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REFERENCES

- [1] J. Schönheim, On maximal systems of k -tuples, *Studia Sci. Math. Hung.*, 1 (1966), 363–368.
- [2] M.K. Fort – G.A. Hedlund, Minimal coverings of pairs by triples, *Pacific J. Math.*, 8 (1958), 709–719.
- [3] W.H. Mills, On the covering of pairs by quadruples I, *J. Comb. Theory Ser. A*, 13 (1972), 55–78.
- [4] W.H. Mills, On the covering of pairs by quadruples II, *J. Comb. Theory Ser. A*, 15 (1973), 138–166.
- [5] W.H. Mills, On the covering of triples by quadruples, *Proc. of the Fifth South. Conf. on Comb., Graph Theory and Comp.*, Florida Atlantic University, Boca Raton, Florida, 1974, 563–581.
- [6] W.H. Mills, Covering designs I: Coverings by a small number of subsets, *Ars Combinatoria*, 8 (1979), 199–315.
- [7] R.G. Stanton – J.G. Kalbfleisch – R.C. Mullin, Covering and packing designs, *Proc. of the Second Chapel Hill Conf. on Comb. Math. and Appl.*, University of N.C., 1970, 428–450.
- [8] H. Hanani, Balanced incomplete block designs and related designs, *Discrete Math.*, 11 (1975), 255–369.

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FOR $t \geq 3$, EVERY t -DIMENSIONAL PARTIAL ORDER CAN BE EMBEDDED IN A $t + 1$ -IRREDUCIBLE PARTIAL ORDER

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ABSTRACT

The dimension of a partial order X is the least integer t for which there exist linear extensions X_1, X_2, \dots, X_t so that $x_1 \leq x_2$ in X if and only if $x_1 \leq x_2$ in X_i for each $i = 1, 2, \dots, t$. For an integer $t \geq 2$, a partial order is t -irreducible if it has dimension t and every proper nonempty subpartial order has dimension less than t . The only 2-irreducible partial order is a 2-element antichain. There are infinitely many 3-irreducible partial orders, and they may be conveniently grouped into 9 infinite families with 18 odd examples left over. There are many 2-dimensional partial orders which cannot be embedded in a 3-irreducible partial order, for example, any 2-dimensional partial order whose length and width both exceed 4. However, when $t \geq 3$, we prove that every t -dimensional partial order can be embedded in a $t + 1$ -irreducible partial order.

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1. INTRODUCTION

In this paper, we prove that if $t \geq 3$, every t -dimensional partially ordered set can be embedded in a $t + 1$ -irreducible partially ordered set. This result is false when $t = 2$ since there are many 2-dimensional posets which cannot be embedded in 3-irreducible posets. The construction used to establish this theorem has its origins in chromatic graph theory, and an elementary version had been previously used by the authors [12] to show that for each $t \geq 2$, every t -irreducible poset is a subposet of a $t + 1$ -irreducible poset. For the sake of completeness, we present in Section 2 the fundamental definitions and summarize some preliminary material on the dimension of partially ordered sets. For additional background material, we refer the reader to the survey articles [6] and [10]. In particular, [6] contains an extensive bibliography of papers on dimension theory.

2. NOTATION, TERMINOLOGY AND PRELIMINARY RESULTS

A *partially ordered set (poset)* is a set X equipped with a reflexive, antisymmetric and transitive binary relation \leq . If $x_1, x_2 \in X$, $x_1 \not\leq x_2$ and $x_2 \not\leq x_1$, then x_1 and x_2 are *incomparable* and we write $x_1 \parallel x_2$. For each point $x_1 \in X$, we let $D_X(x_1) = \{x_2 \in X: x_2 < x_1\}$, $U_X(x_1) = \{x_2 \in X: x_1 < x_2\}$, and $I_X(x_1) = \{x_2 \in X: x_1 \parallel x_2\}$. We let $I_X = \{(x_1, x_2): x_1 \parallel x_2\}$. X is a *linear order* if $I_X = \emptyset$. If X_1 and X_2 are partial orders on the same set and $x_1 < x_2$ in X_2 whenever $x_1 < x_2$ in X_1 , we say X_2 is an *extension* of X_1 ; if X_2 is a linear order and an extension of X_1 , then X_2 is called a *linear extension* of X_1 . Dushnik and Miller [1] defined the *dimension* of a poset X , denoted $\dim(X)$, as the least positive integer t for which there exist t linear extensions X_1, X_2, \dots, X_t of X so that $x_1 \leq x_2$ in X if and only if $x_1 \leq x_2$ in X_i for each $i = 1, 2, \dots, t$.

If X_1 and X_2 are posets and the point set of X_1 is a subset of the point set of X_2 , the poset X_1 is called a *subposet* of X_2 when $x_1 \leq x_2$ in X_1 if and only if $x_1 \leq x_2$ in X_2 for all $x_1, x_2 \in X_1$. For each point $x \in X$, we let $X - \{x\}$ denote the subposet of X whose point set contains all points in X except x . Of course, $\dim(X - \{x\}) \leq \dim X$ for

each $x \in X$. For an integer $t \geq 2$, a poset X is *t-irreducible* if $\dim(X) = t$ and $\dim(X - \{x\}) < t$ for each $x \in X$. A poset has dimension one if and only if it is a linear order (a chain), so the only 2-irreducible poset is a two point antichain. There are infinitely many 3-irreducible posets, and a complete listing of these posets has been made by Trotter and Moore [10] and by Kelly [4]. These posets can be conveniently grouped into 9 infinite families with 18 odd examples left over.

In Section 3, we will make extensive use of one of the infinite families of 3-irreducible posets. In Section 4, we will use D. Kelly's dimension product construction to obtain for each $t \geq 4$ a particular family of t -dimensional posets. But before we proceed to those results, we need to develop additional material for working with the dimension of a poset.

