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A Theory of Recursive Dimension for Ordered Sets*

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Abstract. The classical theorem of R. P. Dilworth asserts that a partially ordered set of width n can be partitioned into n chains. Dilworth's theorem plays a central role in the dimension theory of partially ordered sets since chain partitions can be used to provide embeddings of partially ordered sets in the Cartesian product of chains. In particular, the dimension of a partially-ordered set never exceeds its width. In this paper, we consider analogous problems in the setting of recursive combinatorics where it is required that the partially ordered set and any associated partition or embedding be described by recursive functions. We establish several theorems providing upper bounds on the recursive dimension of a partially ordered set in terms of its width. The proofs are highly combinatorial in nature and involve a detailed analysis of a 2-person game in which one person builds a partially ordered set one point at a time and the other builds the partition or embedding.

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0. Introduction

The *dimension* of a countable ordered set (A, R) is the least ordinal d such that (A, R) can be embedded into Q^d, where Q is the set of rational numbers with the usual order. We are interested in making this notion effective. Recall that, loosely speaking, a *partial recursive function* is a function that can be computed by an algorithm and a *recursive set* is a set whose characteristic function is a partial recursive function. A precise definition of these concepts can be found in Rogers [16]. The domain of a partial recursive function f may not be recursive; if it is we shall call f a *recursive function*. This is a slight variance from the normal definition. An ordered set (A, R) is *recursive* if both A and R are recursive sets. Thus recursive ordered sets are equipped with algorithms that, upon input of any two points, will determine whether the points are in the domain of the ordered set and, if so, the nature of the comparability between them or else declare them incomparable.

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We define the *recursive dimension* of (A, R) to be the least ordinal d such that (A, R) can be recursively embedded into Q^d.

The theory of dimension informs us of the connection between dimension, chain covering number, and width. A *(recursive) chain cover* of (A, R) is a tuple $(C_0, \ldots, C_{c-1})$ where $A = C_0 \cup \ldots \cup C_{c-1}$, $C_i \cap C_j$ is empty if $i \neq j$ and C_i is a (recursive) chain. The *(recursive) chain covering number* of (A, R) is the infimum of the sizes of the (recursive) chain covers of (A, R). The *width* of (A, R) is the supremum of the sizes of the antichains of (A, R). Dilworth's theorem [4] tells us that if the width of (A, R) is finite, then so is the chain covering number of (A, R), and, moreover, they are equal. It is easily seen that the dimension of (A, R) is bounded by the chain covering number of (A, R) and thus by the width of (A, R).

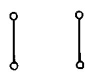
This paper focuses on the connections between width, dimension, recursive chain covering number, and recursive dimension for recursive ordered sets with finite width. Manaster and Rosenstein [12] have shown that there exist recursive ordered sets whose dimension is less than their recursive dimension. Kierstead [7] has shown that an ordered set with finite width w many not have recursive covering number equal to w, but nevertheless has recursive covering number at most $(5^w - 1)/4$. In this paper we prove the following theorems.

THEOREM 0. *There is a recursive ordered set of width* 3 *and recursive chain covering number* 4 *which has no finite recursive dimension.*

THEOREM 1. (a) *Every recursive ordered set with recursive chain covering number no more than* 3 *has recursive dimension no more than* 6. (b) *There is a recursive ordered set with recursive chain covering number* 3 *and recursive dimension* 6.

THEOREM 2. *Every recursive ordered set of width no more than 2 has recursive dimension no more than 5.*

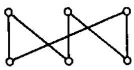
In view of Theorem 0, conditions on a recursive ordered set (A, R), which are sufficient to insure a finite recursive dimension, are of interest. (A, R) is an *interval order* provided it has no (induced) suborder with a Hasse diagram of the form:



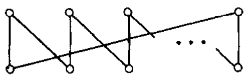
The points in an interval order can be construed as open intervals on the real line where the ordering is interpreted as one 'interval' lying completely to the left of another; incomparable points correspond to overlapping intervals. Kierstead and Trotter [10] have shown that every recursive interval order of width w has recursive chain covering number no more than $3w$-2 and that this bound is sharp. Hopkins [6] proved that every recursive interval order of width 2 has recursive dimension no more than 3. Hopkins [6] also established the general result that the recursive dimension of an interval order of width w is no more than $4w$-4. We add the following theorem, which gives a better bound than Hopkins [6], only when the recursive chain covering number is close to the width.

THEOREM 3. *Every recursive interval order with recursive chain covering number c has recursive dimension no more than 2c.*

The construction to be carried out in the proof of Theorem 0 produces an ordered set with many induced suborders of the form:



which is a 3-crown. More generally, a *crown* is a finite ordered set of height 1 with at least 6 points and a Hasse diagram of the form:



An ordered set is *crown-free* provided none of its induced suborders is a crown. Each interval order and each ordered set of width no more than 2 is crown-free, but the class of crown-free ordered sets is much richer.

The next theorem is our main result.

THEOREM 4. (a) *Every crown-free recursive ordered set with recursive chain covering number c has recursive dimension no more than c!* (b) *For each $c > 2$ there is a recursive crown-free ordered set with recursive chain covering number c, width c, but recursive dimension at least $c\binom{c-1}{t}$, where $t = \lfloor (c-1)/2 \rfloor$.*

The first part of Theorem 4 can be combined with Kierstead's bound on the recursive chain covering number to obtain the following result.

COROLLARY. *Every crown-free recursive ordered set of width w has recursive dimension at most $\left((5^w - 1)/4\right)!$.*

Of course, any improvement in the bound on the recursive chain covering number of a recursive ordered set of width w yields an improved bound in this corollary. Conversely, there are conditions under which the recursive dimension of an ordered set can be used to obtain an improved bound on the recursive chain covering number.

THEOREM 5. *Every recursive ordered set with recursive dimension d and width w can be covered by $\binom{w+1}{2}^{d-1}$ recursive chains.*

Associated with each of the results above is a companion theorem that has to do with the existence of winning strategies for certain two-player perfect information games. Just as there is a recursive setting for discrete properties like dimension, there is a game-theoretic setting as well. This game theoretic setting is the subject of Section 1 below and appears to be of independent interest. Section 2 is devoted to the proofs of the six theorems stated above.

This paper belongs to a growing area of research: recursive combinatorics. Other contributions to this area that may interest the reader are Bean [1, 2], Schmerl [18, 19], Kierstead [7–9], Kierstead and Trotter [10], Hopkins [6], McNulty [13], Manaster and Remmel [11], Manaster and Rosenstein [12], and Remmel [15]. A chapter in Rosenstein [17] is devoted to the theory of recursive linear orders. The theorems proved here were announced in McNulty [13].

