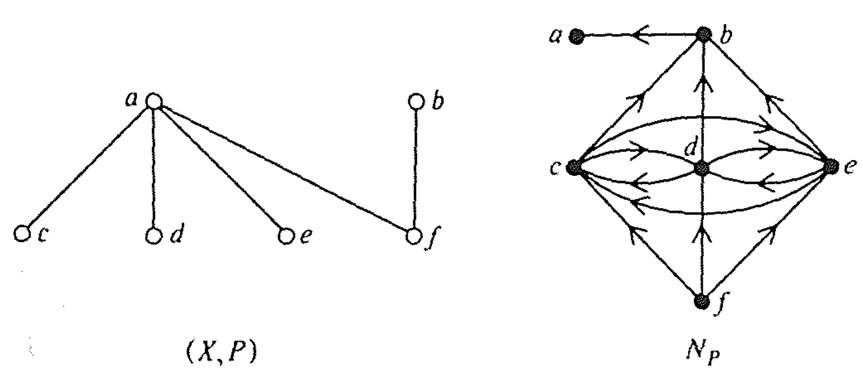


Fig. 29

Then $D_1 = \{(e, b), (b, a), (f, e)\}$ and $D_2 = \{(e, c), (c, d), (d, e)\}$ are both critical digraphs. Note that D_2 is a directed circuit in N_P, but that D_1 is not. We begin our development of critical digraphs by characterizing those which contain directed circuits:

Theorem 7.1. *Let D be a critical digraph of an irredundant realizer of R of a poset (X, P). If D contains a directed circuit, then it has no other arcs.*

Proof. Suppose that $C = v_1 v_2 \ldots v_n$ is a directed circuit in D, and that D has an arc xy not in C. If L is the linear extension in R which reverses the non-forced pair (x, y) but no other pair in D, then in L we must have $v_1 < \ldots < v_n < v_1$, which is impossible. Thus D can contain no arcs other than those of C. ||

To illustrate this theorem, we consider the n-element antichain on $\{1, 2, \ldots, n\}$. The n linear extensions L_i: $[i, i+1, \ldots, n, 1, \ldots, i-1]$ form an irredundant realizer, and the n pairs $(1, 2), (2, 3), \ldots, (n-1, n), (n, 1)$ form a critical digraph.

We know from Section 4 that the rank of a $2n$-element antichain is at least n^2, so that, if $n > 2$, no critical digraph for an irredundant realizer of maximum order can be a directed circuit. We now consider the circumstances under which a critical digraph can be a directed circuit.

Given a poset **X**, we define a subposet **Y** to be **partitive** if it has the following two properties:

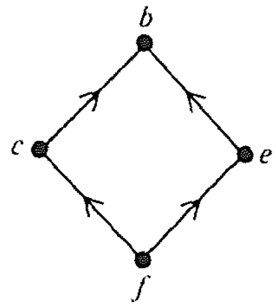
(*i*) if $x \in X - Y$, and if $x > y$ for some $y \in Y$, then $x > y$ for all $y \in Y$;

(*ii*) if $x \in X - Y$, and if $x < y$ for some $y \in Y$, then $x < y$ for all $y \in Y$.

Trivially, a single point is partitive in any poset, as is the entire poset. The 3-element antichain $\{c, d, e\}$ is a non-trivial partitive subposet of the poset **X** in Fig. 29.

Theorem 7.2. *Let (X, P) be a poset. Then the vertices of any directed circuit in the digraph N_P of non-forced pairs form a partitive antichain in (X, P).* ||

Critical digraphs also have an important property involving paths. A digraph is called **unipathic** if there is at most one directed path from one given vertex to another. Although a critical digraph need not be unipathic, it must satisfy a weaker condition—a subgraph H of N_P is called ***P*-unipathic** if the existence of two directed paths from x to y in H implies that (x, y) is not a non-forced pair. For example, consider the poset in Fig. 29 and the digraph H of non-forced pairs in Fig. 30. Although H is not unipathic, it is P-unipathic since (f, b) is not a non-forced pair. Note that, if N_P has a directed path from x to y, then (x, y) is in either P or N_P, since the relation $P \cup N_P$ is transitive.



