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[W6] D.B. West, "Poly-unsaturated partitions": The Greene-Kleitman Theorem is best possible. *J. Comb. Th. (A)* (submitted).

[W7] D.B. West, L.H. Harper, and D.A. Daykin, Some remarks on normalized matching. *J. Comb. Th (A)* 35(1983), 301-308.

[W8] D.B. West and D.J. Kleitman, Skew chain orders and sets of rectangles. *Disc. Math.* 27(1979), 99-102.

[W9] D.B. West and D.B. Shmoys, Recognizing graphs with fixed interval number is NP-complete. *Disc. Appl. Math.* 8(1984), 295-305.

[W10] D.B. West and C.A. Tovey, Semiantichains and unichain coverings in the direct products of partial orders. *SIAM J. Alg. Disc. Meth.* 2(1981), 295-305.

[W11] D. B. West, W. T. Trotter, Jr., G. W. Peck, and P. Shor, Regressions and monotone chains: a Ramsey-type extremal problem for partial orders. *Combinatorica,* 4(1984), 117-119.

[W12] A.T. White and L.W. Beineke, Topological graph theory. in *Selected Topics in Graph Theory* (L.W. Beineke and R.W. Wilson, eds.) Academic Press (1978), 15-49.

[W13] R. Wille, Lexicographic decomposition of ordered sets (graphs), *Fachbereich Mathematik,* Tech. Hochschule Darmstadt, Report No. 705.

[W14] H.S. Witsenhausen, On intersections of interval graphs. *Disc. Math.* 31(1980), 211-216.

[W15] D.R. Woodall, Menger and Konig systems. in *Theory and Applications of Graphs, Kalamazoo 1976* (Y. Alavi and D.R. Lick, eds.) Springer-Verlag Lect. Notes 642 (1978), 620-635.

[Y1] K. Yamamoto, Logarithmic order of free distributive lattices. *J. Math. Soc. Japan* 6(1954), 343-353.

[Y2] M. Yannakakis, The complexity of the partial order dimension problem. *SIAM J. Alg. Disc. Meth.* 3(1982), 351-358.

[Y3] A.C. Yao, some complexity questions related to distributive computing. *Proc. 11th ACM Symp. Th. Comp. (1979), 209-213.*

GRAPHS AND ORDERS IN RAMSEY THEORY AND IN DIMENSION THEORY

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ABSTRACT

The purpose of this paper is to present a concise and relatively self contained treatment of recent results linking partially ordered sets with topics more traditionally associated with graph theory and combinatorics: Ramsey theory and chromatic graph theory. In particular, we will present the major theorems of Nešetřil and Rödl ([31], [33], and [34]) concerning Ramsey theory for partially ordered sets in a new setting which will allow nonspecialists to appreciate the power and beauty of these results. Other Ramsey theoretic results for partially ordered sets will be discussed briefly and some directions for future research will be indicated.

We will also present a concise treatment of the constructions of Ross and Trotter ([53], [54]) for irreducible partially ordered sets utilizing familiar concepts from chromatic graph (and hypergraph) theory. When combined with the complexity theorems of Yannakakis [57], these constructions show that partially ordered sets can simultaneously exhibit both mathematical elegance and awkward pathology.

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1. Notation, Terminology, and Preliminary Results.

Throughout this paper, we will use the notation and terminology of [50] for partially ordered sets (posets, ordered sets, and dimension). For the sake of completeness we give here the central definitions, notations, and conventions. Formally a poset $\underset{\sim}{X}$ consists of a pair (X,P) where X is a nonempty set and P is a reflexive, antisymmetric, and transitive relation on X. P is called a partial order on X. The notations $(x,y) \in P$, xPy, $x \leq y$ in P, and $x \leq y$ in $\underset{\sim}{X}$ are used interchangeably. Similarly, we write $x < y$ in P or $x < y$ in $\underset{\sim}{X}$ when $x \leq y$ in P and $x \neq y$. When neither (x,y) nor (y,x) is in P, we say x and y are incomparable and write xIy in P or xIy in $\underset{\sim}{X}$. We denote the set of all incomparable pairs by I_P or $I_{\underset{\sim}{X}}$. A poset (X,P) is called an antichain if xIy in P for every $x,y \in X$ with $x \neq y$.

If (X,P) and (Y,Q) are posets, then we say (X,P) is isomorphic to (Y,Q) when there exists a function $f\colon X \xrightarrow[\text{onto}]{1-1}$ so that $(x_1,x_2) \in P$ if and only if $(f(x_1),f(x_2)) \in Q$. In this paper, we do not distinguish between isomorphic posets and write $\underset{\sim}{X} = \underset{\sim}{Y}$ when $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ are isomorphic. Similarly, we say $\underset{\sim}{Y}$ is contained in $\underset{\sim}{X}$ (or $\underset{\sim}{X}$ contains $\underset{\sim}{Y}$) and write $\underset{\sim}{Y} \subseteq \underset{\sim}{X}$ when Y is isomorphic to a subposet of $\underset{\sim}{X}$.

A partial order L on a set X is called a linear order (also total order or simple order) when $I_L = \phi$, and the poset (X,L) is called a chain. We denote an n-element chain by $\underset{\sim}{n}$. We denote the chain consisting of the positive integers by $\underset{\sim}{N}$ and the chain consisting of the real numbers by $\underset{\sim}{R}$. We denote the free sum (also called disjoint sum) of poset $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ by $\underset{\sim}{X} + \underset{\sim}{Y}$ and the cartesian product by $\underset{\sim}{X} \times \underset{\sim}{Y}$. The notation $\underset{\sim}{X}^t$ means the product of t copies of $\underset{\sim}{X}$.

If P and Q are partial orders on a set X and $P \subseteq Q$, then Q is called an extension of P. A linear order L which is an extension of P is called a linear extension of P. A theorem of Szpilrajn [43] asserts that for any partial order P, the collection C of all linear extensions of P is nonempty and $\cap C = P$. The dimension [7] of a poset (X,P), denoted $\dim(X,P)$ or $\dim(\underset{\sim}{X})$, is the smallest positive integer t for which there exist linear extensions $L_1, L_2, \ldots, L_t$ of P whose intersection is P. A poset has dimension one if and only if it is a chain.

Alternately, the dimension [35] of (X,P) is the smallest positive integer t for which there exists a function f which assigns to each $x \in X$ a sequence $f(x)(1), f(x)(2), \ldots, f(x)(t)$ of real numbers so that $x \leq y$ in P if and only if $f(x)(i) \leq f(y)(i)$ in $\underset{\sim}{R}$ for $i = 1,2,\ldots,t$. The function f is called an embedding of (X,P) in $\underset{\sim}{R}^t$.

If (X,P) is a poset and Y is a nonempty subset of X, then the restriction of P to Y, denoted $P(Y)$, is defined by $P(Y) = P \cap (Y \times Y)$. It is clear that $P(Y)$ is a partial order on Y, and we say $(Y,P(Y))$ is a subposet of (X,P). Obviously $\mathrm{Dim}(Y,P(Y)) \leq \mathrm{Dim}(X,P)$ whenever $\phi \neq Y \subseteq X$. The subposet $(Y,P(Y))$ is called a proper subposet of (X,P) when $\phi \neq Y \neq X$.

Part I: RAMSEY THEORY FOR POSETS

2. Ramsey Theory - Background Material

Intuitively speaking, Ramsey theory is the formal study of generalizations of the elementary result known as the "Pigeon Hole Principle" which asserts that if $(m-1)\,n + 1$ pigeons are placed in n distinct holes, then there is (at least) one hole into which (at least) m pigeons have been placed. In order to simplify the formal statements of the Ramsey theoretic results which follow, we pause to introduce some notation and terminology.

Throughout this section, we will be discussing functions of a set X to the set $\{1,2,\ldots,r\}$ of the first r positive integers. The integers $\{1,2,\ldots,r\}$ are called colors, and the function is called a coloring of X. We will write $\psi\colon X \to \{1,2,\ldots,r\}$ and we call ψ an r-coloring if we also want to specify the number r of colors.

For a set S and a positive integer m, we let $\binom{S}{m}$ denote the set of all m-element subsets of S.