We end this section with a discussion of some technical details concerning dimension and recursive functions that will be needed for our proofs. Dushnik and Miller [5] and Ore [14] showed that a countable ordered set (A, R) can be embedded in Q^d if and only if R is the intersection of d linear orderings of A. Manaster and Rosenstein [12] observed that in the case $d = 2$ this is effective. Their argument can be easily generalized to show that for every recursive ordered set (A, R), (A, R) can be recursively embedded into Q^d if and only if R is the intersection of d recursive linear orderings of A. This characterization of recursive dimension will form the basis for our proofs. If $L_0, \ldots, L_{d-1}$ are (recursive) linear orderings of A and $R = L_0 \cap \ldots \cap L_{d-1}$, we call $(L_0, \ldots, L_{d-1})$ a *(recursive) realizer* for (A, R). Thus the determination of the (recursive) dimension of (A, R) leads to the search for (recursive) realizers of minimal size.

It is possible to enumerate all algorithms. Let ϕ_e be the partial recursive function computed by the eth entry in this list. The execution of an algorithm on a particular input occurs in a step-by-step manner. $\phi_e^n(x) = y$ means that if x is input to the eth algorithm, then, after no more than n steps of computation, the program reaches its conclusion and outputs y. It is possible to construct the list $\phi_0, \phi_1, \ldots$ so that the 4-ary relation $\phi_e^n(x) = y$ is itself recursive. All this is made precise in Rogers [16].

1. The $\mathscr{K}$–$\mathscr{L}$ Expansion Game

A number of theorems of discrete mathematics have the following form: Every structure with a given property may be expanded by the adjunction of more structure so that the resultant structure enjoys another given property. Among the theorems of this kind are Dilworth's Theorem and the Four Color Theorem. In essence, Dilworth's Theorem says: If (A, R) is an ordered set of width w, then there is a chain cover $(C_0, C_1, \ldots, C_{w-1})$ of (A, R). In this case, the chain cover is the adjoined 'structure'.

Likewise the Four Color Theorem asserts: If (V, E) is a planar graph, then there is (C_0, C_1, C_2, C_3), which is a partition of V into independent sets. The theory of general algebraic structures provides the language to formulate this kind of phenomenon.

A *(relational) structure* is a system (A, F) where A is a nonempty set and F is a sequence of relations on A of finite rank (e.g. if $R \subset A^n$, then R has *rank* n). For our purposes F will always be a finite sequence. Two structures $(A, R_0, R_1, \ldots, R_{n-1})$ and $(B, S_0, S_1, \ldots, S_{m-1})$ are *similar* provided $n = m$ and the ranks of R_i and S_i are the same for each $i < n$. The structure $(A, R_0, \ldots, R_{n-1}, Q_0, \ldots, Q_{p-1})$ is an *expansion* of the structure $(A, R_0, \ldots, R_{n-1})$. Let $\mathscr{K}$ be a class of similar structures and let $\mathscr{L}$ be another class of similar structures. A *$\mathscr{K}$–$\mathscr{L}$ expansion theorem* is an assertion of the form: Every structure in $\mathscr{K}$ has an expansion in $\mathscr{L}$.

Dilworth's Theorem, The Four Color Theorem, and several of the theorems proven below are $\mathscr{K}-\mathscr{L}$ expansion theorems for the appropriate classes $\mathscr{K}$ and $\mathscr{L}$.

The *$\mathscr{K}-\mathscr{L}$ expansion game* is played by two players, the $\mathscr{K}$-player and the $\mathscr{L}$-player, in the following manner: the play alternates between the two players with the $\mathscr{K}$-player having the first play. At the end of any round of play, a finite structure in $\mathscr{L}$ will have been constructed which is an expansion of a structure in $\mathscr{K}$. So on his turn the $\mathscr{K}$-player is confronted with

$$(B, R_0, \ldots, R_{n-1}, Q_0, \ldots, Q_{m-1}) \in \mathscr{L}$$

such that $(B, R_0, \ldots, R_{n-1}) \in \mathscr{K}$ where B is finite. The $\mathscr{K}$-player must add a new point p and define new relations $R_0^+, \ldots, R_{n-1}^+$ such that

$$(B \cup \{p\}, R_0^+, \ldots, R_{n-1}^+) \in \mathscr{K}$$

and

$$R_i \text{ is the restriction of } R_i^+ \text{ to } B \text{ for all } i < n.$$

The $\mathscr{L}$-player must define new relations $Q_0^+, \ldots, Q_{m-1}^+$ such that

$$(B \cup \{p\}, R_0^+, \ldots, R_{n-1}^+, Q_0^+, \ldots, Q_{m-1}^+) \in \mathscr{K}$$

and

$$Q_j \text{ is the restriction of } Q_j^+ \text{ to } B \text{ for all } j < m.$$

The $\mathscr{K}$-player *wins* a $\mathscr{K}-\mathscr{L}$ expansion game if the play reaches a structure for which the $\mathscr{L}$-player has no legal response. Otherwise the $\mathscr{L}$-player *wins* the game. Thus, the $\mathscr{K}$-player is building a structure in $\mathscr{K}$ one point at a time with the ambition of frustrating the $\mathscr{L}$-player's attempts to expand the $\mathscr{K}$ structures to $\mathscr{L}$ structures.

A *winning strategy* for the $\mathscr{K}$-player is a function $\mathscr{S}$ which assigns to each finite $\mathscr{L}$ structure $B = (B, R_0, \ldots, R_{n-1}, Q_0, \ldots, Q_{m-1})$ a $\mathscr{K}$ structure $\mathscr{S}(\mathscr{B}) = (B \cup \{p\}, R_0^+, \ldots, R_{n-1}^+)$ such that, regardless of how the $\mathscr{L}$-player plays, the $\mathscr{K}$ player will always win the expansion game by playing the structure $\mathscr{S}(\mathscr{B})$ whenever he is confronted with the structure $\mathscr{B}$. Likewise, a winning strategy for the $\mathscr{L}$-player is a function which assigns to each finite $\mathscr{K}$-structure an expansion in $\mathscr{L}$ so that the $\mathscr{L}$-player wins by responding with the value of the function. As a consequence of the König Infinity Lemma, if the $\mathscr{K}$-player has a winning strategy for the $\mathscr{K}-\mathscr{L}$ expansion game, then there is a fixed n such that regardless of how the $\mathscr{L}$-player plays, the $\mathscr{K}$-player will always win in at most n plays. Thus, the winning strategy for the $\mathscr{K}$-player, if it exists, is a finite function and, hence, a recursive function. For the $\mathscr{L}$-player, it is conceivable that he may have a winning strategy, but no partial recursive winning strategy. However, almost any winning strategy for the $\mathscr{L}$-player *which can be precisely described* is likely to be a partial recursive function. In any case, this applies to all the winning strategies used in the next section.

