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DIMENSION OF THE CROWN S_n^k

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Abstract. In 1941, Dushnik and Miller introduced the concept of the dimension of a poset (X, P) as the minimum number of linear extensions of P whose intersection is exactly P. Although Dilworth has given a formula for the dimension of distributive lattices, the general problem of determining the dimension of a poset is quite difficult. An equally difficult problem is to classify those posets which are dimension irreducible, i.e., those posets for which the removal of any point lowers the dimension. In this paper, we construct for each $n \geq 3$, $k \geq 0$, a poset, called a crown and denoted S_n^k, for which the dimension is given by the formula $\{2(n+k)/(k+2)\}$. Furthermore, for each $t \geq 3$, we show that there are infinitely many crowns which are irreducible and have dimension t. We then demonstrate a method of combining a collection of irreducible crowns to form an irreducible poset whose dimension is the sum of the crowns in the collection. Finally, we construct some infinite crowns possessing combinatorial properties similar to finite crowns.

1. Introduction

In 1941, Dushnik and Miller [6] introduced the concept of the dimension of a poset (X, P) as the minimum number of linear extensions of P whose intersection is exactly P. Equivalently, Ore [10] defined the dimension of (X, P) as the smallest positive integer k for which (X, P) can be embedded in $\mathbf{R}^k$. Hiraguchi [7, 3] showed that the dimension of (X, P) is $\leq \frac{1}{2}|X|$ and Komm [9] showed that the dimension of the poset consisting of all subsets of an n element set ordered by inclusion is n. Dilworth [5] showed that the dimension of the distributive lattice $L = 2^X$ is the width of X.

A poset (X, P) is said to be irreducible if the dimension of P restricted to any proper subset of X is less than the dimension of (X, P). Hiraguchi observed that for $n \geq 3$, the poset (denoted S_n^0 in this paper) consisting

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of the singleton sets and the $n-1$ element subsets of an n element set ordered by inclusion is irreducible and has dimension n. Bogart and Trotter proved [4] that for $n \geq 4$, any poset on $2n$ points with dimension n is isomorphic to S_n^0. Recently, Kimble and Trotter have independently proved that for $n > 4$, any poset on $2n+1$ points with dimension n contains a subposet isomorphic to S_n^0. Kimble's proof of this theorem will appear in his thesis [8]. An infinite family of irreducible posets (denoted S_3^k, $k > 0$, in this paper) is used by Baker, Fishburn and Roberts [1] to show that the collection of all posets of dimension $\leq n$ is not axiomatizable by a sentence of first order logic.

For an arbitrary poset (X, P), the determination of its dimension is quite difficult. Even more difficult is the problem of determining an irreducible subposet of (X, P) with the same dimension. In this paper, we construct for each $n > 3$, $k > 0$, a poset called a crown and denoted S_n^k whose dimension is given by the formula $\lceil 2(n+k)/(k+2) \rceil$. For each $t > 3$, we show that there are infinitely many crowns which are irreducible and have dimension t. Given a collection of irreducible crowns, we construct an irreducible poset whose dimension is the sum of the dimensions of the crowns in the collection.

Finally, we construct some infinite crowns possessing combinatorial properties similar to the finite crowns.

2. Preliminary development

For a poset (X, P), the notations $(x, y) \in P$ and $x \leq y$ are used interchangeably. If $(x, y) \in P$ and/or $(y, x) \in P$, we say x and y are comparable and write $x\,C\,y$. If neither (x, y) nor (y, x) is in P, we say x and y are incomparable and write $x \,\mathrm{I}\, y$. If $x \neq y$ but $(x, y) \in P$, we say x is under y in P (also y is over x in P). Given a poset (X, P) and a subset $Y \subseteq X$, the poset $(Y, P \cap (Y \times Y))$ is called a subposet of (X, P) and $P \cap (Y \times Y)$ is called the restriction of P to Y. A partial order L on a set X is called a linear order when $x, y \in X$ imply $(x, y) \in L$ and/or $(y, x) \in L$. For a finite set X, it is useful to visualize a linear order L on X as simply a vertical listing of the elements of X with $(x, y) \in L$ and $x \neq y$ iff x is under y in the listing. When space requires, we use the notation $L = [x_1, x_2, x_3, \ldots, x_n]$ to indicate the linear order L on the set $X = \{x_i : i \leq n\}$ defined by $(x_j, x_i) \in L$ iff $j \leq i$, i.e., larger elements are

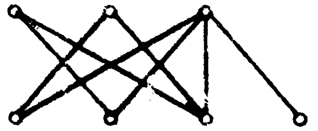

Fig. 1.

listed first. A linearly ordered subposet of a poset (X, P) is called a chain.

If P and Q are partial orders on X and $P \subseteq Q$, then Q is called an extension of P; if Q is a linear order on X, then Q is a linear extension of P. As first observed by Szpilrajn [11] for every poset (X, P), the collection C of all linear extensions of P is nonempty and $\bigcap C = P$. The dimension of a poset (X, P), denoted $\dim(X, P)$, is then the minimum number of linear extensions of P whose intersection is P. Later in this paper some examples involving infinite posets are presented; in this case, obvious modifications of both our original definition and subsequent observations are required. For the time being, we restrict our attention to finite sets where $\dim(X, P)$ can be defined equivalently as the smallest positive integer k such that (X, P) can be embedded in $\mathbf{R}^k$. For example, if $A = \{a_1, a_2, ..., a_n\}$ and (X, P) is the poset consisting of all subsets of A ordered by inclusion, then the identification of elements of X (subsets of A) with characteristic sequences is an embedding of (X, P) in $\mathbf{R}^n$ and thus $\dim(X, P) \leq n$.

A poset (X, P) is said to be dimension irreducible or simply irreducible if the dimension of any proper subposet of (X, P) is less than the dimension of (X, P). In Figs. 1 and 2, the Hasse diagrams of two non-isomorphic irreducible posets of dimension 3 are given.

For a poset (X, P), X is completely determined by P so we may write $\dim P$ for $\dim(X, P)$. On the other hand, there are many posets consisting of a familiar set X with a standard partial order P, and in such cases

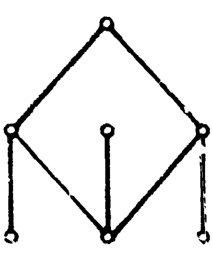

Fig. 2.

we may write dim X for dim(X, P). For example, the standard partial order on $\mathbf{R}^n$ is the product ordering and with this ordering dim $\mathbf{R}^n = n$.