For an r-coloring $\psi\colon \binom{S}{m} \to \{1,2,\ldots,r\}$, we call a subset $H \subseteq S$ monochromatic (or homogeneous) if ψ assigns every $A \in \binom{H}{m}$ to the same color. If this color is the integer i, we will say $\binom{H}{m}$ is monochromatic in color i. The fundamental question is to determine when such monochromatic sets necessarily exist. The following, now classic, theorem of F.P. Ramsey [38] is the starting point for this section.

Theorem 1: Let m and r be positive integers and let $n_1,n_2,\ldots,n_r$ be integers with $n_i \geq m$ for $i = 1,2,\ldots,r$. Then there exists an integer n_0 (whose value depends on $m,r,n_1,n_2,\ldots,n_r$) so that for every set S with $|S| \geq n_0$ and for every r-coloring $\psi\colon \binom{S}{m} \to \{1,2,\ldots,r\}$ of the set of m-element subsets of S, there exists a color i and an n_i-element monochromatic subset $H_i \subseteq S$ so that ψ assigns every m-element subset of H_i to color i, i.e., H_i is monochromatic in color i. □

We will find it convenient to utilize Ramsey theoretic results stated in terms of infinite sets. The infinite version of the preceding result is:

Theorem 1': Let m and r be positive integers and let $\psi\colon \binom{\underset{\sim}{N}}{m} \to \{1,2,\ldots,r\}$ be an r-coloring. Then there exists a color i and an infinite subset $H \subseteq \underset{\sim}{N}$ so that H is monochromatic in color i. □

Well known generalizations of this theorem have been developed by Schur [41], Rado [36], Van de Waerden [55], Folkman [9], and Hales, Jewett [16]. We refer the reader to the survey articles by Graham [11], [17] and the book by Graham, Rothschild and Spencer [15] for additional background material on the subject. For a concise discussion of the latest developments on the frontiers of Ramsey theory, we refer the reader to [13].

In this paper, we will require two well known Ramsey theoretic results. The first of these is a special case of the so called Product Ramsey theorem [15]. Let k be a positive integer and let $S_1, S_2, \ldots, S_k$ be sets. For a positive integer m, we refer to the elements of $\binom{S_1}{m} \times \binom{S_2}{m} \times \ldots \times \binom{S_k}{m}$ as <u>grids</u>.

<u>Theorem 2</u>: Let k and m be positive integers and let $S_1, S_2, \ldots, S_k$ be infinite subsets of N, the set of positive integers. Then let $\psi: \binom{S_1}{m} \times \binom{S_1}{m} \times \ldots \times \binom{S_1}{m} \to \{1,2,\ldots,r\}$ be an r-coloring of grids. Then for each $j = 1,2,\ldots,k$, there exists an infinite subset $H_j \subseteq S_j$ and a color i so that ψ assigns every grid in $\binom{H_1}{m} \times \binom{H_2}{m} \times \ldots \times \binom{H_k}{m}$ to color i. □

Let S be a set and let s be a positive integer. Then we denote by $\Pi(S,s)$ the set of all partitions of the form $S = S_1 \cup S_2 \cup \ldots \cup S_s$ with $S_i \neq \phi$ for $i = 1,2,\ldots,s$. We write $P:[S = S_1 \cup S_2 \cup \ldots \cup S_s]$ to indicate the precise form of partition $P \in \Pi(S,s)$. The sets $S_1, S_2, \ldots, S_s$ are called the <u>parts</u> of the partition P. Also, we say P is a <u>partition of S into s parts</u>. A partition differs from a coloring in that the parts of the partition are not ordered.

When $|S| = n$, we let $\Pi_n(S) = \bigcup_{s=1}^{n} \Pi(S,s)$. A natural partial order on $\Pi_n(S)$ is defined by setting $P_1 \le P_2$ when every part of P_2 is the union of one or more of the parts of the partition P_1.

The next theorem is a very natural extension of Ramsey's theorem to partitions and was first proved by Rothschild [40]. We encourage the reader to investigate the important paper of Graham and Rothschild [14], in which the concept of an n-parameter set is introduced. Many of the well known Ramsey theoretic results follow as easy corollaries to the principal result of [14].

<u>Theorem 3</u>: Let s and t be positive integers with $s \le t$. Then there exists an integer k_0 so that if $|S| = k \ge k_0$ and $\psi: \Pi(S,s) \to \{1,2,\ldots,r\}$ is any r-coloring of the partitions of S into s parts, then there exists a partition $P_0 \in \Pi(S,t)$ and a color i so that ψ assigns every partition $P \in \Pi(S,s)$ with $P_0 \le P$ to color i. □

Many readers may be more familiar with the "finite unions" theorem [9] which follows as an easy corollary to the preceding Ramsey theorem for partitions. For a set S, let $\mathcal{P}(S)$ denote the set of nonempty subsets. If $\mathcal{F} = \{F_1, F_2, \ldots, F_t\}$ is a family of t pairwise disjoint sets of $\mathcal{P}(S)$, we call a set $A \in \mathcal{P}(S)$ a <u>finite union</u> of $\mathcal{F}$ if there is a nonempty subset $B \subseteq \{1,2,\ldots,t\}$ so that $A = \bigcup_{i \in B} F_i$.

<u>Theorem 4 [9]</u>: For every pair r,t of positive integers, there exists an integer so that if S is any set with $|S| \ge k$ and $\psi: \mathcal{P}(S) \to \{1,2,\ldots,r\}$ is any r-coloring of $\mathcal{P}(S)$, then there exists a family $\mathcal{F} = \{F_1, F_2, \ldots, F_t\}$ of t pairwise disjoint sets in $\mathcal{P}(S)$ and a color i so that ψ assigns all finite unions of $\mathcal{F}$ to color i. □

3. <u>Ramsey Theory for Partially Ordered Sets</u>.

The <u>width</u> of a poset is the maximum size of an antichain. The classical theorem of R.P. Dilworth [6] asserts that if the width is finite, there exists a partition of the poset into width number of chains.

<u>Theorem 5</u>: If (X,P) is a poset of finite width n, then (X,P) can be partitioned into n chains. □

An immediate consequence of this theorem is the following Ramsey theoretic result (It also follows from the dual version of Dilworth's theorem - an elementary result.)

<u>Theorem 6</u>: If (X,P) is a poset and $|X| \ge (n_1-1)(n_2-1) + 1$, then (X,P) contains an n_1-element antichain or an n_2-element chain. □

It follows immediately from Theorem 1 that for every pair n_1, n_2 of positive integers there exists a number n_0 so that if (X,P) is any poset with $|X| \ge n_0$, then (X,P) contains an n_1-element chain or an n_2-element antichain. To see this we need only observe that the partial order P determines a 2-coloring of $\binom{X}{2}$: Assign a pair (x,y) to color i if (x,y) is a comparable pair and to color 2 when it is an incomparable pair. Of course, Theorem 6 allows the surprising conclusion that $n_0 = (n_1-1)(n_2-1) + 1$ works. This value is considerably smaller than what would be guaranteed by appealing only to Theorem 1.

Another topic which yields Ramsey theoretic results for partially ordered sets is the study of regressions. Let (X,P) be a poset. Then a map $f: X \to X$ is called a <u>regression</u> if $f(x) \le x$ in P for every $x \in X$. Note that a regression need not be order preserving. On the other hand, it is of interest to investigate conditions which force a regression to be order preserving on some chain of specified size. A chain $C = \{x_1 < x_2 < \ldots < x_k\}$ is called a monotonic k-chain if $f(x_1) \le f(x_2) \le \ldots \le f(x_k)$ in P.

It is straightforward to conclude from Theorem 1 that for every

w, k, there exists a number n_0 so that if (X,P) is a poset of width at most w and $|X| \geq n_0$, then for every regression f on X, there exists a monotonic k-chain. As was the case with Theorem 5, the combinatorial elegance of partially ordered sets allows a precise determination of n_0 and its value is considerably smaller than the one guaranteed by appealing only to Ramsey theory. The result is due to Peck, Schor, Trotter, and West [56].