So there are three types of $\mathscr{K}-\mathscr{L}$ expansion theorems:

CLASSICAL THEOREM. *Every structure in $\mathscr{K}$ can be expanded to a structure in $\mathscr{L}$.*

GAME THEORETIC THEOREM. *The $\mathscr{L}$-player has a winning strategy for the $\mathscr{K}-\mathscr{L}$ expansion game.*

RECURSIVE THEOREM. *Every recursive structure in $\mathscr{K}$ can be expanded to a recursive structure in $\mathscr{L}$. (Where the notion of a recursive structure is the obvious one.)*

A key to the proofs in the next section is the following lemma.

LEMMA. *Suppose $\mathscr{K}$ is a class of similar structures closed under the formation of substructures and that $\mathscr{L}$ is a class of similar structures closed under the formation of direct limits. If the $\mathscr{L}$-player has a partial recursive winning strategy for the $\mathscr{K}-\mathscr{L}$ game, then every recursive structure in $\mathscr{K}$ can be expanded to a recursive structure in $\mathscr{L}$.*

Proof. Let A be a recursive structure in $\mathscr{K}$. The recursive expansion A^+ of A is obtained from a run of the $\mathscr{K}-\mathscr{L}$ expansion game. The $\mathscr{K}$-player merely enumerates the points of A recursively, playing the induced substructures; while the $\mathscr{L}$ player, of course, uses his partial recursive winning strategy. After a (possibly infinite) run of the $\mathscr{K}-\mathscr{L}$ game in this manner, all of the points in A will have been enumerated. The structure A^+ obtained as the direct limit of the structures played by the $\mathscr{L}$-player is the desired expansion of A. Since $\mathscr{L}$ is closed under direct limits, we conclude that $A^+ \in \mathscr{L}$. By allowing the game to run until all points in any finite set have been listed, we can determine whether or not any of the relations of A^+ hold among the points. Hence A^+ is recursive as desired. □

In the event that the $\mathscr{K}$-player has a winning strategy for the $\mathscr{K}-\mathscr{L}$ expansion game, it is sometimes possible to construct a recursive structure in $\mathscr{K}$ that has no recursive expansion in $\mathscr{L}$. Roughly speaking, such a structure is spliced together from an infinite number of finite pieces – each piece designed by the $\mathscr{K}$-player's strategy to defeat a particular partial recursive attempt at expansion. Such a construction invokes Cantor's diagonal argument in a fashion familiar from recursion theory and also seems to depend on algebraic features of $\mathscr{K}$ to insure that the resulting structure does indeed belong to $\mathscr{K}$. If $\mathscr{K}$ is closed under the formation of substructures and direct limits and also possesses the amalgamation property, then such an argument could be carried out. For our purposes, the amalgamation property is too strong and so, in the next section we take a more *ad hoc* approach.

2. The Proofs

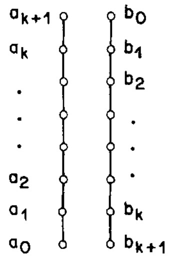
All of the proofs depend on the analysis of one kind of $\mathscr{K}-\mathscr{L}$ expansion game. $\mathscr{K}$ will always be a class of structures of the form $(A, R, C_0, C_1, \ldots, C_{c-1})$ where (A, R) is a certain kind of ordered set and $(C_0, C_1, \ldots, C_{c-1})$ is a chain cover of (A, R). $\mathscr{L}$ will always be a class of structures of the form $(A, R, C_0, \ldots, C_{c-1}, L_0, \ldots, L_{n-1})$ where $(A, R, C_0, \ldots, C_{c-1}) \in \mathscr{K}$ and $(L_0, \ldots, L_{n-1})$ is a realizer of (A, R). Thus, the variations in the $\mathscr{K}-\mathscr{L}$ expansion games we will consider depend only on the numbers c and n and on the conditions imposed on the ordered sets (A, R). The key to the proofs is the construction of winning strategies. To establish Theorems 0, 1(b), and 4(b) we produce winning strategies for the $\mathscr{K}$-player. To prove Theorems 1(a), 2, 3, 4(a), and 5 we produce partial recursive winning strategies for the $\mathscr{L}$-player.

PROOF OF THEOREM 0. Let $\mathscr{K}$ be the class of all structures of the form $(A, R, C_0, C_1, C_2, C_3)$ where (A, R) is an ordered set of width $\leqslant 3$ and (C_0, C_1, C_2, C_3) is a chain cover of (A, R). For each natural number n, let $\mathscr{L}_n$ be the class of all structures of the form $(A, R, C_0, \ldots, C_3, L_0, \ldots, L_{n-1})$ where $(A, R, C_0, \ldots, C_3) \in \mathscr{K}$ and $(L_0, \ldots, L_{n-1})$ is a realizer of (A, R). Our first step is to provide a winning strategy for the $\mathscr{K}$-player in each $\mathscr{K}-\mathscr{L}_n$ expansion game.

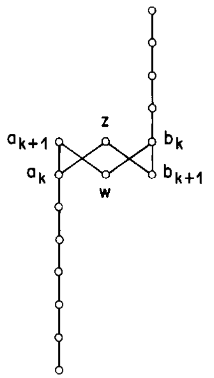
Let $\mathscr{A} \in \mathscr{L}_n$ with $\mathscr{A} = (A, R, C_0, \ldots, C_3, L_0, \ldots, L_{n-1})$. The pairs (x, y) and (u, v) of elements of A behave *pointlike* in A provided that xRy, and uRv, and for each $j < n$ either vL_jx or yL_ju. The first stage of the $\mathscr{K}$-player's strategy is to build an ordered set consisting of two incomparable chains, until the $\mathscr{L}_n$-player is forced to declare two pairs of distinct points to be pointlike. In the second stage of his strategy the $\mathscr{K}$-player adds two new points, declaring them incomparable and placing them on two new chains, in such a way that the $\mathscr{L}$-player is forced to resign. During the first stage the $\mathscr{K}$-player proceeds as follows:

> At play $2i$ he introduces a new point a_i and places it at the top of chain C_0.
> At play $2i + 1$ he introduces a new point b_i and places it at the bottom of chain C_1.
> Before each play he checks the $\mathscr{L}_n$-player's response to see if there are any pointlike pairs; if so he goes to stage II; if not he repeats the first stage.