A set $R = \{X_1, X_2, \dots, X_t\}$ of t (not necessarily distinct) linear extensions of a poset X is called a *realizer* of X if $x_1 \leq x_2$ in X if and only if $x_1 \leq x_2$ in X_i for each $i = 1, 2, \dots, t$. The dimension of X is then the minimum size of a realizer of X . Let I_X denote the set of all incomparable pairs of X . Then it is easy to see that a collection $R = \{X_1, X_2, \dots, X_t\}$ of linear extensions of X is a realizer of X if and only if for each $(x_1, x_2) \in I_X$, there exists some $X_i \in R$ so that $x_2 < x_1$ in X_i . If $I \subseteq I_X$ and $R = \{X_1, X_2, \dots, X_t\}$ is a collection of linear extensions of X , then we say R *reverses* I if for every $(x_1, x_2) \in I$, there exists some $X_i \in R$ with $x_2 < x_1$ in X_i . The dimension of a poset X with $I_X \neq \emptyset$ is then the minimum number of linear extensions of X required to reverse all incomparable pairs.

For a binary relation R , we let $\hat{R} = \{(r_1, r_2): (r_2, r_1) \in R\}$. \hat{R} is called the *dual* or *reverse* of R . When X is not a chain, it follows that $\dim(X)$ is the least t for which there exists a partition $\hat{I}_X = I_1 \cup I_2 \cup \dots \cup I_t$ so that for each $i = 1, 2, \dots, t$, there exists a linear extension X_i of X with $x_1 < x_2$ in X_i for every $(x_1, x_2) \in I_i$. It is therefore natural to consider the following question:

If $I \subseteq I_X$, under what conditions does there exist a linear extension X_0 of X with $x_1 < x_2$ in X_0 for every $(x_1, x_2) \in I$?

The answer to this question is easy to provide. A set $\{(c_i, d_i): 1 \leq i \leq m\} \subseteq I_X$ is called a *TM-cycle of length m* when $d_i \leq c_{i+1}$ for $i = 1, 2, \dots, m-1$ and $d_m \leq c_1$. It is easy to show that these sets provide an answer to this question (see [11], for example).

Lemma 1. *Let X be a poset and let $I \subseteq I_X$. Then there exists a linear extension X_0 of X with $x_1 < x_2$ in X_0 for every $(x_1, x_2) \in I$ if and only if I does not contain a TM-cycle. ■*

In many cases it is convenient to have a somewhat more technical version of this result at our disposal. A TM-cycle $\{(c_i, d_i): 1 \leq i \leq m\}$ is said to be *strong* if $d_i \leq c_j$ if and only if $j = i + 1$ for $1 \leq i \leq m - 1$, and $d_m \leq c_1$ if and only if $j = 1$. It is straightforward to verify that if $I \subseteq I_X$ and I contains a TM-cycle, then I also contains a strong TM-cycle. Furthermore, if I is a strong TM-cycle, then no proper subset of I contains a TM-cycle.

Lemma 2. *Let X be a poset and let $I \subseteq I_X$. Then there exists a linear extension X_0 of X with $x_1 < x_2$ in X_0 for every $(x_1, x_2) \in I$ if and only if I does not contain a strong TM-cycle. ■*

In view of the preceding result, it is natural to associate with a poset X a hypergraph G_X so that the dimension of X is the same as the chromatic number of G_X . Here we use the definition of the chromatic number of G_X as the least number of colors required to assign colors to the vertices of G_X so that no edge of G_X has all of its vertices assigned the same color. The scheme for defining G_X is immediate. The vertex set of G_X is the set I_X of incomparable pairs and a subset $I \subseteq I_X$ is an edge if its reverse \hat{I} is a strong TM-cycle.

From a practical view point, the hypergraph G_X contains too many vertices to be of much value in determining the dimension of X . However, there is a natural way to determine a subhypergraph H_X of G_X so that H_X and G_X have the same chromatic number, and in many cases the combinational structure of H_X is more readily analyzed.

An incomparable pair $(x_1, x_2) \in I_X$ is called a *nonforced pair* if $x_3 < x_1$ implies $x_3 < x_2$ for all $x_3 \in X$ and $x_2 < x_4$ implies $x_1 < x_4$

for all $x_4 \in X$. We let N_X denote the set of all nonforced pairs. It is customary to treat N_X as both a binary relation and a directed graph. In the latter interpretation, we draw an edge from x_2 to x_1 whenever $(x_1, x_2) \in N_X$. In Figures 1a and 1b we show a 2-dimensional poset X and its digraph N_X of nonforced pairs.

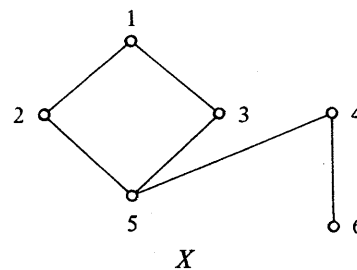


Figure 1a

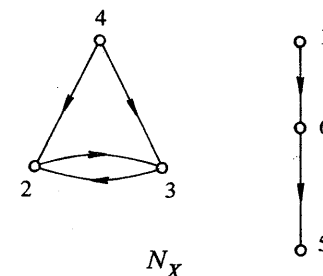


Figure 1b

The graph theoretic properties of the digraph N_X are central to the theory of rank for partially ordered sets, and we refer the reader to [6], [7], [8] and [9] for additional material on this topic. In this paper we will require some elementary properties of N_X . We state these results without proof and refer the reader to [7] for details.

The initial advantage gained from considering the set of nonforced pairs is that they are useful in identifying realizers.

Lemma 3. *Let $R = \{X_1, X_2, \dots, X_t\}$ be a set of linear extensions of a poset X . Then R is a realizer of X if and only if for each $(x_1, x_2) \in N_X$, there exists some $X_i \in R$ so that $x_2 < x_1$ in X_i . ■*

It follows immediately from the preceding lemma that if X is not a chain, then the dimension of X is the minimum number of linear extensions of X required to reverse the nonforced pairs of X . This observation allows us to determine a subhypergraph H_X of G_X which has the same chromatic number as G_X . The vertex set of H_X is the set N_X of nonforced pairs of X . A subset $N \subseteq N_X$ is an edge in H_X if and only if \hat{N} is a TM-cycle. It is often the case that the hypergraph H_X has relatively simple structure; in particular, it is frequently a simple graph whose coloring properties can be easily determined.