Theorem 7: Let w and k be positive integers and let (X,P) be any poset with $|X| \geq (w+1)^{k-1}$ and width at most w. Then every regression on (X,P) has a monotonic k-chain. □

In order to force long monotonic chains, it is of course necessary that a poset have long chains. Furthermore, either the width must be bounded or we must impose some additional conditions on the structure of the poset such as is the case in the following theorems.

Theorem 8: (Rado [37] Harzheim [17], [18]): For every k, there exists an integer n_0 so that $n \geq n_0$ and if f is any regression on the Boolean algebra $\underline{2}^n$, then f has a monotonic k-chain □

The following result was announced by K. Leeb at the 1982 Conference on Ordered Sets in Banff, Canada.

Theorem 9 (Leeb [26]): For every k, there exists an integer n_0 so that if $n \geq n_0$, $S = \{1,2,\ldots,n\}$, and f is any regression on the partition lattice $\Pi_n(S)$, then f has a monotonic k-chain. □

In another direction, it is of interest to investigate Ramsey theoretic theorems where there are close analogies between the result for graphs and partially ordered sets. For example, consider the following two theorems of Rival and Sands [39].

Theorem 10: Every infinite graph $G = (V,E)$ contains an infinite subset $S \subseteq V$ so that every vertex of G is adjacent to precisely none, one, or infinitely many of the elements of S. Furthermore, every vertex of S is adjacent to none or infinitely many of the vertices of S. □

Theorem 11: Every infinite poset (X,P) of finite width contains an infinite chain C so that every point of P is comparable with none or infinitely many of the points in C. Moreover, if X is countable, then C can be chosen so that every point in X is either comparable with none of points in C or is comparable with all but finitely many elements of C.

In comparing these two results, the reader should note the slightly stronger form of Theorem 11 which requires the special structure of posets.

4. The Nesetřil-Rödl Theorems

In this section, we present the elegant theory developed by Nesetřil and Rödl in a series of important papers [31], [32], [33], and [34]. For nonspecialists, these papers can be overwhelming and our primary goal for this section will be to present straightforward combinatorial proofs of the major results of this theory. Some notation will be required.

Let $\underset{\sim}{X} = (X,P)$ and $\underset{\sim}{Y} = (Y,Q)$ be posets. We denote by $\binom{\underset{\sim}{X}}{\underset{\sim}{Y}}$ the set of all subposets of $\underset{\sim}{X}$ which are isomorphic to $\underset{\sim}{Y}$. A poset $\underset{\sim}{Y}$ is called a Ramsey poset if for every $r \geq 2$ and every poset $\underset{\sim}{X}$, there exists a poset $\underset{\sim}{Z}$ so that for every r-coloring ψ of $\binom{\underset{\sim}{Z}}{\underset{\sim}{Y}}$, there exists a subposet $\underset{\sim}{X}' \subseteq \underset{\sim}{Z}$ with $\underset{\sim}{X}'$ isomorphic to $\underset{\sim}{X}$ so that ψ sends every poset in $\binom{\underset{\sim}{X}'}{\underset{\sim}{Y}}$ to the same color.

A finite poset $\underset{\sim}{Y} = (Y,Q)$ is called a weak order [42] if there exists a function $f\colon Y \to \underset{\sim}{R}$ so that $y_1 < y_2$ in Q if and only if $f(y_1) < f(y_2)$ in $\underset{\sim}{R}$. Equivalently [42], $\underset{\sim}{Y}$ is a weak order if and only if $\underset{\sim}{Y}$ does not contain a subposet isomorphic to $\underline{2} + \underline{1}$. Also, a weak order is the ordinal sum of antichains.

Let $\underset{\sim}{Y} = (Y,Q)$ and let $\mathcal{Q}$ denote the set of all linear extensions of Q. Define an equivalence relation $\sim$ on $\mathcal{Q}$ by setting $M_1 \sim M_2$ exactly when there exists an isomorphism $g\colon \underset{\sim}{Y} \to \underset{\sim}{Y}$ so so that $y_1 < y_2$ in M_1 if and only if $g(y_1) < g(y_2)$ in M_2. It is straightforward to verify that $\sim$ is an equivalence relation on $\mathcal{Q}$. The following result is an easy exercise.

Lemma 12: A poset $\underset{\sim}{Y} = (Y,Q)$ is a weak order if and only if $M_1 \sim M_2$ for every pair of linear extensions of Q. □

Let $\underset{\sim}{Y} = (Y,Q)$ and $\underset{\sim}{Y}' = (Y',Q')$ be isomorphic posets. If L and L' are linear extensions of Q and Q' respectively, we extend the preceding definition by defining $L \sim L'$ when there exists an isomorphism $f\colon Y \to Y'$ so that $x \leq y$ in L if and only if $f(x) \leq f(y)$ in L'. With this terminology behind us, we can now state the principal result of the Nesetřil-Rödl theory.

Theorem 13: A poset $\underset{\sim}{Y}$ is a Ramsey poset if and only if it is a weak order. □

Before giving the proof of Theorem 13, we must develop some additional background material. First, let $\underset{\sim}{X} = (X,P)$ be a poset and let $\mathcal{P}$ denote the set of all linear extensions of P. If $L \in \mathcal{P}$ and $L\colon [x_1 < x_2 < \ldots < x_n]$ then we can use L to linearly order $\mathcal{P}$ lexicographically. To be more precise, if $|\mathcal{P}| = t$, we can label the linear extensions in $\mathcal{P}$ as $L_1, L_2, \ldots, L_t$ with $L_i\colon [x_{i_1} < x_{i_2} < \ldots < x_{i_n}]$ so that whenever $1 \leq i < j \leq t$ and α is the least integer for which $x_{i\alpha} \neq x_{j\alpha}$, we always have $x_{i\alpha} < x_{j\alpha}$ in L. Note that this rule implies that $L = L_1$.

If $\underset{\sim}{Y} = (Y,Q)$ is a subposet of $\underset{\sim}{X}$ and $\mathcal{Q}$ denotes the set of all linear extensions of Q, we can use $L = L_1$ to lexicographically order Q as $M_1 < M_2 < \ldots < M_s$. Note that $M_1 = L_1(Y)$, the restriction of L_1 to Y, and the ordering on $\mathcal{Q}$ is the same as would be produced if we used M_1 to lexicographically order $\mathcal{Q}$.

Lemma 14: Let $\underset{\sim}{X} = (X,P)$ be a poset and let $\underset{\sim}{Y} = (Y,Q)$ be a subposet. Then let $\mathcal{P}$ and $\mathcal{Q}$ denote the sets of linear extensions of P and Q respectively. Let L be an arbitrary linear extension from $\mathcal{P}$ and use L to lexicographically order $\mathcal{P}$ and $\mathcal{Q}$ as $L_1 < L_2 < \ldots < L_t$ and $M_1 < M_2 < \ldots < M_s$ respectively. Define a function $f: \{1,2,\ldots,s\} \to \{1,2,\ldots,t\}$ by $f(i) = $ least j for which $L_j(Y) = M_i$. Then f is strictly increasing.

Proof: It is obvious that f is one to one, so it remains to show that f is order-preserving. To prove this, we will consider an algorithm which computes f, or more precisely computes $L_{f(i)}$. Let $M_i = [y_1 < \ldots < y_m]$ be a member of $\mathcal{Q}$, and suppose that $L_{f(i)} = [x_1 < \ldots < x_n]$. That is, $[x_1 < \ldots < x_n]$ is the first linear extension of P which restricts to $[y_1 < \ldots < y_m]$.

The key fact about lexicographic order is that to find the first sequence satisfying certain conditions, one can use a "greedy" approach. That is, first make the first entry as small as possible, then make the second entry as small as possible, and so on. Therefore we can inductively compute x_1, then x_2, then x_3, and so on.

The statement that $[x_1 < \ldots < x_n]$ is a linear extension of P is equivalent to the statement that x_k is minimal in the poset $\underset{\sim}{X} - \{x_1,x_2,\ldots,x_{k-1}\}$ for each $k = 1,2,\ldots,n$. Saying that $[x_1 < \ldots < x_n]$ has $[y_1 < \ldots < y_m]$ as a restriction is the same as saying that each x_k either does not belong to Y, or else x_k is the first element of $[y_1 < \ldots < y_m]$ that does not belong to $\{x_1,x_2,\ldots,x_{k-1}\}$.