To see that the $\mathscr{K}$-player can always force the $\mathscr{L}_n$-player to construct the linear orders $L_0, L_1, L_2, \ldots, L_{n-1}$ so that for some k, the pairs (a_k, a_{k+1}) and (b_{k+1}, b_k) are pointlike in each L_i, we make the following observation. After adding the points a_i and b_i to the two chains the $\mathscr{K}$-player pauses to count the number m_i of linear orders in which the $\mathscr{L}_n$-player defines a_i to be larger than b_i. Note that $1 \leqslant m_i \leqslant n-1$ for all i, since a_i is over b_i at least once, but not always, as they are incomparable. On the other hand, since $a_{i+1} > a_i$ and $b_i > b_{i+1}$ in R, we know that a_{i+1} is over b_{i+1} in any L_j for which a_i is over b_i. Thus $m_i \leqslant m_{i+1}$ for each i. We conclude that there is some k with $k < n$ for which $m_k = m_{k+1}$. In this case, it follows that (a_k, a_{k+1}) and (b_{k+1}, b_k) are pointlike in each L_j. At this stage, the $\mathscr{K}$-player has constructed the following ordered set.



To earn his victory, the $\mathscr{K}$-player only needs to add new points $z \in C_2$ and $w \in C_3$ as shown below.



Since (a_k, a_{k+1}) and (b_{k+1}, b_k) are pointlike, the $\mathscr{L}_n$-player is constrained to place z above w in all the linear orders to insure that they all extend R – making it impossible to have $R = L_0 \cap \ldots \cap L_{n-1}$. Thus, the $\mathscr{K}$-player, playing according to this strategy will always win the $\mathscr{K}$–$\mathscr{L}_n$ expansion game.

Now we are prepared to construct a recursive ordered set $(A, R, C_0, \ldots, C_3)$ in $\mathscr{K}$ that has no recursive expansion in any $\mathscr{L}_n$. This construction is carried out in layers, each of which is labelled with a pair (e, n) of natural numbers. Fix some recursive bijection $\langle\ ,\ \rangle$ from $N \times N$ to N which is increasing in both variables. Whenever a point of our ordered set is inserted in layer (e, n) we will always mean it to be R-smaller than all points in layer (f, p), where $\langle e, n\rangle < \langle f, p\rangle$ and R-larger than all points in layer (d, m) where $\langle d, m\rangle < \langle e, n\rangle$.

We say that $f(x, y, i)$ represents an n-ary realizer $(L_0, \ldots, L_{n-1})$ of (A, R) if for each $i < n$, $f(x, y, i)$ is the characteristic function of L_i. Notice that any recursive realizer will be represented by a recursive function.

In order to destroy the possibility of a finite recursive dimension we invoke the $\mathscr{K}$-player's strategy against the $\mathscr{L}_n$-player in layer (e, n) to meet the requirement.

> R_e: $\phi_e(x, y, i)$ does not represent an n-ary realizer of (A, R).
> where $\phi_e(x, y, i)$ is defined to be $\phi_e(\langle\langle x, y\rangle, i\rangle)$.

Suppose, for the moment, that $\phi_e(x, y, i)$ really is a characteristic function of a linear order, for each $i < n$. By invoking the $\mathscr{K}$-player's strategy, against the $\mathscr{L}_n$-player's attempt to play according to ϕ_e after finitely many steps we see that R_e must hold. However, $\phi_e(x, y, i)$ must be computed many times during the course of play. In general, there is no way to know whether ϕ_e is a characteristic function, or indeed, whether any particular computation of ϕ_e will ever end. As it stands our construction could be stalled forever by such an event. To avoid this kind of deadend, we arrange a recursive pattern that visists each layer (e, n) infinitely often. On each visit, we perform as much of the construction as possible on the basis of allowing ϕ_e to execute an additional step in its computation. After a countably infinite number of stages, all computations in all layers which could possibly conclude will have been finished. Every possible recursive expansion of the resulting structure to a structure in a $\mathscr{L}_n$ will have been defeated in some layer where all computations were finished, while in those layers where computations never ended there was nothing to defeat.

We now provide the details of the algorithm, which proceeds in stages. $A_{e,n}$ will represent the (e, n) level of A. Thus $A = \cup_{e,n \in N} A_{e,n}$

Stage 0. Each of $A, R, C_0, \ldots, C_3$ is empty.

Stage $s + 1$. Suppose that $s = \langle\langle e, n\rangle, k\rangle$. If $k < 2n$ put $2s$ into $A_{e,n}$. If k is even put $2s$ into C_0, make $2s$ R-greater than every other element of $C_0 \cap A_{e,n}$, and R-incomparable to every element of $C_1 \cap A_{e,n}$. If k is odd put $2s$ into C_1, make $2s$ R-smaller than every other element of $C_1 \cap A_{e,n}$, and R-incomparable to every element of $C_0 \cap A_{e,n}$.

Now suppose that $2n \leqslant k$. If $\phi_e^s(x, y, i)$ does not represent an n-ary realizer of $(A_{e,n}, R|A_{e,n})$ then go to the next stage. Note that since at any stage $A_{e,n}$ is finite we can effectively check this condition. Otherwise there must be a pair of pointlike pairs (a, b) and (c, d) in $A_{e,n}$ such that $a, b \in C_0$ and $c, d \in C_1$. Let $D = \{x \in A_{e,n} : xRa$ or $xRc\}$ and $U = \{x \in A_{e,n} : bRx$ or $dRx\}$. Add two new points $2s$ and $2s + 1$ to $A_{e,n}$ so that $2s$ is R-incomparable to $2s + 1$, $2sRx$ for all $x \in U$, $xR2s + 1$ for all $x \in D$, $2s$ is R-incomparable to all $x \in D$, $2s + 1$ is R-incomparable to all $x \in U$, $2s \in C_3$, and $2s + 1 \in C_4$.

Finally if $\langle d, m\rangle < \langle e, n\rangle < \langle f, p\rangle$ make any point added at stage $s + 1$ R greater than any point already in $A_{d,m}$ and R smaller than any point already in $A_{f,p}$.