H

Fig. 30

Theorem 7.3. *Let (X, P) be a poset. Then every critical digraph of an irredundant realizer is P-unipathic.* ||

It follows that a critical digraph is either a directed circuit on the vertices of a partitive antichain, or it is P-unipathic and acyclic.

Corollary 7.4. *If the rank of a poset (X, P) is r, then there is either a partitive antichain of order at least r, or an acyclic P-unipathic digraph with r arcs.* ||

We are primarily interested in critical digraphs for irredundant realizers with as many linear extensions as possible—that is, critical digraphs whose arcs are equal in number to the rank of the poset. For convenience, we call these **critical rank digraphs** of the poset. We further define a poset to be **rank-degenerate** if it is a chain or if every critical rank digraph is a directed circuit. We are interested in determining all rank-degenerate posets.

For disjoint posets **X** and **Y**, we let $\mathbf{X} \oplus \mathbf{Y}$ denote the poset obtained from the union of **X** and **Y** by putting $x < y$ for all $x \in \mathbf{X}$ and $y \in \mathbf{Y}$. Consider now the poset $\mathbf{X} = \underline{h} \oplus \bar{2} \oplus \underline{k}$, where h and k are any positive integers, and where the subposet $\bar{2}$ consists of v and w. Then **X** has only two linear extensions, and the only critical digraph consists of vw and wv. Hence, **X** is rank-degenerate. Similarly, consider the poset $\mathbf{Y} = \underline{h} \oplus \bar{3} \oplus \underline{k}$, where $\bar{3}$ consists of u, v and w. Then the digraph of non-forced pairs has just the six arcs joining these three vertices. Also, **Y** is 2-dimensional and has six linear extensions, and there are two irredundant realizers of order 3, each having a directed circuit as its critical digraph. Then **Y** is also rank-degenerate. In fact, it can be shown that this completes the list of rank-degenerate posets:

Theorem 7.5. *A poset is rank-degenerate if and only if it is a subposet of $\underline{h} \oplus \bar{3} \oplus \underline{k}$, for some positive integers h and k.* ||

We now consider posets **X** which are not rank-degenerate—that is, where **X** has a critical rank digraph which is acyclic and P-unipathic. The next theorem is powerful in that it enables one to get realizers from maximal acyclic P-unipathic digraphs:

Theorem 7.6. *Let* $\mathbf{X} = (X, P)$ *be a poset which is not rank-degenerate, and let* D *be a maximal acyclic* P*-unipathic digraph. Then* D *is a critical digraph for* $\mathbf{X}$. ||

We remark that the primary difficulty in proving this theorem is in deciding when to reverse those non-forced pairs in N_P which do not belong to D. However, once this theorem is proved, we have a graph-theoretical formula for the rank as an immediate consequence:

Theorem 7.7. *The rank of a poset* $\mathbf{X} = (X, P)$ *which is not rank-degenerate is the maximum number of arcs in any acyclic* P*-unipathic digraph of* $\mathbf{X}$. ||

The computation of the rank using this theorem can often be simplified by first getting rid of directed circuits in N_P. This we do by choosing a linear order L, and then defining the **acyclic digraph** N_P^* **of non-forced pairs** by

$$N_P^* = \{(x, y) \in N_P: (y, x) \notin N_P \quad \text{or} \quad (x, y) \in L \cap N_P\}.$$

Strictly speaking, N_P^* depends on L, but it is easy to see that any two linear orders determine isomorphic subgraphs of N_P.

Corollary 7.8. *The rank of a poset* $\mathbf{X} = (X, P)$ *which is not rank-degenerate is the maximum number of arcs in a* P*-unipathic subgraph of* N_P^*. ||

Since N_P^* is acyclic, it is an AO-graph, and so we can employ the conventions introduced in Section 5 for order diagrams. For example, Fig. 31 shows N_P^* for the poset $\mathbf{X}$ of Fig. 29, with N_P^* determined by L: $[a, b, c, d, e, f]$. To see this, consider the arcs in the triangle $T = \{(c, d), (d, e), (c, e)\}$. No P-unipathic subgraph can contain all three of these arcs, since they form two disjoint paths from c to e and $(c, e) \in N_P$. There are six other triangles in N_P^* for which similar statements hold. The arcs in T each belong to three of these triangles, whereas the other arcs in N_P^* belong to at most two triangles. It follows that no P-unipathic subgraph of N_P^* can contain eight of the ten arcs in N_P^*, and hence that rank $\mathbf{X} \leqslant 7$. On the other hand, removing the arcs (c, d), (d, e) and (c, e) from N_P^* leaves a