If $I \subseteq I_X$, we abuse notation somewhat and denote by $X \cup I$ the binary relation of the point set of the poset X defined by $(x_1, x_2) \in X \cup I$ if and only if $x_1 < x_2$ in X or $(x_1, x_2) \in I$. Note that an incomparable pair $(x_1, x_2) \in I$ is a nonforced pair if and only if $X \cup \{(x_1, x_2)\}$ is transitive.

Lemma 4. *If X is a poset and N_X is its set of nonforced pairs, then the binary relation $X \cup N_X$ is transitive. ■*

A subset Y of a poset X is said to be *partitive* (in X) if the following two conditions are satisfied:

- (i) If $x \in X - Y$ and $x > y$ for some $y \in Y$, then $x > y$ for all $y \in Y$.
- (ii) If $x \in X - Y$ and $x < y$ for some $y \in Y$, then $x < y$ for all $y \in Y$.

A partitive subset Y in a poset X is nontrivial when $2 \leq |Y| < |X|$. The following result is a special case of the formula for the dimension of an ordinal sum (see [3] or [7]).

Lemma 5. *If Y is a nontrivial partitive subposet of a poset X and $y_0 \in Y$, then $\dim(X) = \max\{\dim(X - (Y - \{y_0\})), \dim(Y)\}$. ■*

In particular, it follows that if $t \leq 2$ and X is t -irreducible, then X contains no nontrivial partitive subposets.

Lemma 6. *Let X be a poset and N_X the set of nonforced pairs of X . If the binary relation $X \cup N_X$ contains a directed cycle $\{(x_i, x_{i+1}) : 1 \leq i < m\} \cup \{(x_m, x_1)\}$ where $m \geq 2$, then the subposet $\{x_1, x_2, \dots, x_m\}$ is a partitive antichain in X . ■*

The only 2-irreducible poset is a two element antichain. For this poset $I_X = N_X = X \cup N_X$ and each of these binary relations is a directed cycle of length two. But for $t \geq 3$, no such pathology can occur.

Lemma 7. *Let $t \geq 3$, let X be a t -irreducible poset, and let N_X be the set of nonforced pairs of X . Then the binary relation $X \cup N_X$ is acyclic — that is, it contains no directed cycles. ■*

For any poset X for which $X \cup N_X$ is acyclic, it is therefore permissible to consider $X \cup N_X$ as a partially ordered set. With this interpretation, $X \cup N_X$ is an extension of the poset X . For such posets, a linear extension $X_0 = \{x_1 < x_2 < x_3 < \dots < x_n\}$ of $X \cup N_X$ is said to be *consistent* if $i < j$ whenever $x_i \in D_X(x_n)$ and $x_j \in I_X(x_n)$. A maximal element x of the poset $X \cup N_X$ is called a *strongly maximal* element of X .

Lemma 8. *Let X be a poset and let N_X be the set of nonforced pairs of X . If $X \cup N_X$ is acyclic and x is a strongly maximal element of X , then there exists a consistent linear extension $X_0 = \{x_1 < x_2 < x_3 < \dots < x_n\}$ of $X \cup N_X$ with $x = x_n$. ■*

If X is a t -dimensional poset and X_0 is a consistent linear extension of $X \cup N_X$, then X_0 cannot belong to any realizer of size t of X since X_0 reverses no nonforced pairs. On the other hand, we will frequently make minor modifications in a consistent linear extension to obtain one which does belong to a realizer of size t . Here is one such instance; others will be discussed in Section 3.

If $X_0 = \{x_1 < x_2 < x_3 < \dots < x_n\}$ is a consistent linear extension of $X \cup N_X$, where $D_X(x_n) = \{x_1, x_2, \dots, x_s\}$, and $I_X(x_n) = \{x_{s+1}, x_{s+2}, \dots, x_{n-1}\}$, then the linear order

$$X_0^* = \{x_1 < x_2 < x_3 < \dots < x_s < x_n < x_{s+1} < x_{s+2} < \dots < x_{n-1}\}$$

is called the *reverse* of X_0 . Note that X_0^* is a linear extension of X but that X_0^* is not in general a linear extension of $X \cup N_X$. The following lemma shows that X_0^* belongs to a realizer of size t for X when X is t -irreducible. The result is a special case of the theorem due to Hiraguchi [3] which states that the removal of a point from a poset decreases the dimension by at most one.

Lemma 9. *Let $t \geq 3$ and let X be a t -irreducible poset. Also let X_0 be a consistent linear extension of $X \cup N_X$. Then let x be the strongly maximal element of X which is the greatest element in X_0 .*

If $\{X'_1, X'_2, \dots, X'_{t-1}\}$ is a realizer of $X - \{x\}$ and for each $i = 1, 2, \dots, t-1$, we form a linear extension X_i of X by adding x to X'_i as the largest element, then $\{X_0^*, X_1, X_2, \dots, X_{t-1}\}$ is a realizer of X . ■

3. THE EMBEDDING POSET WHEN $t = 3$

In this section, we present an infinite family of 3-irreducible posets $\{X(n, 3): n \geq 1\}$ whose special properties will be particularly useful in the proof of our principal theorem in the case $t = 3$. It is of secondary importance that these posets are irreducible. What actually matters is that they each have a consistent linear extension (which of course cannot belong to any realizer of size 3) such that if any one of a large number of minor modifications is made, the resulting linear extension belongs to a realizer of size 3.

For each $n \geq 1$, the poset $X(n, 3)$ is a 3-irreducible poset for which the linear extension $X_0 = \{x_1, x_2, x_3, \dots, x_{2n+5}\}$ is consistent. In Figure 2, we show a diagram for $X(n, 3)$; for clarity, only the subscripts are shown.