It is clear from the construction that (A, R) is an ordered set with width 3 and $(C_0, \ldots, C_3)$ is a chain cover of (A, R). We now check that $A, R, C_0, \ldots, C_3$ are all recursive. First note that each stage of the construction is effective. Also, if $x \in A$, then x is put into A at the $\lfloor (x/2) + 1 \rfloor$st stage. Thus we can effectively determine whether or not $x \in A$ by carrying out the first $(x/2) + 1$ stages of the construction. So A is recursive. Similar arguments show that $R, C_0, \ldots, C_3$ are recursive.

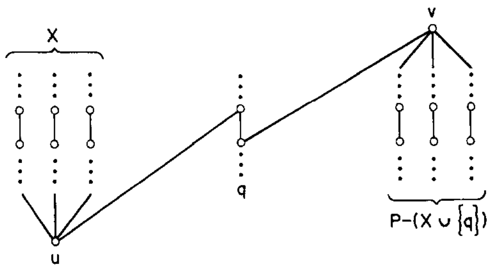
Finally suppose that $(L_0, \ldots, L_{n-1})$ is a recursive realizer of (A, R). Then $(L_0, \ldots, L_{n-1})$ is represented by some recursive function $\phi_e(x, y, i)$. Let A_0 be the finite set of elements in $A_{e,n}$ by stage $\langle\langle e, n\rangle, 2n\rangle$. There exists a $p \geqslant 2n$ such that for all $x, y \in A_0$ and $i < n$, $\phi_e(x, y, i)$ gives an output in at most p steps. Then after stage $\langle\langle e, n\rangle, p\rangle$ $\phi_e^p(x, y, i)$ represents an n-ary realizer of $A_{e,n}$. But the construction at stage $\langle\langle e, n\rangle, p\rangle + 1$ ensures that $\phi_e(x, y, i)$ will not represent an n-ary realizer of (A, R). □

PROOF OF THEOREM 4(b): Let $\mathscr{K}$ be the class of structures $(A, R, C_0, \ldots, C_{c-1})$ where (A, R) is of width $\leqslant c$ and $(C_0, \ldots, C_{c-1})$ is a chain cover of (A, R). Let $\mathscr{L}_n$ be the associated class as in the proof of Theorem 0, where $n = c\binom{c-1}{t} - 1$. We will produce a winning strategy for the $\mathscr{K}$-player. The first step is to see that the $\mathscr{K}$-player can force a position in which for all $i \neq j$ all points in C_i are incomparable to all points in C_j and in each chain C_i there exists a pair p_i of adjacent points such that for all $i \neq j$, p_i and p_j are point-like.

Observe that if (x, y) and (u, v) are point-like and $xRx'Ry'Ry$ and $uRu'Rv'Rv$, then

(x', y') and (u', v') are point-like. First the $\mathscr{K}$-player uses the strategy described in the proof of Theorem 0 to produce two point-like pairs (x_0^1, y_0^1) and (x_1^1, y_1^1) such that $x_0^1, y_0^1 \in C_0$ and $x_1^1, y_1^1 \in C_1$. Next the $\mathscr{K}$-player uses the above observation and the Theorem 0 strategy to produce pairwise point-like pairs (x_0^2, y_0^2), (x_1^2, y_1^2), and (x_2^2, y_2^2) such that $x_i^2, y_i^2 \in C_i$, for $i < 3$ and $x_i^1 \, R \, x_i^2 \, R \, y_i^2 \, R \, y_i^1$ for $i < 2$. Continuing in this manner the $\mathscr{K}$-player can produce the desired pairs.

Any of the linear orders L_i induces a linear order on $P = \{p_0, p_1, \ldots, p_{c-1}\}$ since the p's behave point-like. For each $i < n$, let q_i be the $(t+1)$st L_i-largest element of P and let $X_i = \{p_j \in P : q_i L_i p_j \text{ and } q_i \neq p_j\}$. Since there are $c\binom{c-1}{t}$ ways to choose an element $q \in P$ and a t-element subset $X \subset P - \{q\}$, and $n < c\binom{c-1}{t}$, we can choose (q, X) as above so that $(q, X) \neq (q_i, X_i)$ for any $i < n$. Now the $\mathscr{K}$-player adds two new points u and v as follows to assure his win. He declares that u is R-smaller than all elements of C_j where $p_j \in X$ and R-smaller than the top element of q and v is R-greater than the bottom element in q and v is R-greater than all elements in C_k, where $p_k \in P - (X \cup \{q\})$. The points u and v belong to the chains C_i and C_j where p_i is the least element of X and p_j is the least element of $P - (X \cup \{q\})$. The resulting set might look like:



Confronted with the incomparability of u and v, the L_n-player is forced to put u above v in at least one of the linear orders L_i. But this is impossible since it would force $(q, X) = (q_i, X_i)$. Thus, the $\mathscr{K}$-player has a winning strategy for the $\mathscr{K}$–$\mathscr{L}_n$ expansion game.

It remains to note that the winning strategies of the $\mathscr{K}$-player can be recursively spliced together just as in the proof of Theorem 0. A moment's reflection reveals that the layering done in the construction will produce no crowns. □

PROOF OF THEOREM 1(b). Applying Theorem 4(b) to $c = 3$ we obtain a recursive ordered set whose chain covering number is 3 and whose recursive dimension is at least $3\binom{2}{1} = 6$. □

We now turn to situations in which the $\mathscr{L}$-player has a winning strategy. Recall that the task of the $\mathscr{L}$-player is to fit the new point, just introduced by the $\mathscr{K}$-player, into each of a number of linear orders in such a way that the resulting linear orders constitute a realizer of the ordered set built by the $\mathscr{K}$-player. Thus, after a number of plays the $\mathscr{L}$-player is confronted with the following situation:

(*S*): $A^+ = A \cup \{p\}$ where $p \notin A$, R^+, an order on A^+ extending the order R on A, $(C_0, \ldots, C_{c-1})$ is a chain cover of (A^+, R^+), $p \in C_k$, for exactly one k with $0 \leqslant k < c$, $(L_0, \ldots, L_{n-1})$ a realizer of (A, R).

In each L_i there is an interval into which p can be inserted: the upper bound of this interval is the L_i-least element of A that is R^+-larger than p (if such a bound exists), while the lower bound is the L_i-largest element of A that is R^+-smaller than p (if such a bound exists). Positioning p anywhere in this interval will produce an extension of both R^+ and L_i. So our strategies for the $\mathscr{L}$-player will involve tie-breaking schemes for deciding on the appropriate positions in this interval in which to insert the new point. In every case, the scheme will involve the subscripts on the chains which make up the chain cover.