A third formulation of the dimension of a poset (X, P) uses the observation that if C is a collection of linear extensions of P, $\bigcap C = P$ and $x \parallel y$ in P, then there must exist L_1 and L_2 in C with x over y in L_1 and y over x in L_2. On the other hand, if we have a collection D of extensions of P (not necessarily linear extensions of P) which satisfy this property, then $\bigcap D = P$. And if we let C be a collection of linear extensions of P obtained by extending each partial order in D to a linear order, then $\bigcap C = P$. Thus we see that dim(X, P) is the minimum number of extensions of P whose union contains the complement of P in $X \times X$. This formulation of dimension will prove quite useful in the arguments appearing in this paper.

For a finite poset, the height is defined as one less than the maximum number of points in a chain. For example, the height of the poset in Fig. 1 is one while the height of the poset in Fig. 2 is two.

3. Definition of the crown S_n^k

For each $n \geq 3$, $k \geq 0$, we define the crown S_n^k as the poset of height 1 with $n + k$ maximal elements $a_1, a_2, \ldots, a_{n+k}$ and $n + k$ minimal elements $b_1, b_2, \ldots, b_{n+k}$. Each b_i is incomparable with $a_i, a_{i+1}, a_{i+2}, \ldots, a_{i+k}$ and under the remaining $n - 1$ maximal elements. Of course, it is necessary to interpret the subscripts in this definition cyclically and hereafter such statements will be made without reminder of the necessary cyclic interpretation. To illustrate this definition, the Hasse diagram for S_4^2 is shown in Fig. 3.

In future arguments concerning S_n^k, we denote the set of maximal elements by A, the minimal elements by B and use P for the partial order. S_n^k is then formally the poset $(A \cup B, P)$.

For each minimal element $b \in B$, the set of maximal elements which are incomparable with b is denoted $I(b)$. For a maximal element $a \in A$, $I(a)$ is defined dually. For each b_i, $I(b_i)$ is a sequence in A: we refer to a_i as the first element in $I(b_i)$ and call $[a_i, a_{i+1}, a_{i+2}, \ldots, a_{i+k}]$ the cycle order on $I(b_i)$. For any subset D of A (or B), there is a linear ordering induced on D by subscripts which we refer to as the subscript order on D. The reverse subscript order on D is defined similarly. For example, in

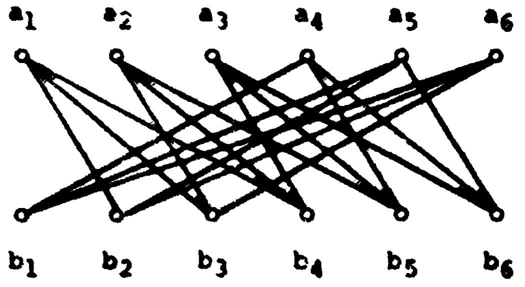

Fig. 3.

S_4^2. $I(b_5) = \{a_1, a_5, a_6\}$; the first element in $I(b_5)$ is a_5; the cycle order on $I(b_5)$ is $[a_5, a_6, a_1]$; the subscript order on $I(b_5)$ is $[a_6, a_5, a_1]$; and the reverse subscript order in $I(b_5)$ is $[a_1, a_5, a_6]$.

4. The dimension of the crown S_n^k

We first observe that any discussion concerning the dimension of crowns can be simplified by constantly appealing to the following lemma and its corollary.

Lemma 4.1. *Let C be a collection of linear extensions of S_n^k. Then the following statements are equivalent:*

(i) $\bigcap C = P$;

(ii) *for each $b \in B$ and $a \in I(b)$, there exists $L \in C$ with b over a in L.*

Proof. In view of our earlier remarks concerning alternate definitions of dimension, it is clear that (i) implies (ii).

On the other hand, to show that (ii) implies (i), we must prove that if $x \parallel y$ in P, then there exist $L_1, L_2 \in C$ with x over y in L_1 and y over x in L_2.

If a and a' are distinct elements of A, then $I(a) - I(a') \neq \emptyset \neq I(a') - I(a)$. Let $b \in I(a) - I(a')$ and $b' \in I(a') - I(a)$; then by statement (ii), there exist $L_1, L_2 \in C$ with b over a in L_1 and b' over a' in L_2. Since each $L \in C$ is an extension of P and a is over b' in P and a' is over b in P, we see that a' is over a in L_1 and a is over a' in L_2.

This argument obviously dualizes to the situation where b and b' are distinct elements of B. Thus it remains only to show that if $b \in B$ and $a \in I(b)$, then there exists $L \in C$ with a over b in L. Since $n \geq 3$,

$A - I(b) \neq \emptyset$, let $a' \in A - I(b)$. Then there exists $L \in C$ with a over a' in L and since a' is over b in P, a is over b in L. The proof of Lemma 4.1 is now complete.

In order to avoid listing all the elements of S_n^k, the following corollary will prove useful.

Corollary 4.2. *The following statements are equivalent:*

(i) $\dim S_n^k \leq m$;

(ii) *there exists a collection D of m linear extensions of subposets of S_n^k such that for each $b \in B$ and $a \in I(b)$, there exists $L \in D$ such that b is over a in L.*

Proof. As before, (i) implies (ii) is trivial. Conversely, it is easy to show that for every poset (X, P), every $Y \subseteq X$ and every linear extension L of P restricted to Y, there is a linear extension M of P such that M restricted to Y is L. Then it is possible to form a collection C of m linear extensions of S_n^k satisfying statement (ii) of the preceding lemma and the proof of our corollary is complete.