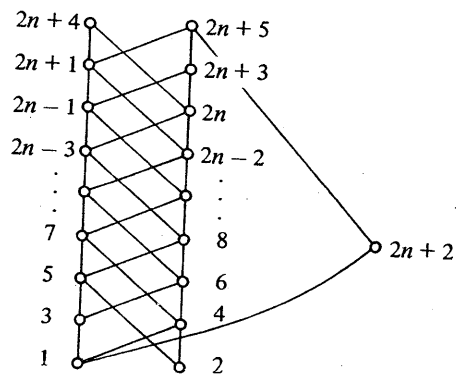


Figure 2

Let $X = X(n, 3)$; then it is straightforward to verify that the set N_X of nonforced pairs is the union of the two sets N'_X and N''_X where

$$N'_X = \{(x_i, x_{i+1}): 1 \leq i \leq 2n, 2n+2 \leq i \leq 2n+4\},$$

$$N''_X = \{(x_2, x_{2n+2}), (x_3, x_{2n+2}), (x_{2n+2}, x_{2n+4})\}.$$

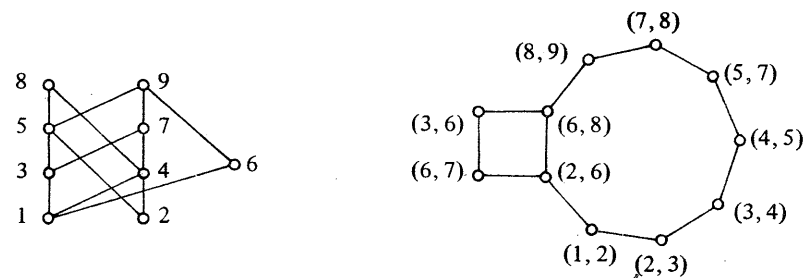
The following lemma gives important information on the structure of the hypergraph H_X .

Lemma 10. *If $n \geq 2$ and $X = X(n, 3)$, then the hypergraph H_X is a simple graph.*

Proof. Let N be an edge in the hypergraph H_X and suppose that N contains at least three vertices of H_X . After relabelling, we may assume that $\hat{N} = \{(c_i, d_i): 1 \leq i \leq m\}$ is a strong TM-cycle in X with $m = |N| \geq 3$. Since $C = \{c_1, c_2, \dots, c_m\}$ and $D = \{d_1, d_2, \dots, d_m\}$ are antichains in X , we may conclude that $m = 3$ and that $x_{2n+2} \in C \cap D$. Without loss of generality, we may assume that $x_{2n+2} = c_1$ and $x_{2n+2} = d_3$. Then since $(d_1, c_1) = (d_1, x_{2n+2}) \in N$, we know that either $d_1 = x_2$ or $d_1 = x_3$. Similarly, either $c_3 = x_{2n+3}$ or $c_3 = x_{2n+4}$. In any case, we would conclude that $d_1 < c_3$, which contradicts the assumption that \hat{N} is a strong TM-cycle. ■

Lemma 11. *If $n \geq 2$ and $X = X(n, 3)$, then the graph H_X is a 3-colorable graph. Furthermore, the removal from H_X of any one of the $2n$ vertices in the set $\{(x_i, x_{i+1}): 1 \leq i \leq 2n\}$ leaves a 2-colorable graph. ■*

We illustrate the preceding lemma when $n = 2$. For clarity only the subscripts are shown.



$X = X(2, 3)$

H_X

Figure 3

The reader may note that Lemma 11 has not been presented in the strongest possible form, but as will become clear, we are only interested in certain vertices in the graph H_X . Recall that the consistent linear extension X_0 reverses no nonforced pairs of X , so there is no realizer of size 3 to which X_0 belongs. Now let i be any integer with $1 \leq i \leq 2n$ and let X_0^i be the linear extension of X obtained by interchanging x_i and x_{i+1} in X_0 that is

$$X_0^i = \{x_1 < x_2 < \dots < x_{i-1} < x_{i+1} < x_i < \\ < x_{i+2} < x_{i+3} < \dots < x_{2n+5}\}.$$

We now show that each X_0^i belongs to a realizer of size 3 of X .

Lemma 12. *Let $n \geq 2$ and let $X = X(n, 3)$. Then for each $i = 1, 2, \dots, 2n$, there exist linear extensions X_1^i, X_2^i so that $\{X_0^i, X_1^i, X_2^i\}$ is a realizer of X .*

Proof. From Lemma 11, we note that for each $i = 1, 2, \dots, 2n$, the graph $H_X - \{(x_i, x_{i+1})\}$ can be 2-colored using the colors $\{1, 2\}$. Then for each $j = 1, 2$, let X_j^i be a linear extension of X which reverses all nonforced pairs which have been assigned color j . Since X_0^i reverses the nonforced pair (x_i, x_{i+1}) , it follows that $\{X_0^i, X_1^i, X_2^i\}$ reverses N_X and these linear extensions are a realizer of X . ■

4. THE EMBEDDING THEOREM WHEN $t = 3$

Suppose that X and Y are disjoint subsets of a poset Z . We say that Y is an *upper filter* of X if the following two conditions are satisfied:

- (i) For every $(x_1, x_2) \in N_X$, there exists $y \in Y$ with $x_1 < y$ and $x_2 \parallel y$.
- (ii) For every $x \in X$ and every $y \in Y$, $y \not\prec x$.

A linear extension Z_0 of Z is called an *injection of X over Y* when $x \in X$, $y \in Y$, and $x \parallel y$ imply that $x > y$ in Z_0 . The concepts of upper filters and injections are related by the following elementary result.