Without further delay, here is the algorithm that computes $L_{f(i)}$, written in pidgin Pascal. More discussion will follow.

```
r: = 1;  (*y_r is the first element of M_i not yet used. *)
for k: = 1 to n do
begin (*Compute x_k. Either x_k = y_r or else x_k is not in Y.*)
      (*First find z, the best we can do with something not in Y.*)
A: = set of minimal elements of the poset X - {x_1,x_2,...,x_{k-1}};
if   A - Y ≠ ϕ then
     z: = least element of  (A-Y, L_1(A-Y));
if   A - Y ≠ ϕ  and  (r > m  or  (z,y_r) ∈ L_1)
  then   x_k: = z
  else begin
         x_k: = y_r;
         r: = r + 1
         end
end
```


A few words of explanation are in order for the else clause. If $A - Y = \phi$, then $A \subseteq Y$. Since A is not empty, it follows that y_r exists, i.e. $r \leq m$. Furthermore, $y_r \in A$. For otherwise there would be an element y_i of A such that $y_i < y_r$ in Q. Since M_i is a linear extension of Q, it follows that $i < r$. But then $y_i \in \{x_1,x_2,\ldots,x_{k-1}\}$, which contradicts the statement $y_i \in A$. Thus, since $y_r \in A$, it is permissible to let $x_k: = y_r$.

The remaining case is that $A - Y$ is nonempty and $(y_r,z) \in L_1$. As argued above, there is no element y_i of Y such that $y_i \in A$ and $y_i < y_r$ in Q. Consequently, if $y_r \notin A$, then there is some element w in $A - Y$ such that $w < y_r$ in P. Since L_1 is a linear extension of P, $w < y_r$ in L_1. But this contradicts the choice of z. Therefore $y_r \in A$.

Now let us use the algorithm above to show that f is order-preserving. Suppose that $M, M' \in \mathcal{Q}$ and M precedes M' lexicographically. Let j be the first place in which M and M' differ. Thus $M = [y_1 < y_2 < \ldots < y_m]$, $M' = [y_1 < \ldots < y_{j-1} < y'_j < \ldots < y'_m]$, and $y_j < y'_j$ in L_1. Imagine two instances of the algorithm above computing $[x_1 < \ldots < x_n]$, the first extension containing M, and $[x'_1 < \ldots < x'_n]$, the first extension of P containing M', in parallel. So long as $r < j$, the two computations will clearly proceed identically. Consider what happens when $r = j$. As long as $A - Y \neq \phi$ and $(z,y_j) \in L_1$, z will be used as both x_k and x'_k. But this can happen only finitely many times. If $A - Y$ is empty, then we have $x_k = y_j$ and $x'_k = y'_j$, so $x_k < x'_k$ in L_1. If $A - Y$ is nonempty and $(y_j,z) \in L_1$, then $x_k = y_j$ and x'_k is either z or y'_j, so again $x_k < x'_k$ in L_1. Thus, if k is the first place in which $[x_1 < \ldots < x_n]$ and $[x'_1 < \ldots < x'_n]$ differ, then $x_k < x'_k$ in L_1. This proves that f is order-preserving. □

At first glance, the preceding lemma may seem hopelessly technical, but it will play a vital role in the proof of Theorem 13, and we trust that its value will then be clear. We now present the proof of the positive part of Theorem 13.

Lemma 15: If $\underset{\sim}{Y} = (Y,Q)$ is a weak order, then $\underset{\sim}{Y}$ is a Ramsey poset.

Proof: Let $\underset{\sim}{X} = (X,P)$ be an arbitrary poset and let r be a positive integer. We show that there exists a poset $\underset{\sim}{Z} = (Z,R)$ so that for every r-coloring of the subposets of $\underset{\sim}{Z}$ which are isomorphic to $\underset{\sim}{Y}$, there exists a subposet $\underset{\sim}{X}'$ of $\underset{\sim}{Z}$ so that $\underset{\sim}{X}' \cong \underset{\sim}{X}$ and all copies of $\underset{\sim}{Y}$ in $\underset{\sim}{X}'$ are assigned to the same color.

Let $\mathcal{P}$ and $\mathcal{Q}$ denote the sets of linear extensions of P and Q respectively. Then let $L \in \mathcal{P}$ and $M \in \mathcal{Q}$. Use L and M to linearly order $\mathcal{P}$ and $\mathcal{Q}$ as $L_1 < L_2 < \ldots < L_t$ and $M_1 < M_2 < \ldots < M_s$ respectively. Let $|X| = n$ and label the points in X so that $L_1: [x_1 < x_2 < \ldots < x_n]$. Similarly, let $m = |Y|$ and label the points in Y so that $M_1: [y_1 < y_2 < \dots < y_m]$. For each $j = 1,2,\dots,s$, let $M_j: [y_{j1} < y_{j2} < \dots < y_{jm}]$. Note that $y_{1\gamma} = y_\gamma$ for each $\gamma = 1,2,\dots,m$.

Next, we apply the Ramsey theorem for partitions to choose a positive integer $k = k_0$ so that if S is any set with $|S| = k$ and $\psi': \Pi(S,s) \to \{1,2,\dots,r\}$ is any r-coloring of the partitions of S into s parts, then there exists a partition $P_0 \in \Pi(S,t)$ and a color δ so that ψ' assigns every $P \in \Pi(S,s)$ with $P_0 \le P$ to color δ.

In order to complete the argument, it suffices to show that we may choose $Z = N^k$. (To simplify the presentation of the argument, we take Z to be an infinite poset. Of course, we can actually choose Z as $\underline{p}^k$ where p is a sufficiently large integer.) Let $\psi: \binom{Z}{Y} \to \{1,2,\dots r\}$ be any r-coloring of the copies of Y in Z. We now proceed to show that there exists a subposet $X' \subseteq Z$ and a color γ so that $X' \cong X$ and every copy of Y in X' is assigned color γ by ψ.

Let $S = \{1,2,\dots,k\}$ and let $\ell = |\Pi(S,s)|$. Label $\Pi(S,s)$ arbitrarily as $P_1,P_2,\dots,P_\ell$. Then set $Z_1 = Z = N^k$ and suppose that for some α with $1 \le \alpha \le \ell$ we have defined Z_α with Z_α isomorphic to N^k. We use the partition P_α to determine a subposet $Z_{\alpha+1}$ of Z_α with $Z_{\alpha+1}$ also isomorphic to N^k. At the same time, we will inductively define an r-coloring $\psi': \Pi(S,s) \to \{1,2,\dots,r\}$.

Let $S_1 = S_2 = \dots = S_k = N$ so that $Z_\alpha = S_1 \times S_2 \times \dots \times S_k$. For each $\beta = 1,2,\dots,k$, let $T_\beta = \{t_{1\beta} < t_{2\beta} < \dots < t_{m\beta}\}$ be an m-element subset of S_β. With the grid $G = T_1 \times T_2 \times \dots \times T_k$ and the partition P_α, we associate a subposet $Y(G,\alpha)$ of Z_α with $Y(G,\alpha)$ isomorphic to Y. This is accomplished as follows. Since P_α is a partition of $\{1,2,\dots,k\}$ into s parts, we can label these parts as $B_{\alpha 1}, B_{\alpha 2},\dots,B_{\alpha s}$ so that the least integer in $B_{\alpha i}$ is less than the least integer in $B_{\alpha j}$ when $1 \le i < j \le s$. Label the points in $Y(G,\alpha)$ as $\{(y'_{i1}, y'_{i2},\dots,y'_{ik}): 1 \le i \le m\}$. For each $\beta = 1,2,\dots,k$ and each $i = 1,2,\dots,m$, $y'_{i\beta}$ is given by: $y'_{i\beta} = t_{\gamma\beta}$ if and only if $\beta \in B_{\alpha j}$ and $y_i = y_{j\gamma}$. With this definition, it follows easily that $Y(G,\alpha)$ is isomorphic to Y by the map $y_i \to (y'_{i1}, y'_{i2},\dots,y'_{ik})$. To see this, we simply observe that for each $\beta = 1,2,\dots,k$, the βth coordinates linearly order the points in $Y(G,\alpha)$ in the same order as M_j orders the corresponding points in Y where $B_{\alpha j}$ is the part in the partition P_α containing the integer β.