Statement (ii) of this corollary suggests a scheme for obtaining an upperbound on the dimension of a crown. It seems that the essential feature of an efficiently chosen collection C of linear extensions of S_n^k is that the extensions in C put as many minimal elements over maximal elements as possible. Keeping this principle in mind we consider the linear extension L_1 of $I(b_1) \cup I(a_1)$ defined by

$$L_1: [b_1, a_{k+1}, b_{n+k}, a_k, b_{n+k-1}, a_{k-1}, \ldots, b_{n+2}, a_2, b_{n+1}, a_1].$$

In this list, b_1 is over every a in $I(b_1)$, b_{n+k} is over every a in $I(b_{n+k}) \cap I(b_1)$, and so on. Now it is not necessary to put b_1 over a maximal element in any other list, but b_{n+k} must be over a_{n+k}, b_{n+k-1} must be over a_{n+k} and a_{n+k-1}, and so on. This suggests constructing a second linear extension

$$L_2: [b_n, a_n, b_{n+1}, a_{n+1}, b_{n+2}, a_{n+2}, \ldots, b_{n+k}, a_{n+k}].$$

Now we see that we have satisfied the requirements of statement (ii) for the subset $D_1 = \{b_n, b_{n+1}, b_{n+2}, \ldots, b_{n+k}, b_1\}$ with these two extensions

L_1 and L_2, i.e. if $b \in D_1$ and $a \in I(b)$, then b is over a in L_1 or L_2. Furthermore, D_1 consists of a sequence of $k + 2$ minimal elements. If we simply add $k + 2$ to the subscript of each element of L_1 and L_2, we obtain two new linear extensions of subposets of S_n^k. If we denote these extensions by L_3 and L_4, then it is clear that the requirements of statement (ii) are satisfied for the set $D_2 = \{b_2, b_3, \ldots, b_{k+3}\}$ by L_3 and L_4. Continuing this process, we obtain a sequence of subsets $D_1, D_2, D_3, \ldots, D_t$, where $t = \lceil (n + k)/(k + 2) \rceil$, which cover B. And we have obtained a collection $C = \{L_1, L_2, L_3, \ldots, L_{2t}\}$ of linear extensions of subposets S_n^k which satisfies the requirements of statement (ii) for B. Thus we have established an upperbound on the dimension of crowns.

Theorem 4.3. $\operatorname{Dim} S_n^k \leq 2\lceil (n + k)/(k + 2) \rceil$.

Example 4.4. $\operatorname{Dim} S_9^3 \leq 2\lceil 12/5 \rceil = 6$ and the six linear extensions of subposets of S_9^3 produced by Theorem 4.3 are:

L_1: $[b_1, a_4, b_{12}, a_3, b_{11}, a_2, b_{10}, a_1]$,
L_2: $[b_9, a_9, b_{10}, a_{10}, b_{11}, a_{11}, b_{12}, a_{12}]$,
L_3: $[b_6, a_9, b_5, a_8, b_4, a_7, b_3, a_6]$,
L_4: $[b_2, a_2, b_3, a_3, b_4, a_4, b_5, a_5]$,
L_5: $[b_{11}, a_2, b_{10}, a_1, b_9, a_{12}, b_8, a_{11}]$,
L_6: $[b_7, a_7, b_8, a_8, b_9, a_9, b_{10}, a_{10}]$.

In this construction process, we observe that in the first $2t - 2$ lists there is no duplication among the lists in accomplishing the requirements of statement (ii) and each of these lists appears to be constructed in a reasonably efficient manner. These observations suggest that the upperbound on the dimension of crowns given in Theorem 4.3 is fairly close to the actual dimension. Moreover, they suggest a very natural approach for establishing a lower bound on the dimension of crowns.

We begin by defining the weight of a crown S_n^k, denoted $W(S_n^k)$, as the total number of pairs (b, a) with $b \in B$ and $a \in I(b)$. Since there are $n + k$ minimal elements and $|I(b)| = k + 1$ for every $b \in B$, we see that $W(S_n^k) = (n + k)(k + 1)$. If $Y \subseteq A \cup B$ and L is a linear extension of P restricted to Y, then we define the weight of L, denoted $W(L)$, as the number of pairs (b, a), where $b \in B \cap Y$, $a \in I(b) \cap Y$ and b is over a in L. If $b \in B \cap Y$, we define the weight of b in L, denoted $W_L(b)$, as the number of pairs (b, a) with $a \in I(b) \cap Y$ and b over a in L.

We now make the following elementary observations concerning these definitions.

Fact 1:

$$W(L) = \sum_{b \in B \cap Y} W_L(b).$$

Fact 2: If C is a collection of linear extensions of subposets of S_n^k which satisfies the requirements of statement (ii), then

$$\sum_{L \in C} W(L) \geq W(S_n^k).$$

An easy computation shows that the weights of the linear extensions constructed in the proof of Theorem 4.3 are all $\frac{1}{2}(k+1)(k+2)$.

This suggests the following result.

Lemma 4.5. *If L is a linear extension of a subposet of S_n^k, then $W(L) \leq \frac{1}{2}(k+1)(k+2)$.*

Proof. We may assume without loss of generality that L is a linear extension of S_n^k, otherwise we extend L to all of S_n^k which cannot decrease the weight of L. Now L restricted to B is a linear ordering on the minimal elements. Let b^1 be the largest minimal element in this ordering, b^2 the next largest, and so on. Since L is a linear extension of S_n^k, if a is under b^i, then $a \in I(b^1) \cap I(b^2) \cap \ldots \cap I(b^i)$ and $\{b^1, b^2, \ldots, b^i\} \subseteq I(a)$. Since $|I(a)| = k+1$, we see that $W(b^i) = 0$ for every i satisfying $k+2 \leq i \leq n+k$. To bound $W(b^i)$ for $i \leq k+1$, we make the following observation. If S is a finite set of points on a line or a circle and $A_1, A_2, \ldots, A_m$ are distinct subsets of S each consisting of p consecutive points, then the number of points in $A_1 \cap A_2 \cap \ldots \cap A_m$ is at most $p+1-m$. Hence it follows that $W(b^i) \leq k+2-i$ for every i such that $1 \leq i \leq k+1$.

Now we have

$$W(L) = \sum_{b \in B} W_L(b) = \sum_{i=1}^{n+k} W_L(b^i) = \sum_{i=1}^{k+1} W_L(b^i)$$
$$\leq \sum_{i=1}^{k+1} k+2-i = \sum_{i=1}^{k+1} i = \tfrac{1}{2}(k+1)(k+2).$$

Theorem 4.6. $\lceil 2(n+k)/(k+2)\rceil \le \dim S_n^k \le 2\lceil (n+k)/(k+2)\rceil$.