Our first theorem has a relatively simple tie-breaking scheme.

PROOF OF THEOREM 3. So suppose that the $\mathscr{L}$-player is confronted with the situation (S) above.

The $\mathscr{L}$-player's response will be as follows. For each $i = 0,1, \ldots, c-1$, with $i \neq k$, we insert p in the highest possible position in L_{2i} and in the lowest possible position in L_{2i+1}. We insert p as low as possible in L_{2k} and as high as possible in L_{2k+1}. In some sense, the linear order L_{2i} is an effort by the $\mathscr{L}$-player to place a point p over all points in C_i with which it is incomparable. (It is a simple task to accomplish this goal in a nonrecursive setting, but in general, it is impossible in this expansion game.) A dual statement holds for L_{2i+1}.

If the $\mathscr{L}$-player has followed this strategy at each stage of the game, it remains to show that $(L_0^+, L_1^+, \ldots, L_{2c-1}^+)$ is a realizer of (A^+, R^+). To accomplish this, it suffices to show that if (x, y) is an arbitrary incomparable pair in (A^+, R^+), then there is at least one j for which yL_j^+x. By induction, we may assume that one of x and y is the new point p.

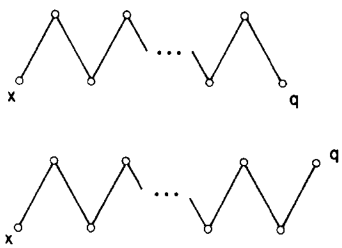
Let us suppose first that $p = x$ and that $y \in C_j$. We show that yL_{2j}^+x. To see that this is true, we suppose to the contrary that xL_{2j}^+y. Since x was inserted as high as possible in L_{2j}, we know that if z is the point immediately over x in L_{2j}^+, then xRz. Similarly, if we let w denote the point immediately under y in L_{2j}^+, then wRy. Since x and y are incomparable in R and zL_{2j}^+w, z is incomparable to w in R. However, this violates the definition of an interval order since it implies that R contains two incomparable two-point chains. The contradiction shows that yL_{2j}^+x. A similar argument shows that if $p = y$ and $x \in C_k$, then yL_{2k+1}^+x. With this observation, the proof is complete. □

Our next theorem will require a more elaborate tie-breaking scheme. When $(C_0, C_1, C_2, \ldots, C_{c-1})$ is a chain cover of (A, R) and $x \in A$, we let $\alpha(x)$ denote the unique i for which $x \in C_i$. Now let σ be a linear oder (a permutation) of $\{0, 1, 2, \ldots, c-1\}$. Then we may define a recursive linear extension $L = L_\sigma$ of R using σ as a tie-breaker. Suppose at the time the new point p is given L_σ has placed the points preceeding p in the order $w_0 L_\sigma w_1 L_\sigma \ldots L_\sigma w_t$. To extend L_σ to include p, we will place p before w_0, after w_t or between w_i and w_{i+1} for some i. The rule is as follows. Let j be the least k such that for all $i \geqslant k$ not $w_i R p$. Find the least $i \geqslant j$, if it exists, for which either pRw_i or $\alpha(p)\, \sigma\, \alpha(w_i)$ and insert p immediately before w_i. If no such i exists, p is placed after w_t.

PROOF OF THEOREM 4(a). We assume that $(C_0, C_1, \ldots, C_{c-1})$ is a recursive chain cover of (A, R). Then for each linear order σ of $\{0, 1, 2, \ldots, c-1\}$, we form the recursive linear extension L_σ of R as defined above. There are $c!$ such extensions. We then proceed to show that this collection is a recursive realizer for R. It suffices to choose an arbitrary incomparable pair (x, y) and show that there is at least one σ for which $yL_\sigma x$. Without loss of generality, we assume that $\alpha(y) = 0$ and $\alpha(x) = c - 1$.

The determination of the desired order σ requires the construction of an auxilary partial order on $\{1, 2, 3, \ldots, c-2\}$. We proceed as follows.

F is an *up*-fence *from* x *to* q provided F is an ordered set of height 1 with one of the following kinds of Hasse diagrams:



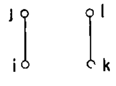
Next, we define a binary relation Δ on $\{1, 2, \ldots, c-2\}$ by the following rule: $i \,\Delta\, j$ if, and only if, there exist $p \in C_i$, $q \in C_j$ with p R^+-smaller than y and there is an up-fence F from x to q such that each point of F is R^+-incomparable with both p and y.

The reader should note that the relation Δ is defined in terms of representatives of the chains C_i and C_j. The conditions need not be satisfied by every pair of points from these chains. We now proceed to show that the relation Δ is an irreflexive partial order and in fact is an interval order. These results follow easily from a lemma whose proof requires crown-freeness at a crucial stage.

LEMMA. *If* $i \,\Delta\, j$ *and* $k \,\Delta\, l$, *then either* $i \,\Delta\, l$ *or* $k \,\Delta\, j$.

Proof. For the sake of contradiction suppose $i \,\Delta\, j$ and $k \,\Delta\, l$ and both $i \not\Delta\, l$ and $k \not\Delta\, j$. Choose points $p \in C_i$, $q \in C_j$, $p' \in C_k$, $q' \in C_l$ and up-fences F from x to q and F' from x to q' such that p, q and F witness $i \,\Delta\, j$ while p', q' and F' witness $k \,\Delta\, l$. Since $i \not\Delta\, l$ it follows that p must be R^+-comparable with some point of F'. Since y is R^+-larger than p and R^+-incomparable with all of F', it follows that p is R^+-smaller than some point of F'. Let r' be the point on F' closest to x which is R^+-larger than p. Pick r on F to be closest to x and R^+-larger than p'. Note that r and r' are R^+-incomparable since otherwise r is R^+-greater than p or r' is R^+-greater than p'. Also r and r' are connected by a path no interior point of which is R^+-comparable to y, p, or p'. Hence, r and r' are connected by a fence H which also has this property. But then y, p, r', H, r, p', y constitutes a crown in (A^+, R^+), which was forbidden. □

Now Δ is clearly an irreflexive relation. Both anti-symmetry and transitivity follow immediately from the lemma and the irreflexivity of Δ. The lemma itself asserts that there are no suborders with Hasse diagrams like



so Δ is an interval order. Now adjoin 0 as a least element and $c - 1$ as a greatest element. The resulting order Δ_1 is an irreflexive interval order of $\{0, 1, \ldots, c-1\}$. Fix any representation of this order by intervals from the real line with distinct left end points. Let σ_0 be the linear extension of Δ_1 'by left endpoints'; that is, $i \sigma_0 j$ if, and only if, the left endpoint of the interval representing i is less than the left endpoint of the interval representing j.