We can now define an r-coloring $\psi_\alpha: \binom{S_1}{m} \times \binom{S_2}{m} \times \binom{S_k}{m} \to \{1,2,\dots,r\}$ of the grids in $\binom{S_1}{m} \times \binom{S_2}{m} \times \dots \times \binom{S_k}{m}$ by the rule $\psi_\alpha(G) = \psi(Y(G,\alpha))$. Next, we apply the Product Ramsey theorem. For each $i = 1,2,\dots,s$, choose an infinite subset $H_i \subseteq S_i$ and a color γ, so that ψ_α assigns every grid in $\binom{H_1}{m} \times \binom{H_2}{m} \times \dots$

$\times \binom{H_k}{m}$ to color γ. We set $\psi'(P_\alpha) = \gamma$ and $Z_{\alpha+1} = H_1 \times H_2 \times \dots \times H_k$. Observe that $Z_{\alpha+1} \cong N^k$ as was desired.

We repeat this process inductively until the poset $Z_{\ell+1}$ and the r-coloring ψ' of $\Pi(S,s)$ have been obtained. Next, we apply the Ramsey theorem for partitions and conclude that there exists a partition $P_0 \in \Pi(S,t)$ and a color δ_0 so that ψ' assigns every partition $P \in \Pi(S,s)$ with $P_0 \le P$ to color δ_0. We label the parts of P_0 as $A_1,A_2,\dots,A_t$ so that whenever $1 \le i < j \le t$, the least integer in A_i is less than the least integer in A_j.

Let $Z_{\ell+1} = S_1 \times S_2 \times \dots \times S_k$. For each $\beta = 1,2,\dots,k$, choose an n-element subset $H_i = \{h_{1\beta} < h_{2\beta} < \dots < h_{n\beta}\} \subseteq S_i$. Define a subposet $X' \subseteq H_1 \times H_2 \times \dots \times H_k$ with $X' = \{(x'_{i1}, x'_{i2},\dots, x'_{ik}): 1 \le i \le n\}$ by the rule: $x'_{i\beta} = h_{\gamma\beta}$ if and only if $\beta \in A_j$ and $x_i = x_{j\gamma}$. As was the case before, it follows easily that the mapping $x_i \to (x'_{i1}, x'_{i2},\dots,x'_{ik})$ is an isomorphism from X to X'. This statement follows from the observation that for each $\beta = 1,2,\dots,k$, the βth coordinates linearly order the points in X' in the same order as L_j orders the corresponding points in X, where A_j is the part in the partition P_0 containing the integer β.

To complete the argument, it remains only to show that every copy of Y in X' is assigned to color δ_0 by the r-coloring ψ. Let Y'' be any subposet of X' with Y' isomorphic to Y. Label the points in Y'' as $y''_1, y''_2,\dots, y''_m$ so that the map $y_i \to y''_i$ is an isomorphism. Order the s linear extensions of Y'' lexicographically as $M''_1, M''_2,\dots,M''_s$, using $M''_1: [y''_1 < y''_2 < \dots < y''_m]$. Note that for all i_1, i_2, j with $1 \le i_1 < i_2 \le m$ and $1 \le j \le s$, we have $y_{i_1} < y_{i_2}$ in M_j if and only if $y''_{i_1} < y''_{i_2}$ in M''_j.

We next determine a partition P of $s = \{1,2,\dots,k\}$ into s parts (actually, P depends on Y'') by the rule: $\beta \in B_j$ if the βth coordinates order the points in Y'' in the same order as M''_j. Of course, we have $P_0 \le P$ so that ψ' assigns color δ_0 to the partition P. Now we appeal to the technical lemma preceding this theorem to conclude that if $1 \le j_1 < j_2 \le s$, the least integer in B_{j_1} is less than the least integer in B_{j_2}. To see this, we choose an arbitrary pair j_1, j_2 with $1 \le j_1 < j_2 < s$ and let β_1 and β_2 be the least integer in B_{j_1} and B_{j_2} respectively. In the partition P_0, let β_1 and β_2 belong to A_{γ_1} and A_{γ_2} respectively. Then β_1 is the least integer in A_{γ_1} and β_2 is the least integer in A_{γ_2}. Note that A_{γ_1} consists of those β for which the β^{th} coordinates order the points in X' in the same order as L_{γ_1} orders the corresponding points in X. It follows from Lemma 14 that $\gamma_1 < \gamma_2$ and thus the least integer in A_{γ_1} is less than the least integer in A_{γ_2}. But clearly for each $i = 1,2$, the least integer in A_{γ_i} is the least integer in B_{j_i}

and the desired conclusion has been reached.

Now choose the unique integer α with $1 \le \alpha \le \ell$ for which $P = P_\alpha$. It follows that $Y'' = Y(G,\alpha)$ for the grid $G = T_1 \times T_2 \times \dots \times T_k$ where for each $\beta = 1,2,\dots,k$, T_β is the m-element subset of H_β formed by the βth coordinates of the points in Y''. It follows that $\psi(Y'') = \psi(Y(G,\alpha)) = \psi_\alpha(G) = \psi'(P_\alpha) = \delta_0$, i.e. ψ assigns the subposet Y'' to color δ_0. Since Y'' was arbitrary, the proof of the claim is complete. □

To show that a poset Y which is not a weak order is not a Ramsey poset requires the development of an intermediate theorem of independent interest. However, the proof technique is amazingly similar to the preceding result.

<u>Theorem 16</u>: Let $Y = (Y,Q)$ be a poset and let M be an arbitrary linear extension of Q. Then there exists a poset $X = (X,P)$ so that if L is an arbitrary linear extension of P, then there exists a subposet $Y' = (Y',Q')$ of (X,P) and an isomorphism $f: Y \cong Y'$ so that $y_1 < y_2$ in M if and only if $f(y_1) < f(y_2)$ in $L(Y')$.

<u>Proof</u>: We assume without loss of generality that $\dim(Y) = s \ge 3$. Otherwise embed Y as a subposet of a poset of larger dimension. Let $s = \dim(Y)$ and then let $\{M_1,M_2,\dots,M_s\}$ be a realizer of Q with $M = M_1$. (Note: this collection need only be a realizer of Q. It is not necessarily a list of all linear extensions of Q.) We label Y so that M_1: $[y_1 < y_2 < \dots < y_m]$. For each $i = 1,2,\dots,s$, we also let M_i: $[y_{i1} < y_{i2} < \dots < y_{im}]$.

Next, we apply the Ramsey theorem for partitions with $t=s$, $s=2$, and $r=2$, i.e., we choose an integer k so that if S is any set with $|S| \ge k$ and $\psi: \Pi(S,2) \to \{1,2\}$ is a 2-coloring of the partitions of S into two parts, then there exists a partition $P_0 \in \Pi(S,s)$ and a color δ so that if $P \in \Pi(S,2)$ and $P_0 \le P$, then P is assigned by ψ to color δ.

We will now proceed to show that we may take $X = N^k$. (As was the case in the previous proof, we find it convenient to use infinite posets. We can actually take $X = p^k$ where p is a sufficiently large integer.) Let L be an arbitrary linear extension of X.

Let $S = \{1,2,\dots,k\}$ and let $\ell = |\Pi(S,2)|$. Then let $P_1, P_2, \dots, P_\ell$ be any listing of the partitions in $\Pi(S,2)$. Set $X_1 = X$. Suppose that for some α with $1 \le \alpha \le \ell$, we have defined a subposet X_α of X with X_α also isomorphic to N^k. We now proceed to determine $X_{\alpha+1}$. At the same time, we inductively define a 2-coloring $\psi: \Pi(S,2) \to \{1,2\}$ of the partition of S into two parts.

For each $\beta = 1,2,\dots,k$, let $S_\beta = N$ so that $X_\alpha = S_1 \times S_2 \times \dots \times S_k$. Define a 2-coloring $\psi_\alpha: \binom{S_1}{2} \times \binom{S_1}{2} \times \dots \times \binom{S_n}{2} \to \{1,2\}$ of grids as follows.