Proof. If $\dim S_n^k = t$ and $\{L_1, L_2, \ldots, L_t\}$ is a collection of linear extensions of S_n^k which satisfy the requirements of statement (ii), then

$$W(S_n^k) \le \sum_{i=1}^{t} W(L_i),$$

and thus $(n+k)(k+1) \le \frac{1}{2}t(k+1)(k+2)$ and $2(n+k)/(k+2) \le t$. The theorem follows since t is an integer.

The upper and lower bounds on the dimension of crowns given in Theorem 4.6 are either equal or differ by one depending on the size of the remainder r in the equation $n + k = (k+2)q + r$, where $0 \le r < k+2$. If $r = 0$, both bounds yield $\dim S_n^k = 2q$ and if $\frac{1}{2}(k+2) < r < k+2$, then we have $\dim S_n^k = 2q+2$, but if $0 < r \le \frac{1}{2}(k+2)$, then the lower bound is $2q+1$ and the upper bound is $2q+2$. If $r = 1$, we can see that $\dim S_n^k = 2q+1$ since the last list L_{2t} constructed by the scheme used in the proof of Theorem 4.3 may be omitted. In fact, the only necessary role performed by L_{2t-1} is to put b_{n-1} over every a in $I(b_{n-1})$. We illustrate this argument with the appropriate linear extensions for S_3^2.

Example 4.7.
$L_1 = [b_1, a_3, b_5, a_2, b_4, a_1]$,
$L_2 = [b_3, a_1, b_4, a_4, b_5, a_5]$,
$L_3 = [b_2, a_2, a_3, a_4]$.

However, it is clear that this line of reasoning will not settle the question when $2 \le r \le \frac{1}{2}(k+2)$.

Theorem 4.8. $\operatorname{Dim} S_n^k = \lceil 2(n+k)/(k+2)\rceil$.

Proof. Let $n + k = q(k+2) + r$, where $0 \le r < k+2$. Then we may assume that $0 < r \le \lfloor \frac{1}{2}(k+2)\rfloor$, otherwise, as noted before, the bounds of Theorem 4.6 are equal.

We first partition the set A of maximal elements into $q+1$ subsets $A_1, A_2, A_3, \ldots, A_{q+1}$ as follows:

(1) For each $j \le q$, let $i = (k+2)(j-1)$ and then define $A_j = \{a_{i+1}, a_{i+2}, a_{i+3}, \ldots, a_{i+k+2}\}$.

(2) Let $A_{q+1} = \{a_i : q(k+2) < i \leq n+k\}$.

We observe that each A_j is an interval in A, the collection $A_1, A_2, \ldots, A_{q+1}$ partitions A, $|A_1| = |A_2| = \ldots = |A_q| = k+2$, and $|A_{q+1}| < \lfloor \frac{1}{2}(k+2) \rfloor$.

Now we further partition A into $2q+1$ subsets $I_1, I_2, I_3, \ldots, I_{2q+1}$ as follows:

(1) For each $j \leq q$, let I_{2j-1} be the subset of A_j containing the first $\lfloor \frac{1}{2}(k+2) \rfloor$ elements as determined by the reverse subscript order on A_j.

(2) For each $j \leq q$, let I_{2j} be the subset of A_j containing the last $\lceil \frac{1}{2}(k+2) \rceil$ elements as determined by the reverse subscript order on A_j.

(3) Let $I_{2q+1} = A_{q+1}$.

We now observe that each I_j is an interval in A, the collection $I_1, I_2, \ldots, I_{2q+1}$ partitions A, and most importantly $|I_j \cup I_{j+1}| \leq k+2$ for every $j \leq 2q+1$.

Example 4.9. For S_{18}^5:

$A_1 = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$; $I_1 = \{a_1, a_2, a_3\}$; $I_2 = \{a_4, a_5, a_6, a_7\}$;
$A_2 = \{a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}\}$; $I_3 = \{a_8, a_9, a_{10}\}$;
$I_4 = \{a_{11}, a_{12}, a_{13}, a_{14}\}$;
$A_3 = \{a_{15}, a_{16}, a_{17}, a_{18}, a_{19}, a_{20}, a_{21}\}$; $I_5 = \{a_{15}, a_{16}, a_{17}\}$;
$I_6 = \{a_{18}, a_{19}, a_{20}, a_{21}\}$;
$A_4 = \{a_{22}, a_{23}\}$; $I_7 = \{a_{22}, a_{23}\}$.

Now we are ready to construct a collection $L_1, L_2, \ldots, L_{2q+1}$ of linear extensions of subposets of S_n^k which satisfy the requirements of statement (ii). If we denote the element of I_i with smallest subscript by a^i, then L_i will be a linear extension of $I_i \cup I_{i-1} \cup I(a^i)$. To construct L_i we order I_i in subscript order, I_{i-1} in reverse subscript order and then place every element in I_{i-1} over every element of I_i. Then place the minimal elements of $I(a_i)$ as high in this list of $I_i \cup I_{i-1}$ as the ordering P will permit; this process produces an extension P_i' of P restricted to $I_i \cup I_{i-1} \cup I(a^i)$. Then let L_i be any linear extension of P_i'.

To complete the proof of Theorem 4.8, it remains only to show that this collection of $2q+1$ linear extensions of subposets of S_n^k satisfies the requirements of statement (ii). Suppose $b \in B$ and $a \in I(b)$. Then $a \in I_i$ for some $i \leq 2q+1$, and if b is not over a in L_i, then it follows that b will be over a in L_{i+1}.

Example 4.10. For S_{18}^5, the seven extensions are:

L_1: $[b_{22}, b_{21}, a_{22}, b_{23}, a_{23}, b_1, a_3, b_{20}, a_2, b_{19}, a_1]$,
L_2: $[a_1, b_2, a_2, b_3, a_3, b_4, a_7, b_1, a_6, b_{23}, a_5, b_{22}, a_4]$,
L_3: $[a_4, b_5, a_5, b_6, a_6, b_7, a_7, b_8, a_{10}, b_4, a_9, b_3, a_8]$,
L_4: $[a_8, b_9, a_9, b_{10}, a_{10}, b_{11}, a_{14}, b_8, a_{13}, b_7, a_{12}, b_6, a_{11}]$,
L_5: $[a_{11}, b_{12}, a_{12}, b_{13}, a_{13}, b_{14}, a_{14}, b_{15}, a_{17}, b_{11}, a_{16}, b_{10}, a_{15}]$,
L_6: $[a_{15}, b_{16}, a_{16}, b_{17}, a_{17}, b_{18}, a_{21}, b_{15}, a_{20}, b_{14}, a_{19}, b_{13}, a_{18}]$,
L_7: $[b_{18}, a_{18}, b_{19}, a_{19}, b_{20}, a_{20}, b_{21}, a_{21}, b_{22}, a_{23}, b_{17}, a_{22}]$.