Lemma 13. *Let X and Y be disjoint subsets in a poset Z , and let Y be an upper filter of X . If $\dim(X) = t$ and Z_0 is a linear extension of Z which is an injection of X over Y , then Z_0 cannot belong to a realizer of size t for Z .*

Proof. Any realizer of Z must reverse all incomparable pairs in Z and must therefore reverse the incomparable pairs in N_X . Now let $(x_1, x_2) \in N_X$; choose an element $y \in Y$ so that $x_1 < y$ and $x_2 \parallel y$ in Z . Since Z_0 is an injection of X over Y , we must have $x_1 < y < x_2$ in Z_0 . Thus Z_0 reverses no pairs in N_X . If R is a realizer of size t for Z and $Z_0 \in R$, then the restrictions of the other $t - 1$ linear extensions to X must reverse N_X . Since $\dim X = t$, this is impossible. ■

If X and Y are disjoint subsets of a poset Z , we write $X < Y$ when $x < y$ for every $x \in X$ and $y \in Y$. Similarly, we write $X \parallel Y$ when $x \parallel y$ for every $x \in X$ and $y \in Y$.

We are now ready to present the proof of our principal theorem for the case $t = 3$.

Theorem 14. *Let P be a poset with $\dim P \leq 3$. Then there exists a 4-irreducible poset R containing P as a subposet.*

Proof. We first construct a 4-dimensional poset S containing P as a subposet. In general, S will not be irreducible, but we will prove that S contains a 4-irreducible subposet R which also contains P as a subposet. The poset S is the union of five disjoint subsets P, X, Y, U , and V , with $X \parallel U$, $X < V$, $U < (P \cup Y)$, $P < Y$, and $V \parallel (P \cup Y)$. Furthermore, $P \cup Y$ will be an upper filter of X and V will be an upper filter of U . The posets X and U will be 3-irreducible posets.

Since $\dim P \leq 3$, there exist linear extensions P_1, P_2 , and P_3 of P so that $p_1 \leq p_2$ in P if and only if $p_1 \leq p_2$ in P_i for $i = 1, 2, 3$. These linear extensions need not be distinct. We suppose that $|P| = m$ and let $P_1 = \{p_1 < p_2 < p_3 < \dots < p_m\}$. Next choose an integer n so that $n \geq 2$ and $2n \geq m$. Then the subposet X is $X(n, 3)$.

The subposet Y is a chain containing $2n + 4 - m$ points $\{y_{m+1} < y_{m+2} < \dots < y_{2n+4}\}$. For each $j = 1, 2, \dots, m$, $x_j < p_j$ in S if and

only if $i \leq j$. Also, for each $j = m + 1, m + 2, \dots, 2n + 4$, $x_i < y_j$ in S if and only if $i \leq j$. To see that $P \cup Y$ is an upper filter of X , we consider an arbitrary nonforced pair $(x_i, x_j) \in N_X$. Since X_0 is consistent, we know that $i < j$. If $1 \leq i \leq m$, then $p_i > x_i$ and $p_i \parallel x_j$; if $m + 1 \leq i \leq 2n + 4$, then $y_i > x_i$ and $y_i \parallel x_j$. Therefore, $P \cup Y$ is an upper filter of X is claimed.

The subposet U is the standard example of a 3-irreducible poset. The linear extension $U_0 = \{u_1 < u_2 < u_3 < u_4 < u_5 < u_6\}$ is consistent and the subposets $\{u_1, u_2, u_3\}$ and $\{u_4, u_5, u_6\}$ are antichains. Furthermore, $u_i < u_{3+j}$ if and only if $i \neq j$ for each $i, j = 1, 2, 3$.

The subposet V consists of a single point $\{v\}$ with $u_i < v$ if and only if $i = 1, 2, 3$. Note that $N_U = \{(u_i, u_{3+i}) : i = 1, 2, 3\}$, so that V is an upper filter of U . This completes the definition of the poset S .

We now show that $\dim S \geq 4$. Suppose to the contrary that $\dim S \leq 3$. Since S contains the 3-dimensional poset X , we conclude that $\dim S = 3$. Then let $\mathcal{S} = \{S_1, S_2, S_3\}$ be a realizer of S . We know that these three linear extensions reverse all incomparable pairs in S , but we are particularly interested in how they reverse the incomparable pairs in the following two subsets of I_S :

$$N_1 = \{(x, z) \in I_S : x \in X, z \in P \cup Y\}$$

$$N_2 = \{(u_i, v) \in I_S : i = 4, 5, 6\}.$$

It is easy to see that no linear extension of \mathcal{S} can reverse a pair from N_1 and a pair from N_2 . For if $(x, z) \in N_1$, and $z < x$ in S_i , then $u < z < x < v$ in S_i for every $u \in U$ and $v \in V$. Similarly, if $(u, v) \in N_2$ and $v < u$ in S_i , then $x < v < u < z$ for every $x \in X$ and $z \in P \cup Y$. On the other hand, no linear extension in \mathcal{S} can reverse N_1 for this would imply that it is an injection of X over $P \cup Y$ in violation of Lemma 13. Similarly, no linear extension in \mathcal{S} can reverse N_2 . We conclude that at least two linear extensions in \mathcal{S} reverse pairs in N_1 , and at least two linear extensions in \mathcal{S} reverse pairs in N_2 . But this is clearly impossible. The contradiction shows that $\dim S \geq 4$.

We next show that $\dim(S - \{p_i\}) = 3$ for every $p_i \in P$. To accomplish this, we choose an arbitrary point $p_i \in P$ and construct a realizer $\{S'_1, S'_2, S'_3\}$ of the subposet $S - \{p_i\}$. First, let $\{U_0^*, U_1, U_2\}$ be a realizer of U where U_0^* is the reverse of the consistent linear extension U_0 and U_i is formed from U'_i by adding U_6 as the largest element. Then let $(U \cup V)_0^*$ be the linear extension of $U \cup V$ defined by $(U \cup V)_0^* = \{u_1 < u_2 < u_6 < u_3 < v < u_4 < u_5\}$. The restriction of $(U \cup V)_0^*$ to U is U_0^* ; furthermore $(U \cup V)_0^*$ reverses (u_4, v) and (u_5, v) but not (u_6, v) .