For each $\beta = 1,2,\dots,k$, choose a 2-element subset $T_\beta = \{t_{1\beta} < t_{2\beta}\}$ of S_β and let G be the grid $T_1 \times T_2 \times \dots \times T_k$. We associate with the grid G and the partition P_α a two element antichain $\{u_1,u_2\}$ of X_α by the following rules: Label the parts of P_α as A_1 and A_2 so that the least integer in A_1 is less than the least integer in A_2. Then let $u_1 = (u_{11},u_{12},\dots,u_{1k})$ and $u_2 = (u_{21}, u_{22},\dots, u_{2k})$ where

$$u_{1\beta} = t_{1\beta} \quad \text{if } \beta \in A_1$$
$$u_{1\beta} = t_2 \quad \text{if } \beta \in A_2$$
$$u_{2\beta} = t_2 \quad \text{if } \beta \in A_1$$
$$u_{2\beta} = t_1 \quad \text{if } \beta \in A_2$$

It is clear that $\{u_1,u_2\}$ is an antichain since the β^{th} coordinate of u_1 is less than the β^{th} coordinate of u_2 when $\beta \in A_1$ while the reverse statement holds when $\beta \in A_2$.

Now set $\psi_\alpha(G) = 1$ if $u_1 < u_2$ in L and $\psi_\alpha(G) = 2$ if $u_2 < u_1$ in L. Next, we apply the product Ramsey theorem to choose an infinite subset $H_\beta \subseteq S_\beta$ for each $\beta = 1,2,\dots,k$ and a color γ so that ψ_α assigns each grid in $\binom{H_1}{2} \times \binom{H_2}{2} \times \dots \times \binom{H_k}{2}$ to color γ. Set $X_{\alpha+1} = H_1 \times H_2 \times \dots \times H_k$ and $\psi(P_\alpha) = \gamma$. Note that $X_{\alpha+1}$ is isomorphic to N^k as was desired.

Repeat this process until the poset $X_{\ell+1}$ has been obtained and the 2-coloring ψ has been completely determined. We now apply the Ramsey theorem for partitions to obtain a partition P_0 of S into s parts and a color δ_0 so that ψ assigns every partition $P \in \Pi(S,2)$ with $P_0 \le P$ to color δ_0. From Ramsey theory alone, we only know that either $\delta_0 = 1$ or $\delta_0 = 2$. However, using special properties of posets, we will conclude that δ_0 must be 1.

We label the parts of P_0 as $A_1,A_2,\dots,A_s$ so that whenever $1 \le i < j \le s$, the least integer in A_i is less than the least integer in A_j.

Next, let $X_{\ell+1} = S_1 \times S_2 \times \dots \times S_k$ and for each $\beta = 1,2,\dots,k$, let $H_\beta = \{h_{1\beta} < h_{2\beta} < \dots < h_{m\beta}\}$ be an m-element subset of S_β. Define a subposet Y' of $X_{\ell+1}$ so that the point set of Y' is $\{y'_i: 1 \le i \le m\}$ with $y'_i = (y'_{i1}, y'_{i2},\dots,y'_{ik})$ where for each $i = 1,2,\dots,m$ and each $\beta = 1, 2, \dots, k$, we have $y'_{i\beta} = h_{\gamma\beta}$ if $\beta \in A_j$ and $y_i = y_{j\gamma}$, i.e., the β^{th} coordinate of y'_i occupies the same position in the m-element chain H_β that y_i occupies in the m-element chain M_j where A_j is the part of the partition P_0 containing the integer β. It follows easily that the map $y_i \to y'_i$ is an isomorphism from Y to Y'. Note that $1 \in A_1$ so that $y'_i = h_{\gamma 1}$ if and only if $y_i = y_{1\gamma}$. It follows that the linear ordering M'_1 of Y' by first coordinates is M'_1: $[y'_1 < y'_2 < \dots < y'_m]$. Recall that $M = M_1$ and that M_1: $[y_1 < y_2 < \dots < y_m]$.

Now let $\{y'_{i_1}, y'_{i_2}\}$ be a 2-element antichain in Y' where $i_1 < i_2$. Then we may associate with this antichain a partition P of S into two parts B_1 and B_2 where B_1 consists of those integers β for which the βth coordinates place y'_{i_1} less than

y'_{i_2} and B_2 consists of the remaining values of β. By definition, the least integer of B_1 (which is 1) is less than the least integer of B_2. Choose α with $1 \le \alpha \le \ell$ so that $P = P_\alpha$. Note that $P_0 \le P$. Also note that the coordinates of y'_{i_1} and y'_{i_2} determine a grid G and we may therefore conclude that ${}^1\psi_\alpha(G) = {}^2\psi(P_\alpha) = \delta_0$.

Now suppose that $\delta_0 = 1$. Then $y'_{i_1} < y'_{i_2}$ in L. It follows that the map f defined by $f(y_i) = y'^{1}_{i}$ is an isomorphism from $\mathbf{Y}$ to $\mathbf{Y}'$ and that $y_{i_1} < y_{i_2}$ in ${}^1M$ if and only if $f(y_{i_1}) < f(y_{i_2})$ in $L(\mathbf{Y}')$ and we are done. So it remains to consider the case where $\delta_0 = 2$. In this case, we define a linear extension M'_2 of $\mathbf{Y}'$ by $M'_2 = L(Y')$. Then consider the pair $\{M'_1, M'_2\}$. Note that if $y_{i_1} < y_{i_2}$ in Q, then $y'_{i_1} < y'_{i_2}$ in $M'_1 \cap {}^1M'_2$. However, if y'_{i_1} and $y'^{2}_{i_2}$ are incomparable in Q^2 and $y'_{i_1} < y'_{i_2}$ in M'_1, then $y'^{1}_{i_2} < y'_{i_1}$ in M'_2. This implies that $\{M'_1, M'^{1}_2\}$ is a realizer of $\mathbf{Y}'^2$ and thus $\dim(Y') \le 2$. The contradiction shows that $\delta_0 \ne 2$. With this observation, the proof is complete □

We are now ready to complete the proof of Theorem 13.

Lemma 17: If $\mathbf{Y} = (Y,Q)$ is not a weak order, then $\mathbf{Y}$ is not a Ramsey poset.

Proof. We assume that $\mathbf{Y}$ is a Ramsey poset and proceed to a contradiction. Since $\mathbf{Y}$ is not a weak order, there exists a pair M_1, M_2 of linear extensions so that $M_1 \not\approx M_2$. We label Y so that $M_1: [y_1 < y_2 < \dots < y_m]$. Next, let $\mathbf{Y}_1 = \mathbf{Y}_2 = \mathbf{Y}$ and let $\mathbf{Y}_3 = \mathbf{Y}_1 \oplus \mathbf{Y}_2$, the ordinal sum of $\mathbf{Y}_1$ and $\mathbf{Y}_2$, i.e., in $\mathbf{Y}_3$, each point of $\mathbf{Y}_1$ is less than each point of $\mathbf{Y}_2$. Let $\mathbf{Y}_3 \cong (Y_3, Q_3)$. Then set $M_3 = M_1 \oplus M_2$. To distinguish between the points in Y_1 and Y_2, we let $Y_i \cong \{y_{i1}, y_{i2}, \dots, y_{im}\}$ for $i = 1,2$ so that the map $y_i \to y_{ij}$ is an isomorphism and $M_3(Y_1)$: $[y_{11} < y_{12} < \dots < y_{1m}]$. We then apply Theorem 15 to the poset $\mathbf{Y}_3$ and the linear extension M_3 to obtain a poset $\mathbf{X} = (X,P)$ so that whenever L is a linear extension of P, there exists a subposet $\mathbf{Y}'_3 = (Y'_3, Q'_3) \subseteq \mathbf{X}$ and an isomorphism $f: \mathbf{Y}_3 \to \mathbf{Y}'_3$ so that $y_1 < y_2$ if and only if $f(y_1) < f(y_2)$ in $L(Y'_3)$.