The reader may observe that an obvious modification of the above construction can be used to establish the upper bound $\lceil 2(n+k)/(k+2)\rceil$ directly. We have included the apparently weaker result given in Theorem 4.3 since it motivates the arguments of both Theorem 4.6 and 4.8, and in arguments given later in this paper concerning irreducible crowns, the construction process of Theorem 4.3 will be of value.

5. Irreducible crowns

If we reconsider Example 4.7 given in the preceding section, we remember that the only requirement for L_3 was that it put b_2 over all maximal elements in $I(b_2)$. This suggests that if we remove b_2 from S_3^2, then the lists L_1 and L_2 would be sufficient to determine the restriction of P to $S_3^2 - \{b_2\}$, i.e., $\dim S_3^2 - \{b_2\} = 2$. However, the reader may verify that while it is true that $\dim S_3^2 - \{b_2\} = 2$, some care must be taken in producing two linear extensions of $S_3^2 - \{b_2\}$ whose intersection is the restriction of P to $S_3^2 - \{b_2\}$ since arbitrary extensions of L_1 and L_2 may not be sufficient. For example, the linear extensions M_1: $[a_4, a_5, b_1, a_3, b_5, a_2, b_4, a_1, b_3]$ and M_2: $[a_1, a_2, b_3, a_3, b_4, a_4, b_5, a_5, b_1]$ of $S_3^2 - \{b_2\}$ are extensions of L_1 and L_2, respectively, but $(a_5, a_4) \in M_1 \cap M_2$. However, it is easy to see that these apparent difficulties can be overcome.

Lemma 5.1. *Let $x \in S_n^k$. Then the following statements are equivalent:*
(i) $\dim S_n^k - \{x\} \leq m$;
(ii) *there exist linear extensions $L_1, L_2, L_3, \ldots, L_m$ of subposets of S_n^k such that if $b \in B - \{x\}$ and $a \in I(b) - \{x\}$, then b is over a in some L_i.*

Proof. It is clear that statement (i) implies statement (ii). We assume now that $L_1, L_2, \ldots, L_m$ satisfy the requirements of statement (ii) and that $x \in A$. Let $M_1, M_2, M_3, \ldots, M_m$ be arbitrary linear extensions of $L_1, L_2, \ldots, L_m$ to all of $S_n^k - \{x\}$. The arguments used to prove Lemma 4.1 show that if a and a' are distinct elements of $A - x$, then a is over a' in some M_i and a' is over a in some M_j. Furthermore, if $b \in B$ and $a \in I(b) - \{x\}$, then we may choose $a' \in A - I(b) - \{x\}$ and if a is over a' in M_i, a is also over b in M_i.

Now suppose b and b' are distinct elements of B. If $I(b) - I(b') - \{x\} \neq \emptyset \neq I(b') - I(b) - \{x\}$, then b is over b' in some M_i and b' is over b in some M_j. The only difficulty arises then when one of the sets $I(b) - I(b') - \{x\}$ and $I(b') - I(b) - \{x\}$ is empty. There are only two cases where this can happen. If we assume that $x = a_i$, then the pairs of minimal elements with which we must be concerned are (b_i, b_{i+1}) and (b_{i-k-1}, b_{i-k}). Since $b_{i+1} \| a_{i+k+1}$ and $b_i < a_{i+k+1}$ in P, b_{i+1} is over a_{i+k+1} in some M_{j_1}. Similarly, b_{i-k-1} is over a_{i-k-1} in some M_{m_1} and since M_{j_1} and M_{m_1} are extensions of P restricted to $S_n^k - \{a_i\}$, b_{i+1} is over b_i in M_{j_1} and b_{i-k-1} is over b_{i-k} in M_{m_1}. By the argument given above, there exist lists M_{j_2}, M_{m_2} such that a_{i+k+1} is over b_{i+1} in M_{j_2} and a_{i-k-1} is over b_{i-k-1} in M_{m_2}. If b_i is under b_{i+1} in M_{j_2}, move it to a position immediately above b_{i+1}; the resulting linear ordering of $S_n^k - \{a_i\}$ is an extension of P restricted to $S_n^k - \{a_i\}$. Similarly, if b_{i-k} is under b_{i-k-1} in M_{m_2}, we move it immediately above b_{i-k-1}.

Now it is clear that this collection of linear extensions of $S_n^k - \{a_i\}$ intersects to give the restriction of P to $S_n^k - \{a_i\}$. The argument when $x \in B$ is dual.

We can now see that Example 4.7 is merely one example of the following theorem.

Theorem 5.2. *If $n + k = q(k + 2) + 1$, then S_n^k is irreducible and has dimension $2q + 1$.*

Proof. We form $2q$ linear extensions $L_1, L_2, \ldots, L_{2q}$ as given in the proof of Theorem 4.3; these lists satisfy the requirements of statement (ii) for $S_n^k - \{b_{n-1}\}$ and thus $\dim S_n^k - \{b_{n-1}\} \leq 2q$. But Theorem 4.8 states that $\dim S_n^k = 2q + 1$. Since $\dim S_n^k - \{x\} = \dim S_n^k - \{b_{n-1}\}$ for every $x \in S_n^k$, the proof is complete.

Example 5.3. If $n = 3$ and $q = 1$, we have $3 + k = 1(k + 2) + 1$ and k is arbitrary. The family $\{S_3^k : k > 0\}$ are the original crowns as employed by Baker, Fishburn and Roberts [1].

Example 5.4. When $k = 0$, we obtain the standard maximal dimensional posets S_{2q+1}^0 of odd dimension as discussed by Bogart and Trotter [4]. It is interesting that the family $\{S_{2q}^0 : q \geq 2\}$ was shown to be irreducible by Hiraguchi [6] but they are not given by Theorem 5.2. However, this family will appear as a special case in Theorem 5.6.