The order σ_0 has two very important properties.

(*) if $\alpha(v) \Delta_1 \alpha(w)$ and $\alpha(w) \sigma_0 \alpha(z)$, then $\alpha(v) \Delta_1 \alpha(z)$

(**) if both z and w are R^+-incomparable with both y and v, vR^+y, wR^+z, and $\alpha(v) \Delta \alpha(w)$, then $\alpha(v) \Delta \alpha(z)$.

The property (*) is a distinctive property of extension by left-endpoints and it is immediate. To see that (**) is valid choose points $v' \in C_{\alpha(v)}$, $w' \in C_{\alpha(w)}$ and an up-fence F from x to w' which witness $\alpha(v) \Delta \alpha(w)$. Then let $v'' = \max\{v, v'\}$ and let F' be any up-fence from x to z with $F' \subseteq F \cup \{w, z\}$. Then v'', z, and F' witness $\alpha(v) \Delta \alpha(z)$.

Now let us suppose that when the latter of the two points x and y has appeared, the desired property fails to hold and that we have instead $xL_\sigma y$ where $\sigma = \sigma_0$. We proceed to a contradiction. Note that we will be concerned only with the finite set of points which have appeared thus far. Consider the sequence of consecutive points $x = u_0, u_1, u_2, \ldots, u_m = y$ in L_σ. Note that for each i, either $u_i R u_{i+1}$, or $\alpha(u_i) \sigma \alpha(u_{i+1})$. We then define a *blocking chain* in L_σ as a suborder $x = v_0, v_1, v_2, \ldots, v_k = y$ of L_σ so that for each i, either $v_i R v_{i+1}$ or $\alpha(v_i) \sigma \alpha(v_{i+1})$. (The concept of a blocking chain arises in analyzing the effect the permutation σ has in its role as a tie-breaker.) Choose a blocking chain $x = v_0, v_1, v_2, \ldots, v_k = y$ where k is as small as possible. Since k is minimal, we cannot have $v_i R v_{i+1} R v_{i+2}$. Similarly, we cannot have $\alpha(v_i) \sigma \alpha(v_{i+1})$ and $\alpha(v_{i+1}) \sigma \alpha(v_{i+2})$. Thus $v_{2i} R v_{2i+1}$, while v_{2i+1} and v_{2i+2} are incomparable with $\alpha(v_{2i+1}) \sigma \alpha(v_{2i+2})$. Whenever $k \geqslant j \geqslant i + 2$ and $i \geqslant 0$, the points v_j and v_i are incomparable, but $\alpha(v_j) \sigma \alpha(v_i)$. Any other possibility results in a shorter blocking chain.

Next, we observe that v_{k-1} and v_1 witness $\alpha(v_{k-1}) \Delta \alpha(v_1)$. So we may then choose the largest integer i for which $\alpha(v_{k-1}) \Delta \alpha(v_i)$. If i is odd, say $i = 2j + 1$, then we know $\alpha(v_i) = \alpha(v_{2j+1}) \sigma \alpha(v_{2j+2})$. Hence, by property (*) noted previously, we would conclude that $\alpha(v_{k-1}) \Delta \alpha(v_{i+1})$. The contradiction forces i to be even, say $i = 2j$. But in this case, we have $v_{2j} R v_{2j+1}$. Here, we conclude by (**) that $\alpha(v_{k-1}) \Delta \alpha(v_{i+1})$. With this contradiction, our proof is complete. □

PROOF OF THEOREM 1(a). The theorem will follow as an easy corollary to the proof of the preceding theorem. We suppose that (C_0, C_1, C_2) is a recursive chain cover of the ordered set (A, R). There are six orders $\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$ of $\{0, 1, 2\}$. We now proceed to show that the recursive linear orders $\{L_{\sigma_i} : 0 \leqslant i \leqslant 5\}$ form a recursive realizer. This conclusion follows from an analysis of the concept of blocking chain utilized in the proof of Theorem 4(a). In particular, we note that a minimal blocking chain requires at least four distinct chains. Since there are only three chains, we see that no blocking chain can exist, and with this observation, the proof is complete. □

PROOF OF THEOREM 2. Let $\mathscr{K}$ be the class of ordered sets, (P, R) of width $\leqslant 2$. Let $\mathscr{L}$ be the class of structures $(P, R, L_0 \ldots L_4)$ where $(P, R) \in \mathscr{K}$ and $(L_0 \ldots L_4)$ is a realizer of (P, R). We must provide a recursive winning strategy for the $\mathscr{L}$-player in the $\mathscr{K}-\mathscr{L}$ expansion game.

In [7] it is shown that any width two recursive ordered set (P, R) can be covered by six recursive chains. We shall need very specific information about this construction for our proof. The assertion of the following lemma, which we state without proof, is implicit in [7]. R-incomparability is denoted by $\parallel$.

LEMMA. *Let $\mathscr{K}$ be the class of ordered sets of width 2 and let $\mathscr{M}$ be the class of structures $(P, R, R^*, S_0, \ldots S_4, B, C_0, \ldots C_4, A)$ such that:*

(i) $(P, R) \in \mathscr{K}$;

(ii) *$(B, C_0, \ldots C_4)$ is a chain cover of (P, R), B is a maximal chain; and $A = C_0 \cup \ldots \cup C_4$;*

(iii) *(A, R^*) is a linear extension of (A, R) and xR^*y if, and only if, $x, y \in A$ and xRy or for all $z \in B$ $z \parallel x$ implies zRy;*

(iv) *if $x \in C_i, y \in C_j, i \neq j, xR^*y, z \in A$, and $x \parallel z \parallel y$ then xR^*zR^*y;*

(v) *if $x \in C_i$, $b \in B$ and $x \parallel b$ then there are at most two numbers $k < 5$ such that $k \neq i$ and for some $c \in C_k$, $xRc \parallel b$.*

(vi) *aS_ib if, and only if, $a \in A$, $b \in B$, $a \parallel b$, and there exists $c \in C_i$ such that $aRc \parallel b$.*

Then the $\mathscr{M}$-player has a recursive winning strategy in the $\mathscr{K}-\mathscr{M}$ expansion game.