Since $1 \leq i \leq m \leq 2n$, there exists by Lemma 12 a realizer $\{X_0^i, X_1^i, X_2^i\}$ of the 3-irreducible poset X where X_0^i is obtained from X_0 by interchanging x_i and x_{i+1} . We let $(X \cup P \cup Y)_0^i$ denote the linear extension of $X \cup P \cup Y - \{p_i\}$ defined by:

$$\begin{aligned} (X \cup P \cup Y)_0^i &= \{x_1 < p_1 < x_2 < p_2 < x_3 < p_3 < \dots \\ &\dots < x_{i-2} < p_{i-2} < x_{i-1} < p_{i-1} < x_{i+1} < \\ &< x_i < p_{i+1} < x_{i+2} < p_{i+2} < \dots \\ &\dots \leq x_m < p_m < x_{m+1} < y_{m+1} < x_{m+2} < y_{m+2} < \dots \\ &\dots < x_{2n+4} < y_{2n+4} < x_{2n+5}\}. \end{aligned}$$

Then it follows that the restriction of $(X \cup P \cup Y)_0^i$ to X is X_0^i , and the restriction of $(X \cup P \cup Y)_0^i$ to $P - \{p_i\}$ is $P_1 - \{p_i\}$. Furthermore, we note that $(X \cup P \cup Y)_0^i$ is an injection of X over $P \cup Y - \{p_i\}$. It also reverses the nonforced pair $(x_i, x_{i+1}) \in N_X$. We then define:

$$S'_1 = U_1 < (X \cup P \cup Y)_0^i < V,$$

$$S'_2 = X_1^i < U_2^* < V < u_6 < P_2 - \{p_i\} < Y,$$

$$S'_3 = X_2^i < (U \cup V)_0^* < P_3 - \{p_i\} < Y.$$

To see that these linear orders form a realizer of $S - \{p_i\}$, we make the following observations:

1. Each S'_j is clearly a linear extension of $S - \{p_i\}$.

2. $U < X$ and $P \cup Y - \{p_i\} < V$ in S'_1 .
3. $X < U$ and $V < P \cup Y - \{p_i\}$ in S'_2 .
4. S'_1 is an injection of X over $P \cup Y - \{p_i\}$.
5. S'_2 and S'_3 reverse N_2 .
6. The restriction of $\{S'_1, S'_2, S'_3\}$ to X is $\{X_0^i, X_1^i, X_2^i\}$.
7. The restriction of S'_j to $P - \{p_i\}$ is $P_j - \{p_i\}$ for $j = 1, 2, 3$.
8. The restriction of S'_j to U is U_j for $j = 1, 2$ and the restriction of S'_3 to U is U_0^* .

At this point, the proof of our theorem is essentially complete. We have shown that $\dim(S) \geq 4$ but that the removal of any point from P leaves a three dimensional poset. Now for every $t \geq 2$, a t -dimensional poset contains a t -irreducible subposet. In the situation at hand, any 4-irreducible subposet R of S must obviously contain P as a subposet. ■

Although the poset S is not 4-irreducible, it does not miss the mark by far. There are exactly two 4-irreducible subposets of S ; they are $S - \{y_{2n+1}\}$ and $S - \{y_{2n+2}\}$.

5. THE EMBEDDING POSETS WHEN $t \geq 4$

In this section, we construct for each $t \geq 4$ an infinite family $\{X(n, t): n \geq 1\}$ of t -dimensional posets each of which possesses a consistent linear extension in which any one of a large number of minor modifications allows the resulting extension to belong to a realizer of $X(n, t)$ of size t . The construction will utilize the concept of a *dimension product* as introduced in [5] by D. Kelly.

If X and Y are posets, then the *cartesian product* $X \times Y$ is the poset whose point set is the set of all pairs (x, y) where $x \in X$ and $y \in Y$ with $(x_1, y_1) \leq (x_2, y_2)$ if and only if $x_1 \leq x_2$ and $y_1 \leq y_2$. The cartesian product of n copies of X is denoted X^n . It is easy to see that $\dim(X \times Y) \leq \dim X + \dim Y$. The following result of Baker [1] gives a sufficient condition for equality to hold.

Theorem 15. *Let X and Y be posets. If X contains poset x_1, x_2 and Y contains point y_1, y_2 so that $x_1 \leq x \leq x_2$ and $y_1 \leq y \leq y_2$ for every $x \in X$ and $y \in Y$, then $\dim(X \times Y) = \dim X + \dim Y$. ■*

For an integer $n \geq 1$, let n denote the n element chain $\{0 < 1 < 2 < \dots < n - 1\}$. Also for a poset X , let \bar{X} denote the poset obtained from X by adding two new points, one larger than every point in X and the other less than every point in X . Then $\dim(\bar{X}) = \dim X$ for every poset X . Furthermore, $\dim(\bar{X} \times \bar{Y}) = \dim X + \dim Y$. Kelly's dimension product identifies a (relatively small) subposet of $\bar{X} \times \bar{Y}$ whose dimension is $\dim X + \dim Y$, and in many cases, Kelly's construction precisely determines an irreducible subposet. Such will be the case in the construction we now present.

For each $n \geq 1$ and each $t \geq 4$, $X(n, t)$ will be a t -dimensional subposet of the cartesian product $\bar{X}(n, 3) \times 2^{t-3}$. We will find it convenient to use notation similar to that employed in Sections 3 and 4 rather than the notation of Kelly.

The poset $X(n, t)$ is the union of four subposets $X'(n), X''(n), A_{t-3}$, and B_{t-3} . The subposets $X'(n)$ and $X''(n)$ are both copies of $X(n, 3)$ labelled $\{x'_i: 1 \leq i \leq 2n + 5\}$ and $\{x''_i: 1 \leq i \leq 2n + 5\}$ respectively. Furthermore, $x''_i \leq x'_j$, $x'_i \leq x'_j$, and $x''_i < x'_j$ if and only if $x_i \leq x_j$ in $X(n, 3)$.