Example 5.5. Some irreducible crowns of dimension 5: S_5^0, S_6^1, S_7^2, S_8^3, S_9^4, etc.; of dimension 7: S_7^0, S_9^1, S_{11}^2, S_{13}^3, S_{15}^4, etc.; of dimension 9: S_9^0, S_{12}^1, S_{15}^2, S_{18}^3, S_{21}^4, etc.

We may expect that other families of irreducible crowns can be obtained by careful examination of the construction process in Theorem 4.8.

Theorem 5.6. *Suppose $n + k = q(k + 2) + [\frac{1}{2}(k + 2)] + 1$. Then S_n^k is irreducible iff $k = 0$ or k is odd.*

Proof. Suppose first that k is odd and let $k = 2t + 1$; then $[\frac{1}{2}(k + 2)] = t + 1$ and $\dim S_n^k = 2q + 2$. We also observe that $|I_{2j+1}| = t + 1$ for $j = 0, 1, 2, ..., q$, and that $I_{2q+2} = \{a_{n+k}\}$. In the construction process outlined in the proof of Theorem 4.8, we make the following modifications.

In forming the extension L_1, place I_{2q+1} on top of I_1 instead of I_{2q+2} on top of I_1 as was originally done. Order $I_{2q+1} \cup I_1 \cup I(a_1)$ as before. The extensions $L_2, L_3, ..., L_{2q+1}$ are unchanged.

Now we verify that this collection $L_1, L_2, ..., L_{2q+1}$ satisfies the requirements of statement (ii) for $S_n^k - \{a_{n+k}\}$. Clearly, we need only be concerned with minimal elements from $I(a_{n+k})$ which are incomparable with at least one but not all maximal elements in I_{2q+1}. But any such minimal element b is incomparable with every maximal element in I_1, otherwise

$$k + 1 = |I(b)| \leq |I_{2q+1}| + |I_{2q+2}| + |I_1| - 2$$
$$= (t + 1) + 1 + (t + 1) - 2 = 2t + 1 = k .$$

Hence b is over a in L_1. This completes the argument for the case when k is odd.

Now suppose that k is even and positive. We must show that S_n^k is not irreducible. To accomplish this we show that $\dim(S_n^k - \{x\}) = \dim S_n^k = 2q + 2$ for every $x \in S_n^k$. Of course, it is sufficient to show that $\dim S_n^k - \{b_1\} = 2q + 2$. Suppose that $\dim S_n^k - \{b_1\} = 2q + 1$ and that $L_1, L_2, \ldots, L_{2q+1}$ are linear orders of $S_n^k - \{b_1\}$ which satisfy the requirements of statement (ii). We proceed now to obtain a contradiction.

If we let $k = 2t > 0$, then $n + k = q(k + 2) + [\frac{1}{2}(k + 2)] + 1$ implies $n + k - 1 = (2q + 1)(t + 1)$. Now there are $(n + k - 1)(k + 1) = (2q + 1)(t + 1)(2t + 1)$ pairs (b, a) with $b \neq b_1$ and $a \in I(b)$ and for each such pair, b must be over a in some L_i. Now each list L_i has weight $\leq \frac{1}{2}(k + 1)(k + 2) = (2t + 1)(t + 1)$. This shows that each L_i must have the maximum allowable weight. Hence the lowest a in each L_i must be over b_1 in P since it has $k + 1$ b's over it in L_i none of which is b_1. Furthermore, if $b \neq b_1$ and $a \in I(b)$, then b is over a in exactly one L_i. Since b_2 is under every maximal element which covers b_1 except a_{k+2}, we see that one of the lists must have a_{k+2} as its lowest maximal element. We may suppose that this occurs in L_1; then we see that b_2 must be the highest b in L_1. The next highest b in L_1 must have weight k and since only b_1 and b_3 have k common incomparable maximal elements with b_2, this b is b_3. Continuing this argument, we see that L_1 restricted to $I(b_2) \cup I(a_{k+2})$ is $[b_2, a_2, b_3, a_3, b_4, a_4, \ldots, b_{k+2}, a_{k+2}]$. We note that this accounts for the weight of L_1 completely. In particular, b_3 is under a_{k+3} in L_1 so b_3 must be over a_{k+3} in some other list, say L_2. In this list, b_3 is over only one maximal element, so a_{k+3} is the lowest maximal element in L_2 and all elements of $I(a_{k+3})$ are over it in L_2. Now the highest b in L_2 has weight $k + 1$, is incomparable with a_{k+3}, is less than a_{k+2} in P and must clearly be b_{k+3}. It is easy to see that L_2 restricted to $I(a_{k+3}) \cup I(b_{k+3})$ is $[b_{k+3}, a_{2k+3}, b_{k+2}, a_{2k+2}, b_{k+1}, a_{2k+1}, \ldots, b_3, a_{k+3}]$ and again this accounts for the weight of L_2.

We observe that in the pair of lists chosen thus far, L_1 and L_2, the only minimal elements which are over maximal elements with which they are incomparable, come from the set of $k + 2$ minimal elements $\{b_2, b_3, b_4, \ldots, b_{k+3}\}$. It is easy to see how this argument can be repeated to distinguish $2q$ of the lists in which only the first $q(k + 2)$ minimal elements after b_1 can be over maximal elements. This leaves only one list to place each of the remaining $\lfloor\frac{1}{2}(k+2)\rfloor = t+1$ minimal elements over all the maximal elements with which it is incomparable. But in any list there can be at most one minimal element with weight $k+1$. Since $t+1 \geq 2$, this is the contradiction which completes the argument when k is even and positive.

When $t = 0$ and $k = 0$, the linear extensions $L_1, L_2, \ldots, L_{n-1}$ of $S_n^0 - \{b_1\}$ defined by

$$L_i:\ [a_2, a_3, \ldots, \widehat{a_{i+1}}, \ldots, a_n, a_1, b_{i+1}, a_{i+1}, b_2, b_3, \ldots, \widehat{b_{i+1}}, \ldots, b_n]$$

satisfy the requirements of statement (ii) and now the proof of Theorem 5.6 is complete.