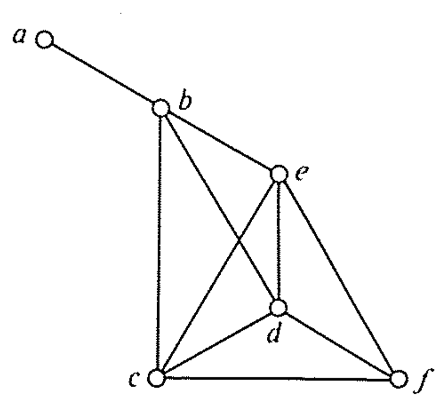


Fig. 31

P-unipathic subgraph, and so rank $\mathbf{X} = 7$. Note that we have just computed the rank of a poset without ever having built a realizer. Thus, Corollary 7.8 has reduced the computation of rank to a digraph extremal problem.

Our next example, due to Maurer and Rabinovitch, yields the rank of any antichain. Let $n \geqslant 4$, and let $\mathbf{X}$ be the antichain on $\{1, 2, \ldots, n\}$ (which is not rank-degenerate). By taking L: $[1, 2, \ldots, n]$, we have $N_P^* = \{(i,j): 1 \leqslant i < j \leqslant n\}$, the complete order on n elements. Now any P-unipathic subgraph H can contain no triangles (since every arc is a non-forced pair). It follows from Turán's theorem that H contains at most $\lfloor \frac{1}{4}n^2 \rfloor$ arcs. On the other hand, if $H_n = \{(i,j): 1 \leqslant i \leqslant \frac{1}{2}n < j \leqslant n\}$, then H_n is P-unipathic (no directed path has length greater than 1), and has $\lfloor \frac{1}{4}n^2 \rfloor$ arcs. Hence, for $n \geqslant 4$, the rank of $\bar{n}$ is $\lfloor \frac{1}{4}n^2 \rfloor$.

A natural question to ask is whether the rank of a poset is determined by its comparability graph. Trotter, Moore and Sumner [**51**] proved that two posets with the same comparability graph have the same dimension, and Stanley [**36**] proved that they also have the same number of linear extensions. It may therefore come as something of a surprise that they need not have the same rank. You may wish to use Corollary 7.8 to verify this for the following example, taken from [**28**]:

For disjoint posets $\mathbf{X}$ and $\mathbf{Y}$, let $\mathbf{X} + \mathbf{Y}$ denote the poset obtained from the union of $\mathbf{X}$ and $\mathbf{Y}$ by taking $x \parallel y$ for all $x \in \mathbf{X}$ and $y \in \mathbf{Y}$. Now let $\mathbf{X}_1 = (1 \oplus \bar{n}) + (1 \oplus \bar{n})$, and $\mathbf{X}_2 = (1 \oplus \bar{n}) + (\bar{n} \oplus 1)$. Then $\mathbf{X}_1$ and $\mathbf{X}_2$ have the same comparability graph, but $\mathbf{X}_1$ has rank $2\lfloor \frac{1}{4}(n+1)^2 \rfloor$, while $\mathbf{X}_2$ has rank $n^2 + 1$.

We now turn our attention to computing the rank of the posets in one special class. The end result will be a directed generalization of Turán's theorem.