Now let $\mathbf{Z} \cong (Z,R)$ be any poset. We describe a method for producing a 2-coloring $\psi: \binom{\mathbf{Z}}{\mathbf{X}} \to \{1,2\}$ so that there is no copy of $\mathbf{X}$ in $\mathbf{Z}$ in which all copies of $\mathbf{Y}$ in $\mathbf{X}$ are assigned the same color by ψ. We begin by choosing an arbitrary linear extension N of R.

Now let $\mathbf{Y}' = (Y',Q)$ be any subposet of Z with $\mathbf{Y} \cong \mathbf{Y}'$. We set $\psi(\mathbf{Y}') = 1$ if $M_1 \sim N(Y')$. If $M_1 \not\sim N(Y')$, we set $\psi(\mathbf{Y}') = 2$

Now suppose that there exists a copy $\mathbf{X}' = (X',P')$ in $\mathbf{Z}$ with $\mathbf{X} = \mathbf{X}'$ and a color $\delta_0 \in \{1,2\}$ so that ψ assigns every copy of $\mathbf{Y}$ in $\mathbf{X}'$ to δ_0. We proceed to a contradiction.

Let $L' = N(X')$. It follows that there exists a subposet $\mathbf{Y}'_3 = (y'_3, Q'_3)$ of $\mathbf{X}'$ and an isomorphism $g: \mathbf{Y}_3 \to \mathbf{Y}'$ so that $u_1 < u_2$ in M_3 if and only if $g(u_1) < g(u_2)$ in $L'(Y'_3)$. Let $M'^{1}_3 = L'(Y'_3)$ and for $i = 1,2$, let $Y'_i \cong g(Y_i)$, $M'_i \cong L'(Y'_i)$ and $\mathbf{Y}'_i = (Y'_i, L'(Y'_i))$.

Similarly, for each $i = 1,2$, label the points in Y'_i as $y'_{i1}, y'_{i2}, \dots, y'_{im}$ so that the isomorphism $g: \mathbf{Y}_3 \to \mathbf{Y}'_3$ is defined by $g(y_{ij}) = y'_{ij}$. It follows easily that $\psi(\mathbf{Y}'_1) = 1$ and $\psi(\mathbf{Y}_2) = 2$. The contradiction completes the proof. □

5. Beyond Partially Ordered Sets

In this section, we develop for graphs and lattices theorems which are analogous to those presented in the preceding section for partially ordered sets. A graph $G_1 = (V_1,E_1)$ is said to be a Ramsey graph if for every $r \le 2$ and every graph $G_2 = (V_2,E_2)$, there exists a graph $G_3 = (V_3,E_3)$ so that for every r-coloring ψ of $\binom{G_3}{G_1}$, the set of all induced copies of G_1 in G_3, there exists an induced copy G'_2 of G_2 in G_3 so that ψ assigns all copies of G_1 in G'_2 to the same color. It follows easily that a graph G is a Ramsey graph if and only if $\bar{G}$, the complement of G, is a Ramsey graph.

Theorem 18 [32]: For every $p \ge 1$, the graph I_p consisting of p vertices and no edges as well as the complete graph K_p on p vertices are Ramsey graphs.

Proof: In view of our remarks concerning complements, we need only show that I_p is a Ramsey graph.

Let $G_2 \cong (V_2,E_2)$ be an arbitrary graph. Choose a set S so that there exists an injection $f: V_2 \xrightarrow{1-1} \mathcal{P}(S)$, where $\mathcal{P}(S)$ is the set of nonempty subsets of S, so that xy is an edge in G_2 if and only if $f(x) \cap f(y) \ne 0$. In other words, G_2 is the intersection graph of a family of nonempty subsets of S. It is trivial to see that if $|V_2| = m$ and we label the vertices of G_2 as $u_1, u_2, \dots, u_m$, then we can take $|S| \le 2m$ since the map which assigns to vertex u_i the set $\{i\} \cup \{j + m: u_iu_j$ is an edge in $G_2\}$ works. Without loss of generality, we may assume f is a bijection, i.e., we may assume that every nonempty subset of S corresponds to a vertex in G_2. Now let $|S| = s$. Next, use the Ramsey Theorem for partitions to choose an integer n so that if T is a set on n elements and $\psi: \Pi(T; p + 1) \to \{1,2,\dots r\}$ is any r-coloring of the partitions of T into $p + 1$ parts, then there exists a color δ_0 and a partition $P_0 \in \Pi(T; s + 1)$ so that ψ assigns every partition $P \in \Pi(T; p + 1)$ with $P_0 \le P$ to color δ_0.

Then let $G_3 = (V_3,E_3)$ be the intersection graph determined by all nonempty sets of $W = \{2,3,\dots,n\}$. Consider a copy of I_p in G_3. This copy corresponds to p pairwise disjoint nonempty subsets $\{A_2, A_3, \ldots, A_{p+1}\}$ of $W = \{2,3,\ldots,n\}$. Let A_1 contain the integer 1 together with those integers in W which do not belong to $A_2 \cup A_3 \cup \ldots \cup A_{p+1}$. Then $\{A_1, A_2, \ldots, A_{p+1}\}$ is a partition of $\{1,2,\ldots,n\}$ into $p+1$ parts. So an arbitrary r-coloring ψ_0 of the copies of I_p in G_3 determines an r-coloring ψ of the partitions of $\{1,2,\ldots,n\}$ into $p+1$ parts. We can then choose a color δ_0 and a partition $P_0 \in \Pi(\{1,2,\ldots,n\}, s+1)$ so that ψ assign every partition $P \in \Pi(\{1,2,\ldots,n\}, p+1)$ with $P_0 \le P$ to color δ_0.

Let the parts of P_0 be $B_1, B_2, \ldots, B_{s+1}$ with $1 \in B_1$. Then let G_2' be the induced subgraph of G_3 determined by those nonempty subsets of $\{2,3,\ldots,n\}$ which are finite unions of $B_2, \ldots, B_{s+1}$. Clearly every copy of I_p in G_2' is mapped by ψ_0 to color δ_0. Furthermore, $G_2 \cong G_2'$. Thus I_p is a Ramsey graph. □

The reader may be interested in comparing this simple proof with the arguments given in [15] for the special cases $p = 1$ and $p = 2$. We will now proceed to show that Theorem 18 is the best possible i.e., if $G_1 = (V_1, E_1)$ in any graph for which neither G_1 nor $\overline{G}_1$ is a complete graph, then G_1 is not a Ramsey graph. To accomplish this, we consider acyclic (but not necessarily transitive) orientations and admissible orderings. An ordering $L: [v_1 < v_2 < \ldots < v_n]$ of the vertex set of a graph G is said to be <u>admissible</u> with respect to an acyclic orientation 0 of the edges of G if $i < j$ whenever $v_i v_j$ is an edge in G and $(v_i, v_j) \in 0$.

<u>Theorem 19 [32]</u>: Let $G_1 = (V_1, E_1)$ be a graph and let 0_1 be an acyclic orientation of the edges of G_1. Then let L_1 be an arbitrary admissible ordering of the vertices of G_1. Then there exists a graph $G_2 = (V_2, E_2)$ and an acyclic orientation 0_2 of the edges of G_2 so that for every admissible ordering L_2 of the vertices in G_2, there exists an induced subgraph $G_1' = (V_1', E_1')$ of G_2 and an isomorphism $f: V_1 \to V_1'$ so that $x < y$ in L_1 if and only if $f(x) < f(y)$ in L_2.

<u>Proof</u>: Let $L_1: [v_1 < v_2 < \ldots < v_n]$. Choose an integer s and a function

$f: V_1 \xrightarrow{1-1} P(S)$ where

$S = \{1,2,\ldots,s\}$ so that:

1. $s + 1 \ge 5$;
2. $v_i v_j$ is an edge in G_1 if and only if $f(v_i) \cap f(v_j) = \emptyset$; and
3. For each i,j with $1 \le i < j \le m$, if $v_i v_j$ is not an edge in G and we set $B_2 = f(v_i) - f(v_j)$, $B_3 = f(v_i) \cap f(v_j)$, and $B_4 = f(v_j) - f(v_i)$, then B_2, B_3, and B_4 are nonempty and the least integer in B_2 is less than the least integer in B_3 which in turn is less than the least integer in B_4.