Example 5.7. Some irreducible crowns of dimension 4: $S_4^1, S_5^3, S_6^5, S_7^7, S_8^9$, etc.; of dimension 6: $S_7^1, S_{10}^3, S_{13}^5, S_{16}^7, S_{19}^9$, etc.; of dimension 8: $S_{10}^1, S_{15}^3, S_{20}^5, S_{25}^7, S_{30}^9$, etc.

We note here that while Theorem 4.8 could easily be modified to include Theorem 4.3, it is not clear that Theorem 5.6 includes Theorem 5.2 as difficulty is encountered in the construction of the first list. However, it is easy to show that the only irreducible crowns arise from Theorem 5.2 and 5.6.

Theorem 5.8. *If $n + k = q(k+2) + r$ with $2 \leq r \leq \lfloor\frac{1}{2}(k+2)\rfloor$, then* $\dim S_n^k = \dim S_n^k - \{x\} = 2q+1$ *for every $x \in S_n^k$. And if $n + k = q(k+2) + \lfloor\frac{1}{2}(k+2)\rfloor + r$ with $2 < r < \lfloor\frac{1}{2}(k+2)\rfloor$, then* $\dim S_n^k = \dim S_n^k - \{x\} = 2q+2$ *for every $x \in S_n^k$.*

Proof. In the first case, $2q$ linear extensions of $S_n^k - \{x\}$ have total weight $\leq 2q\lfloor\frac{1}{2}(k+1)(k+2)\rfloor = q(k+1)(k+2)$ but the weight of

$$\begin{aligned} S_n^k - \{x\} &= (n+k-1)(k+1) = [q(k+2)+(n-1)](k+1) \\ &= q(k+1)(k+2)+(n-1)(k+1) > q(k+1)(k+2)\,. \end{aligned}$$

The second statement is proved similarly.

Fig. 4.

6. Crowns and cartesian products

If X and Y are posets, then $\dim(X \times Y) \leq \dim X + \dim Y$. However, it is easy to prove (see [2]) that if X and Y have universal bounds, then $\dim(X \times Y) = \dim X + \dim Y$. Now suppose $\mathcal{C} = \{C_1, C_2, C_3, ..., C_t\}$ is a finite collection of irreducible crowns with universal bounds added. Then there is an irreducible subposet X of $C_1 \times C_2 \times C_3 \times ... \times C_t$ with $\dim X = \dim C_1 + \dim C_2 + ... + \dim C_t$ and it is very easy to see how to determine X.

For collection $\mathcal{C} = \{(X_i, P_i): i = 1, 2, ..., t\}$ of posets, we define the poset (X, P) called the dimension product of $\mathcal{C}$ and denoted $(X_1, P_1) \otimes (X_2, P_2) \otimes (X_3, P_3) \otimes ... \otimes (X_t, P_t)$ by letting X be the disjoint union $X_1 \cup X_2 \cup ... \cup X_t$ and $P = P_1 \cup P_2 \cup ... \cup P_t \cup \{(b, a): b$ is minimal in some X_i, a is maximal in some X_j and $i \neq j\}$. We illustrate this definition with the Hasse diagrams given in Fig. 4.

We note that if some of the posets (X_i, P_i) contain loose points, i.e. points which are both maximal and minimal, then the definition given in the preceding paragraph may be improper. Since we do not want to exclude such posets, we will partition the loose vertices into disjoint sets M_1 and M_2. Those points in M_1 are then treated as maximal but not minimal and those in M_2 are treated as minimal but not maximal.

Suppose $X_1, X_2, ..., X_t$ is a collection of posets of height one and let Y_i denote the poset formed by adding universal bounds 0_i and 1_i to X_i for each i. We can now define an embedding of the dimension product $X_1 \otimes X_2 \otimes ... \otimes X_t$ into the cartesian product $Y_1 \times Y_2 \times ... \times Y_t$ as follows. If a is a maximal element of $X_1 \otimes X_2 \otimes ... \otimes X_t$, then a is a maximal element of some X_i and we identify this element with $(1_1, 1_2, ..., 1_{i-1}, a, 1_{i+1}, ..., 1_t)$ from $Y_1 \times Y_2 \times ... \times Y_t$. Similarly, we identify a minimal element b from X_i with $(0_1, 0_2, ..., 0_{i-1}, b, 0_{i+1}, ..., 0_t)$. Hence we can conclude that

$$\begin{aligned}\dim(X_1 \otimes X_2 \otimes \ldots \otimes X_t) &\le \dim(Y_1 \times Y_2 \times \ldots \times Y_t)\\ &< \dim Y_1 + \dim Y_2 + \ldots + \dim Y_t\\ &= \dim X_1 + \dim X_2 + \ldots + \dim X_t\,.\end{aligned}$$

Suppose we now extend our definition of the crown S_n^k to include the cases $n = 1, k = 0$ and $n = 2, k = 0$. S_2^1 is the disjoint union of two chains $\{b_1, a_2\}$ and $\{b_2, a_1\}$ but $S_1^0 = \{a_1, b_1\}$ is a two element antichain. So that we can consider $S_1^0 \otimes S_n^k$, we let $M_1 = \{a_1\}$ and $M_2 = \{b_1\}$. With this convention we see that for each $n \ge 2$, S_n^0 is isomorphic to the dimension product $S_1^0 \otimes S_1^0 \otimes \ldots \otimes S_1^0$ of n copies of S_1^0.

However, it is easy to see that if $k > 1$, then S_n^k is not isomorphic to the dimension product of two or more crowns. More generally if (X, P) is isomorphic to the dimension product of two or more crowns of which none have upper parameter $k = 0$ with $n \ge 2$ then this factorization is unique up to the order of the factors.

Theorem 6.1. *Suppose $\mathcal{C} = \{C_1, C_2, \ldots, C_t\}$ is a nonempty collection of irreducible crowns. Then $C_1 \otimes C_2 \otimes \ldots \otimes C_t$ is irreducible and has dimension* $= \dim C_1 + \dim C_2 + \ldots + \dim C_t$.