Let m and n be positive integers with $m \leqslant n$, and let $\mathbf{X}(n, m)$ be the poset on $\{1, 2, \ldots, n\}$ with pairs $\{(i,j): i + m \leqslant j\}$. Figure 32 shows Hasse diagrams

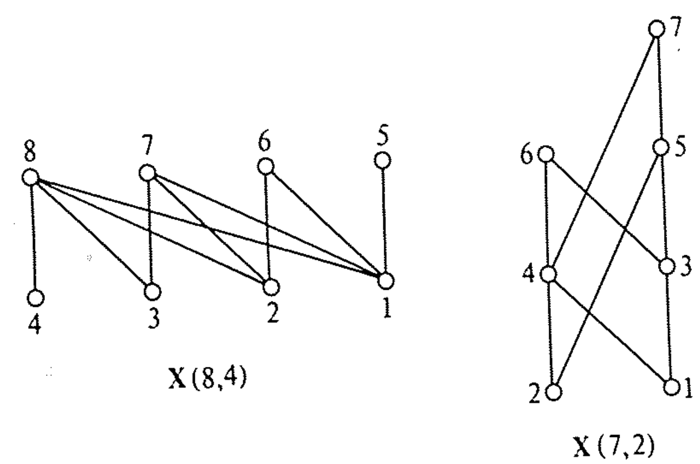


Fig. 32

for $\mathbf{X}(8, 4)$ and $\mathbf{X}(7, 2)$. For a variety of reasons—for example, $\mathbf{X}(2m, m)$ is the "split" of m (see [**46**]), and $\mathbf{X}(2n + 2, 2)$ is the result of removing one point from a 3-irreducible poset shown in Fig. 11—it is natural to consider the rank of $\mathbf{X}(n, m)$. It is easily seen that it is a semi-order and thus, by a theorem of Rabinovitch, has dimension at most 3.

We now look at our extremal problem. Let $\mathbf{L}_n$ denote the digraph of the usual linear order (the transitive tournament) on $\{1, 2, \ldots, n\}$. A subgraph of $\mathbf{L}_n$ is said to be ***m*-locally unipathic** if its restriction to every subset of m consecutive vertices of $\{1, 2, \ldots, n\}$ is unipathic. We let $\mu(n, m)$ denote the maximum number of arcs in an m-locally unipathic subgraph of $\mathbf{L}_n$, and we attempt to determine this number and the corresponding extremal graphs.

Some special cases are already known to us:

(1) $\mu(n, 1) = \mu(n, 2) = \binom{n}{2}$, since $\mathbf{L}_n$ is 2-locally unipathic;

(2) $\mu(3, 3) = 2$, and each of the three subgraphs of $\mathbf{L}_3$ with two arcs is extremal;

(3) $\mu(n, n) = \lfloor \frac{1}{4}n^2 \rfloor$, for $n \geqslant 4$.

In case (3), if $H_n = \{(i, j)\colon 1 \leqslant i \leqslant \frac{1}{2}n < j \leqslant n\}$, and if n is even, then the only extremal graph is H_n, whereas if n is odd, then the only extremal graphs are H_n and $\hat{H}_n$.

For the general case, there are "reasonable" conjectures for the answers, and it turns out that they are correct. Let $V_0, V_1, \ldots, V_{q+1}$, be a partition of $S_n = \{1, 2, \ldots, n\}$ such that

(*i*) each V_i is a set of at least $m - 1$ consecutive integers;

(*ii*) the integers in V_i are less than those in V_{i+1}, for $i \leqslant q$;

(*iii*) when $1 \leqslant i \leqslant q$, V_i contains at least $m - 1$ integers.

Then the subgraph of $\mathbf{L}_n$ with arc-set $\{(x, y)\colon x \in V_i, y \in V_j, i < j\}$ is m-locally unipathic. Furthermore, the members of this family with the most arcs are the digraphs $H(m, q, r)$ defined as follows:

let q and r be integers such that

$$(*) \qquad n = q(m-1) + r, \qquad \text{and} \qquad \lceil \tfrac{1}{2}(m-1) \rceil \leqslant r \leqslant \lceil \tfrac{3}{2}(m-1) \rceil,$$

and let $H(m, q, r)$ be the digraph of the above type with

$$|V_0| = \lfloor \tfrac{1}{2}r \rfloor, \quad |V_{q+1}| = \lceil \tfrac{1}{2}r \rceil, \qquad \text{and} \qquad |V_i| = m - 1, \quad \text{otherwise}.$$

Maurer, Rabinovitch and Trotter [**29**] proved that these are indeed extremal:

Theorem 7.9. *Let $n \geqslant m \geqslant 2$, and let q and r satisfy $(*)$. Then the maximum number of arcs in an m-locally unipathic subgraph H of $\mathbf{L}_n$ is*

$$\mu(n, m) = \binom{q}{2}(m-1)^2 + qr(m-1) + \lfloor \tfrac{1}{4}r^2 \rfloor.$$

Furthermore, H must be $H(m, q, r)$ or $\hat{H}(m, q, r)$ in order to attain this maximum. ‖

We consider two examples:

(*i*) when $n = 16$ and $m = 7$, the only possibility is $q = 2$ and $r = 4$, whence $\mu(16, 7) = 88$, and the unique extremal graph is $H(7, 2, 4)$;

(*ii*) when $n = 10$ and $m = 7$, there are two choices—namely, $q = 1$ and $r = 3$, and $q = 0$ and $r = 10$; hence, $\mu(10, 7) = 25$, and there are three extremal graphs, $H(7, 1, 3)$, $\hat{H}(7, 1, 3)$ and $H(7, 0, 10)$.

Any attempt to prove Theorem 7.9 here would go beyond our space limitations, but we should like to make a few comments on our approach to the problem. For the poset $(X, P) = \mathbf{X}(n, m)$, a subgraph H of N_P is P-unipathic if and only if $H \cup P$ is an m-locally unipathic subgraph of $\mathbf{L}_n$. Since $|P| = \binom{n-m+1}{2}$, we have

$$\text{rank } \mathbf{X}(n, m) = \mu(n, m) - \binom{n - m + 1}{2},$$

when $\mathbf{X}(n, m)$ is not rank-degenerate. The only cases in which $\mathbf{X}(n, m)$ is rank-degenerate occur when $m = 1$, in which case $\mathbf{X}$ is a chain, or when $m = n$ and $n = 2$ or 3, in which case $\mathbf{X}$ is an antichain. Since these cases can be disposed of immediately, we can attack the problem using the theory of rank. The principal weapons in this approach are some "exchange theorems" for arcs in P-unipathic subgraphs of N_P^*.

In conclusion, we note that this graph-theoretical approach to poset rank can be used to obtain a simple proof of the formula for the rank of a distributive lattice, first found by Rabinovitch and Rival [33]. It can also be used to determine all posets with equal rank and dimension (see Maurer, Rabinovitch and Trotter [30]). None the less, much work still remains to be done in exploring the interplay between posets and digraphs.

8. Unsolved Problems

We conclude with a short list of research problems involving the topics discussed in this chapter. As with all such lists, there is an inherent danger that significant problems have been omitted and that uninteresting problems have been included. In presenting this list, it is our intention that it serve only to foster further investigations of general areas.

(1) Is it true that if (X, P) is a poset, and if $|X| \geqslant 3$, then there exists a pair $x, y \in X$ such that $\dim(X, P) \leqslant 1 + \dim(X - \{x, y\}, P_{X-\{x,y\}})$?

(2) Which simple graphs are the Hasse graphs of partially ordered sets?

(3) If H is a hypergraph, under what conditions does there exist a poset $\mathbf{X}$ for which $H = H_{\mathbf{X}}$? What happens if we also require $\mathbf{X}$ to be irreducible? What happens if we require H to be critical?

(4) Under what conditions does the associated hypergraph $H_{\mathbf{X}}$ contain a simple graph G such that $\chi(G) = \chi(H_{\mathbf{X}})$?

(5) Which AO-graphs are planar? Which AO-graphs are zero-join planar?

(6) Which AO-graphs are the digraphs of non-forced pairs of a poset?

(7) To what degree does the digraph of non-forced pairs determine the dimension and rank of a poset?

(8) Do there exist t-irreducible planar posets for all $t \geqslant 3$?

(9) For which posets $\mathbf{X}$ is it true that dim $\mathbf{X}$ = width $\mathbf{X}$?

(10) For each $t \geqslant 1$, construct a $(2t + 1)$-irreducible poset (X, P) containing an antichain A such that t = width $(X - A, P_{X-A})$.

(11) If the maximum valency of a vertex in the comparability graph is k, is the dimension of the poset bounded as a function of k?

(12) Find a forbidden subgraph characterization of circular-arc graphs and rectangle graphs.

(13) (P. Fishburn) What is the maximum interval count of an interval order or an interval graph with n vertices?

(14) What is the maximum dimension of an interval order of length n?

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