Notice that (vi) implies that if $a \in A$ and $b \in B$ have been played so that $a \parallel b$ and there is no $c' \in C_i$ such that $aRc' \parallel b$, then no such c' will ever be played.

The $\mathscr{L}$-player will secretly follow the $\mathscr{M}$-player's strategy while constructing $L_0, \ldots L_4$ so that for $x \parallel y$, if $x \in C_i$, then xL_iy. Also if $x \in B$, then for some $j < 5$, xL_jy. Here is the $\mathscr{L}$-player's strategy:

For each $i < 5$, xL_iy if, and only if, one of the following holds:

(1) xRy;

(2) $x \parallel y$, $y \in A$, and $x \in C_i$;

(3) $x \parallel y$, $x, y \in A$, $y \notin C_i$ and xR^*y;

(4) $x \parallel y$, $y \in B$, and xS_iy;

(5) $x \parallel y$, $x \in B$, and not yS_ix.

First we show that for each $i < 5$, (A, L_i) is a linear order. It is easy to check that L_i is reflexive, antisymmetric, and linear. In order to show transitivity, suppose that $x \, L_i \, y \, L_i \, z$. There is no problem if $x \, L_i \, y$ by the same clause that makes $y \, L_i \, z$. Also we can ignore clauses (4) and (5). This leaves six cases (j, k) to check, where in case (j, k) $x \, L_i \, y$ by clause (j) and $y \, L_i \, z$ by clause (k) for $1 \leqslant j, k \leqslant 3$ and $j \neq k$.

By antisymmetry and linearity it suffices to consider those cases where $j < k$.

Case (1,2). So $xRy \parallel z$ and $y \in C_i$. If x is comparable to z, then xRz and we are done by clause (1); if not and $x \in C_i$ then xL_iz by clause (2); otherwise by (iv) of the lemma, xR^*zR^*y and we are done by clause (3).

Case (1,3). Then $xRyR^*z$ and $z \in C_i$. Thus, by (iii) of the lemma xR^*z and we are done by clause (3).

Case (2,3). Then $x \| y \| z$, $x \in C_i$ and $z \notin C_i$ Thus x is comparable to z, and by (iv) of the lemma, y is R^* – between x and z. By clause (3) yR^*z; thus xR^*yR^*z. Using (iii) of the lemma, xRz; hence, we are done by clause (1).

Next we show that (P, L_i) is a linear order. Again the only problem is transitivity. Suppose that $x\, L_i\, y\, L_i\, z$. By the preceeding argument, we may assume that one of x, y, and z is in A and one is in B. By antisymmetry and linearity, we may assume that either case 1: $x, y \in B$ and $z \in A$, or case 2: that $x \in B$ and $y, z \in A$.

Case 1. If z is comparable to x or y, then by antisymmetry xRz, and we are done by clause (1). Otherwise we have not zS_iy and thus not zS_ix. So we are done by clause (5).

Case 2. First suppose xRy. If x or y is comparable to z, then we are done by clause (1). So suppose $x, y \| z$. Thus zR^*y by (iii) of the lemma and yL_iz by clause (2). Hence $y \in C_i$. Suppose zS_ix. Then there exists $c \in C_i$ such that $zRc \| x$. Clearly, using (ii) of the lemma yRc, which contradicts $c \| x$. Thus not zS_ix, and we are done by clause (5). Now suppose $x \| y$. Then by not clause (4), not yS_ix. If $x \| z$, then yRz and thus not xS_iz. So we are done by clause (5). Suppose x is comparable to z. If yRz, we are done by antisymmetry and clause (1). So suppose $y \| z$. Then, since $y \in C_i$, yL_iz by clause (3). Thus by (iii) of the lemma, xRz.

Finally we check that $(L_0, \ldots L_4)$ is a realizer of (P, R). Suppose $x \| y$ where $x \in C_i$. If $y \in C_j$, then xL_iy and yL_jx. If $y \in B$, then xL_iy. By (v) of the lemma, there exists l such that not xS_ly. So yL_lx. □

Our proof of Theorem 5 relies on the following unpuslihed lemma of J. Schmerl which we include here with his kind permission.

LEMMA (J. Schmerl). *Every recursive ordered set of height* w *can be covered by* $\binom{w+1}{2}$ *recursive anti-chains.*

Proof. Let $\mathscr{K}$ be the class of ordered sets (P, R) of height $\leqslant w$ and $\mathscr{L}$ be the class of structures $(P, R, \ldots A_{i,j} \ldots)_{i+j<w}$, where $(P, R) \in \mathscr{K}$ and $(A_{i,j})_{i+j<w}$ is an anti-chain cover. Notice that there are $\binom{w+1}{2}$ anti-chains $A_{i,j}$. We must provide the $\mathscr{L}$-player a recursive winning strategy for the $\mathscr{K}-\mathscr{L}$ expansion game.

Each time the $\mathscr{K}$ player adds a new point to form (P^+, R^+) the $\mathscr{L}$player should put p into $A^+_{i,j}$, where i is the length of the longest chain in (P^+, R^+) which is entirely below p and j is the length of the longest chain in (P^+, R^+) that is entirely above p. Clearly p is incomparable to all points already in $A_{i,j}$. □

PROOF OF THEOREM 5. Let (A, R) be a recursive ordered set. We prove the theorem by induction on d. If $d = 1$ then R is a linear order; so we are done. For the inductive step, let $d = k + 1$. Suppose that $\{L_1, \ldots L_{k+1}\}$ is a recursive realizer of (A, R). Define $S = L_1 \cap \ldots \cap L_k \cap L^*_{k+1}$ where $L^*_{k+1} = \{(b, a)\colon aL_{k+1}b\}$. Clearly (A, S) is recursive and the height of (A, S) is bounded by the width of (A, R). Thus, by Schmerl's Lemma,

(A, S) can be covered by $\binom{w+1}{2}$ recursive anti-chains. If D is an anti-chain of (A, S), then $R \upharpoonright D = L_1 \upharpoonright D \cap \ldots \cap L_k \upharpoonright D$. Thus, by the inductive hypotheses, each of these $\binom{w+1}{2}$ recursive anti-chains of (A, S) can be covered by $\binom{w+1}{2}^{k-1}$ recursive chains of (A, R). □

Acknowledgement

The authors thank M. Saks for pointing out that the problem of constructing a chain cover for a recursively two-dimensional ordered set of width w is reducible to the problem of constructing an anti-chain cover for a related ordered set of height w.

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