Proof. Let $(X, P) = C_1 \otimes C_2 \otimes C_3 \otimes \ldots \otimes C_t$; then since each C_i is a subposet of X, for every linear extension L of X, we can compute $W_i(L)$, the weight of L restricted to C_i. But if $W_i(L) > 0$, then $W_j(L) = 0$ for every $j \ne i$. This shows that $\dim X \ge \dim X_1 + \dim X_2 + \ldots + \dim X_t$. We have already observed that $\dim X \le \dim X_1 + \dim X_2 + \ldots + \dim X_t$. Furthermore, if $x \in X$, then x belongs to some X_i and $\dim(X_i - x) < \dim X_i$. Since

$$X - x = X_1 \otimes X_2 \otimes \ldots \otimes X_{i-1} \otimes (X_i - x) \otimes X_{i+1} \otimes \ldots \otimes X_t\,,$$

it follows that $\dim X - x < \dim X$ and we conclude that X is irreducible.

Corollary 6.2. *If X is the dimension product of a collection of irreducible crowns, then $S_1^0 \otimes X$ is irreducible and* $\dim S_1^0 \otimes X = 1 + \dim X$.

Example 6.3. Although there are no irreducible posets of dimension 4 on 9 points [8], there are at least two on 10 points: S_4^1 and $S_3^1 \otimes S_1^0$.

Example 6.4. There are at least 28 nonisomorphic irreducible posets of dimension 8 on 50 points:

S_{20}^5, $\quad S_6^5 \otimes S_7^7$, $\quad S_5^3 \otimes S_8^9$, $\quad S_4^1 \otimes S_9^{11}$, $\quad S_{16}^7 \otimes S_2^0$,

$S_{13}^8 \otimes S_3^1$, $\quad S_4^1 \otimes S_3^{16} \otimes S_1^0$, $\quad S_3^1 \otimes S_3^{16} \otimes S_2^0$,

$S_{12}^7 \otimes S_3^3$, $\quad S_5^3 \otimes S_3^{13} \otimes S_1^0$, $\quad S_3^2 \otimes S_3^{15} \otimes S_2^0$,

$S_{11}^6 \otimes S_3^5$, $\quad S_6^5 \otimes S_3^{10} \otimes S_1^0$, $\quad S_3^3 \otimes S_3^{14} \otimes S_2^0$,

$S_{10}^5 \otimes S_3^7$, $\quad S_7^7 \otimes S_3^7 \otimes S_1^0$, $\quad S_3^4 \otimes S_3^{13} \otimes S_2^0$,

$S_9^4 \otimes S_3^9$, $\quad S_8^9 \otimes S_3^4 \otimes S_1^0$, $\quad S_4^5 \otimes S_3^{12} \otimes S_2^0$,

$S_8^3 \otimes S_3^{11}$, $\quad S_9^{11} \otimes S_3^1 \otimes S_1^0$, $\quad S_3^6 \otimes S_3^{11} \otimes S_2^0$,

$S_7^2 \otimes S_3^{13}$, $\quad S_3^7 \otimes S_3^{10} \otimes S_2^0$,

$S_6^1 \otimes S_3^{15}$, $\quad S_3^8 \otimes S_3^9 \otimes S_2^0$,

$S_5^0 \otimes S_3^{17}$.

7. Infinite crowns

When X is infinite, the dimension of (X, P) is the smallest cardinal number $\aleph$ for which there exists a collection $\mathcal{C}$ of linear extensions of P such that $\bigcap \mathcal{C} = P$ and $|\mathcal{C}| = \aleph$. Alternately, $\dim(X, P)$ is the smallest cardinal number $\aleph$ such that (X, P) can be imbedded in the cartesian product of $\aleph$ chains. In this section, we extend the definition of a crown to infinite sets to obtain some interesting examples of dimension for infinite sets.

Let A and B be disjoint copies of $\mathbf{R}$, the set of real numbers, and let $X_1 = X_2 = X_3 = A \cup B$. Now let $\Delta = \{(x, x): x \in A \cup B\}$ and then define:

$P_1 = \Delta \cup \{(b, a): b \in B, a \in A, b \neq a\}$;

$P_2 = \Delta \cup \{(b, a): b \in B, a \in A, a \notin [b, b+1]\}$;

$P_3 = \Delta \cup \{(b, a): b \in B, a \in A, a \in (b-1, b)\}$.

The reader may easily verify that $\dim(X_1, P_1) = c$, $\dim(X_2, P_2) = \aleph_0$ and $\dim(X_3, P_3) = 3$. Furthermore, these examples mirror the behavior of finite sets since for fixed n, $\dim S_n^k$ decreases monotonically from n to 3. Also note that (X_1, P_1) is isomorphic to the dimension product of $2^{\aleph_0}$ copies of S_1^0.

8. Conclusions

The results of this paper show that for each $n \geq 3$, there exist infinitely many nonisomorphic irreducible posets of dimension n. Other families of irreducible posets have been discovered [12], many of which possess a great deal of regularity in their structure.

The author intends to present this work and indicate the relationships between dimension and other invariants such as width, height, and cardinality in subsequent papers. Furthermore, the connections between dimension theory, graph coloring and other combinatorial topics will be established.

References

[1] K. Baker, P. Fishburn and F. Roberts, Partial order of dimension 2, interval orders, and interval graphs, RAND Memorandum, P 4376 (1970).

[2] G. Birkhoff, Lattice Theory, A.M.S. Colloq. Publ. 25 (Am. Math. Soc., Providence, R.I., 1967).

[3] K. Bogart, Maximal dimensional partially ordered sets I, Discrete Math. 5 (1973) 21–31.

[4] K. Bogart and W. Trotter, Maximal dimensional partially ordered sets II, Discrete Math. 5 (1973) 33–43.

[5] R.P. Dilworth, A decomposition theorem for partially ordered sets, Ann. Math. 51 (1950) 161–166.

[6] B. Dushnik and E. Miller, Partially ordered sets, Am. J. Math. 63 (1941) 600–610.

[7] T. Hiraguchi, On the dimension of orders, Sci. Rep. Kanazawa Univ. 4 (1955) 1–20.

[8] R. Kimble, Ph.D. Thesis, M.I.T., Cambridge, Mass.

[9] H. Komm, On the dimension of partially ordered sets, Am. J. Math. 70 (1948) 507–520.

[10] O. Ore, Theory of Graphs, A.M.S. Colloq. Publ. 38 (Am. Math. Soc., Providence, R.I., 1962).

[11] E. Szpilrajn, Sur l'extension de l'ordre partiel, Fund. Math. 16 (1930) 386–389.

[12] W.T. Trotter, Some families of irreducible partially ordered sets, in preparation.