The subposets A_{t-3} and B_{t-3} are both $t - 3$ element antichains labelled $\{a_1, a_2, \dots, a_{t-3}\}$ and $\{b_1, b_2, \dots, b_{t-3}\}$ respectively. Furthermore, $b_i < a_j$ if and only if $i \neq j$ for all $i, j = 1, 2, \dots, t - 3$. In addition $X'(n) > B_{t-3}$ and $X''(n) < A_{t-3}$. This completes the definition of the poset $X(n, t)$.

For graphs G_1 and G_2 , the *join* of G_1 and G_2 , denoted by $G_1 + G_2$, is the graph obtained by taking disjoint copies of G_1 and G_2 and adding an edge between g_1 and g_2 for every vertex $g_1 \in G_1$ and $g_2 \in G_2$. Clearly, the chromatic number of $G_1 + G_2$ is the sum of the chromatic numbers of G_1 and G_2 . For an integer $k \geq 1$, let K_k denote the complete graph on k vertices. It is straightforward to verify the following results concerning the hypergraph associated with $X(n, t)$.

Lemma 16. Let $n \geq 1$ and $t \geq 4$. Also let $X_1 = X(n, 3)$ and $X_2 = X(n, t)$. Then

1. The set N_{X_2} of nonforced pairs of X_2 is the union of two sets N'_{X_2} and N''_{X_2} where $N'_{X_2} = \{(x_i'', x_j') : (x_i, x_j) \in N_{X_1}\}$ and $N''_{X_2} = \{(b_i, a_i) : 1 \leq i \leq t-3\}$.
2. If $n \geq 2$, the hypergraph H_{X_2} is a simple graph.
3. The induced subgraph G_1 of H_{X_2} whose vertex set is N'_{X_2} is isomorphic to H_{X_1} .
4. The induced subgraph G_2 of H_{X_2} whose vertex set is N''_{X_2} is isomorphic to K_{t-3} .
5. $H_{X_2} = G_1 + G_2$.
6. $\chi(H_{X_2}) = \dim(X_2) = t$.
7. For each $i = 1, 2, \dots, 2n$, the removal of the vertex (x_i'', x_{i+1}') from H_{X_2} leaves a graph with chromatic number $t-1$. ■

The reader may note that $X(n, t)$ is not t -irreducible but that the removal of x_1' and x_{2n+5}' leaves a t -irreducible subposet. For symmetry we include these points in the definition of the embedding posets $X(n, t)$. The following linear order is easily seen to be a consistent linear extension of $X(n, t)$:

$$\begin{aligned} X_0 = \{ & b_1 < b_2 < b_3 < \dots < b_{t-3} < \\ & x_1'' < x_1' < x_2'' < x_2' < x_3'' < x_3' < x_4'' < \dots < \\ & x_{2n+3}'' < x_{2n+3}' < x_{2n+4}'' < x_{2n+4}' < x_{2n+5}'' < x_{2n+5}' < \\ & a_1 < a_2 < a_3 < \dots < a_{t-3} \}. \end{aligned}$$

For each $i = 1, 2, \dots, 2n$, we let X_0^i denote the linear extension of X obtained by interchanging $x_i'' < x_i'$ with $x_{i+1}'' < x_{i+1}'$. Note that exactly four entries in X_0 change places in this modification. Note further

that X_0^i reverses the nonforced pair (x_i'', x_{i+1}') and thus by Statement 7 in Lemma 16, there exist linear extensions $X_1^i, X_2^i, \dots, X_{t-1}^i$ so that $\{X_0^i, X_1^i, X_2^i, \dots, X_{t-1}^i\}$ is a realizer of X .

6. THE EMBEDDING THEOREM WHEN $t \geq 4$

The proof of the following theorem is similar to the argument in Section 4 so we will present only the essential steps.

Theorem 17. If $t \geq 4$ and P is a poset whose dimension is at most t , then there exists a $t+1$ -irreducible poset R containing P as a subposet.

Proof. We begin by constructing a $t+1$ -dimensional poset S containing P as a subposet. Suppose P has m elements. Let n be an integer with $2n \geq m$. Let $\{P_1, P_2, \dots, P_t\}$ be a realizer of P with $P_1 = \{p_1 < p_2 < \dots < p_m\}$. Then the poset S is the union of the following disjoint subposets P, X, Y, U, V, C , and D which are defined as follows:

1. $X = X(n, t)$.
2. Y is a $2n+4-m$ element chain labelled $\{y_{m+1} < y_{m+2} < \dots < y_{2n+4}\}$.
3. U is the standard t -dimensional poset labelled $\{u_1, u_2, \dots, u_{2t}\}$ with $u_i < u_{t+j}$ if and only if $i \neq j$ for all $i, j = 1, 2, \dots, t$.
4. $V = \{v\}$ is a one point poset.
5. The subposets $C = \{c_1, c_2, \dots, c_{t-3}\}$ and $D = \{d_1, d_2, \dots, d_{t-3}\}$ are $t-3$ point antichains.

To complete the definition of S , we describe the comparabilities between these subposets:

6. $P < Y$, $(P \cup Y) \parallel A_{t-3}$ and $(P \cup Y) > B_{t-3}$.
7. For each $j = 1, 2, \dots, m$, $x_i'' < x_i' < p_j$ if and only if $i \leq j$.
8. For each $j = m+1, m+2, \dots, 2n+4$, $x_i'' < x_i' < y_j$ if and only if $i \leq j$.

9. $V > \{u_1, u_2, \dots, u_t\}$ and $V \parallel \{u_{t+1}, u_{t+2}, \dots, u_{2t}\}$.
10. For each $i, j = 1, 2, \dots, t-3$, $d_i < c_j$ if and only if $i \neq j$.
11. $(X \cup U) < C$, $(X \cup D) < V$, and $(U \cup D) < (P \cup Y)$.
12. $(P \cup Y) \parallel (V \cup C)$, $V \parallel C$, $X \parallel (U \cup D)$, and $U \parallel D$.

Note that $P \cup Y$ is an upper filter of X and V is an upper filter of U . We now show that $\dim S \geq t+1$. To the contrary, suppose that $\{S_1, S_2, \dots, S_t\}$ is a realizer of S . Then these extensions reverse all nonforced pairs of I_S . In particular they reverse all nonforced pairs in the following three sets:

$$N_1 = \{(x, z): x \in X, z \in P \cup Y, x \parallel z\},$$

$$N_2 = \{(u, v): u \in U, v \in V, u \parallel v\},$$

$$N_3 = \{(d_i, c_i): i = 1, 2, \dots, t-3\}.$$

We then note that no linear extension of S can reverse pairs from any two of these three sets. Furthermore, a linear extension can reverse at most one pair from N_3 . This implies that one of the S_i 's is an injection of X over $P \cup Y$ or an injection of U over V . Neither of these statements can be true, so we conclude that $\dim S \geq t+1$.

We next show that $\dim(S - \{p_i\}) = t$ for $i = 1, 2, \dots, m$. To accomplish this we make the following observations. There is a linear extension $(X \cup P \cup Y)_0^i$ of $X \cup P \cup Y$ such that

- (a) The restriction to X is X_0^i .
- (b) The restriction to $P \cup Y - \{p_i\}$ is $P_1 - \{p_i\} < Y$.
- (c) It is an injection of X over $P \cup Y - \{p_i\}$.

Furthermore, there is a realizer $\{U_0^*, U_1, U_2, \dots, U_{t-1}\}$ of U where U_0^* is the reverse of the consistent linear extension $U_0 = \{u_1 < u_2 < \dots < u_{2t}\}$, and U_i is formed from U_i^i by adding u_{2t} as the largest element. It follows that there is a linear extension $(U \cup V)_0^*$ such that

- (a) The restriction to U is U_0^* .
- (b) It reverses all nonforced pairs in N_2 except (u_{2t}, v) .

Let C_0 and D_0 be arbitrary linear orders on C and D respectively. Then define

$$S'_1 = D_0 < U_1 < (X \cup P \cup Y)_0^i < V < C_0$$

$$S'_2 = U_2 < X_1^i < D_0 - \{d_1\} < c_1 < d_1 < C_0 - \{c_1\} < P_2 - \{p_i\} < Y < V$$

$$S'_3 = U_3 < X_2^i < D_0 - \{d_2\} < c_2 < d_2 < C_0 - \{c_2\} < P_3 - \{p_i\} < Y < V$$

$$S'_{t-2} = U_{t-2} < X_{t-3}^i < D_0 - \{d_{t-3}\} < c_{t-3} < d_{t-3} < C_0 - \{c_{t-3}\} < P_{t-2} - \{p_i\} < Y < V$$

$$S'_{t-1} = D_0 < X_{t-2}^i < U'_{t-1} < V < U_{2t} < P_{t-1} - \{p_i\} < Y < C_0$$

$$S'_t = D_0 < X_{t-1}^i < (U \cup V)_0^* < P_t - \{p_i\} < Y < C_0.$$

It is straightforward to verify that these t linear extensions of $S - \{p_i\}$ form a realizer so we conclude that $\dim(S - \{p_i\}) = t$ for each $i = 1, 2, \dots, m$. It follows that any t -irreducible subposet R of S contains P as a subposet and the proof of our theorem is complete. ■

7. CONCLUDING REMARKS

The reader is encouraged to compare the results and techniques of this paper with [12] where it is proved that every t -irreducible poset is a subposet of a $t+1$ -irreducible poset. In contrast to the situation here, the weaker result presented in [12] requires no specialized knowledge of the nature of particular irreducible posets or such devices as dimension products and upper filters. The authors consider the stronger

embedding theorem presented here to be a surprising result with significant implications in the dimension theory of partially ordered sets. In particular, it implies that the collection of all irreducible posets is far richer than was previously believed. It suggests further that researchers should consider more restrictive properties of irreducible posets in order to make further progress toward understanding these structures.

REFERENCES

- [1] K.A. Baker, Join-independence, and breadth in partially ordered sets, 1961, unpublished.
- [2] B. Dushnik – E.W. Miller, Partially ordered sets, *Amer. J. Math.*, 63 (1941), 600–610.
- [3] T. Hiraguchi, On the dimension of orders, *Sci. Rep. Kanazawa Univ.*, 4 (1955), 1–20.
- [4] D. Kelly, The three-irreducible partially ordered sets, *Canad. J. Math.*, 29 (1977), 367–383.
- [5] D. Kelly, On the dimension of partially ordered sets, *Discrete Math.*, 35 (1981), 135–156.
- [6] D. Kelly – W.T. Trotter, Jr., *An introduction to the dimension theory of partially ordered sets*, in *Ordered Sets*, Ivan Rival, ed., Reidel Publishing, Dordrecht, 1982, 171–211.
- [7] S. Maurer – I. Rabinovitch – W.T. Trotter, Jr., Large minimal realizers of partial order II, *Discrete Math.*, 31 (1980), 297–314.
- [8] S. Maurer – I. Rabinovitch – W.T. Trotter, Jr., A generalization of Turán's theorem to directed graphs, *Discrete Math.*, 32 (1980), 167–189.

- [9] S. Maurer – I. Rabinovitch – W.T. Trotter, Jr., Partially ordered sets with equal rank and dimension, in *Proceedings of the 11th Southeastern Conference on Combinatorics, Graph Theory and Computing, Congressus Numerantium*, 29 (1980), 627–637.
- [10] W.T. Trotter, Jr. – J.I. Moore, Jr., Characterization problems for graphs, partially ordered sets, lattices, and families of sets, *Discrete Math.*, 15 (1976), 361–368.
- [11] W.T. Trotter, Jr. – J.I. Moore, Jr., The dimension of planar posets, *J. Comb. Theory (B)*, 21 (1976), 51–67.
- [12] W.T. Trotter, Jr. – J.A. Ross, Every t -irreducible partial order is a suborder of a $t + 1$ -irreducible partial order, to appear in *Proceedings of the International Colloquium on Graph Theory and Combinatorics, Marseille, 1981*.

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