It is easy to show that the integer s and the desired injection f exists. To see this, let $L_1: [v_1 < v_2 < \ldots < v_m]$ and let $e_1, e_2, \ldots, e_t$ be any labelling of the edges of G_1. Then for each i, let $f(v_i) = \{i, m+t+i\} \cup \{m+j: v_i$ is incident with the edge e_j in $G_1\}$.

Next, we apply the Ramsey Theorem for partitions to choose an integer n so that if T is any set with $|T| = n$ and ψ is any 2-coloring of the partition of T into four parts, then there exists a partition $P_0 \in \Pi(T, s+1)$ so that ψ assigns every $P \in \Pi(T,4)$ with $P_0 \le P$ to the same color.

Let $G_2 = (V_2, E_2)$ be the graph whose vertex set V_2 is the set of nonempty subsets of $W = \{2,3,\ldots,n\}$. Let E_2 consists of those pairs $\{A_1, A_2\}$ from $P(W)$ with $A_1 \cap A_2 = \emptyset$. Let 0_2 be the orientation of E_2 defined by $(A_1, A_2) \in 0_2$ when $A_1 \cap A_2 = \emptyset$ and the least integer in A_1 is less than the least integer in A_2. Now let L_2 be any admissible ordering of G_2. We use this ordering to determine a 2-coloring ψ of the partitions of $\{1,2,\ldots,n\}$ into four parts. Let $P \in \Pi(\{1,2,\ldots,n\},4)$; label the parts of P as B_1, B_2, B_3, B_4 so that the least integer in B_i is less than the least integer in B_j for $1 \le i < j \le 4$. Then the sets $B_2 \cup B_3$ and $B_3 \cup B_4$ are nonadjacent vertices in G_2. Set $\psi(P) = 1$ if $B_2 \cup B_3$ precedes $B_3 \cup B_4$ in L_2; if $B_3 \cup B_4$ precedes $B_2 \cup B_3$ in L_2, then set $\psi(P) = 2$.

Next, choose a color $\delta_0 \in \{1,2\}$ and a partition P_0 from $\Pi(\{1,2,\ldots,n\}, s+1)$ so that ψ assigns every $P \in \Pi(\{1,2,\ldots,n\}, 4)$ with $P_0 \le P$ to color δ_0. We show that δ_0 must be 1. For suppose $\delta_0 = 2$. Label the parts of P_0 as $A_1, A_2, \ldots, A_{s+1}$ so that the least integer in A_i is less than the least integer in A_j where $1 \le i < j \le s+1$. Note that $s + 1 > 5$. Let P' be the partition with parts $B_1' = A_1 \cup A_5 \cup A_6 \cup \ldots \cup A_{s+1}$, $B_2' = A_2$, $B_3' = A_3$, $B_4' = A_4$. Since $P_0 \le P'$, and $\psi(P') = 2$, we know that $A_3 \cup A_4$ precedes $A_2 \cup A_3$ in L_2. Let P'' be the partition with parts $B_1'' = A_1 \cup A_2 \cup A_6 \cup A_7 \cup \ldots \cup A_{s+1}$, $B_2'' = A_3$, $B_3'' = A_4$, and $B_4'' = A_5$. Since $\psi(P'') = 2$, we conclude that $A_4 \cup A_5$ precedes $A_3 \cup A_4$ in L_2. But these statements imply that $A_4 \cup A_5$ precedes $A_2 \cup A_3$ in L_2. However, $(A_4 \cup A_5) \cap (A_2 \cup A_3) = \emptyset$ and $(A_2 \cup A_3, A_4 \cup A_5) \in 0_2$. This contradicts the hypothesis that L_2 was an admissible ordering.

We conclude that the color δ_0 must be 1. The desired embedding of G_1 is now immediate; for each vertex $u_i \in V_1$, we simply take $g(u_i)$ as the union of those A_j's for which j belongs to $f(u_i)$. □

Following along lines which are quite similar to the results for partially ordered sets, it is easy to see that only the complete graphs and their complements have the property that all acyclic orientations and admissible orderings are essentially the same. From this observation, the proof of the following theorem follows immediately.

<u>Theorem 20 [32]</u>: If neither G nor $\overline{G}$ is a complete graph, then G is not a Ramsey graph. □

The role played by acyclic orientations in the preceding Theorem suggests the following definition. A pair (G_1,O_1) consisting of a graph G_1 and an acyclic orientation O_1 is Ramsey if for every $r \le 2$ and every pair (G_2,O_2), there exists a pair (G_3,O_3) so that every r-coloring of the induced copies of (G_1,O_1) in (G_3,O_3) produces a copy of (G_2,O_2) in which all copies of (G_1,O_1) receive the same color. However the following theorem follows easily from a straightforward generalization of Theorem 18.

Theorem 21 [32]: Let G be a graph and let O be any acyclic orientation of the edges of G. Then (G,O) is Ramsey. □

We now turn our attention to lattices. Recall that a finite partially ordered set (X,P) is a lattice when the following conditions are satisfied.

1. For every $x,y \in X$, the set $U(x,y) = \{z: z \ge x$ and $z \ge y$ in $P\}$ is nonempty and has a least element. This element is denoted by $x \vee y$.
2. For every $x,y \in X$, the set $D(x,y) = \{w: w \le x$ and $w \le y$ in $P\}$ is nonempty and has a greatest element. This element is denoted by $x \wedge y$.

In order to develop a Ramsey theory for lattices, it is necessary to introduce a more general (although more technical) class of structures. Following Nesetřil and Rödl [34], we call a poset (X,P) a partial lattice if it satisfies the following weaker conditions:

1' For every $x,y \in X$, if the set $U(x,y) = \{z: z \ge x$ and $z \ge y$ in $P\}$ is nonempty, then it has a least element.
2' For every $x,y \in X$, if the set $D(x,y) = \{w: w \le x$ and $w \le y$ in $P\}$ is nonempty, then it has a greatest element.

Let (X,P) be a poset and let $f: X \to \{1,2,\ldots,k\}$ be a function. We call f a leveling of (X,P) if $f(x) = f(y)$ always implies either $x = y$ or x is incomparable to y in P. Similarly, we call a triple $(\underset{\sim}{X},f,k)$ a leveled partial lattice (abbreviated LP lattice) if $\underset{\sim}{X} = (X,P)$ is a partial lattice and $f: X \to \{1,2,\ldots,k\}$ is a leveling. For each $i = 1,2,\ldots,k$, the set $A_i = f^{-1}(i)$ in an antichain in $\underset{\sim}{X}$ and we call A_i the i^{th} level of $(\underset{\sim}{X},f,k)$.

If $(\underset{\sim}{X}_1,f_1,k_1)$ and $(\underset{\sim}{X}_2,f_2,k_2)$ are LP-lattices then we say $(\underset{\sim}{X},F_1,k_1)$ is contained in $(\underset{\sim}{X}_2,f_2,k_2)$ and write $(\underset{\sim}{X}_1,f_1,k_1) \subseteq (\underset{\sim}{X}_2,f_2,k_2)$ when there exists an embedding $g: \underset{\sim}{X}_1 \to \underset{\sim}{X}_2$ so that

1. $x \le y$ in $\underset{\sim}{X}_1$ if and only if $g(x) \le g(y)$ in $\underset{\sim}{X}_2$.
2. g is cover preserving, i.e., if x covers y in $\underset{\sim}{X}_1$, then $g(x)$ covers $g(y)$ in $\underset{\sim}{X}_2$.
3. $f_1(x) = f_2(g(x))$ for every $x \in X_1$.

The first of these three conditions implies only that $\underset{\sim}{X}_1$ is isomorphic to a subposet of $\underset{\sim}{X}_2$. The second implies that g^{-1} is a lattice embedding in that it must preserve meets and joins when they exist. The third condition requires that the embedding preserve levels in that points at the same level must be mapped to points at the same level.

We can then say that a LP lattice $(\underset{\sim}{X}_1,f_1,k_1)$ is Ramsey if for every $r \ge 2$ and every LP-lattice $(\underset{\sim}{X}_2,f_2,k_2)$, there exists a LP-lattice $(\underset{\sim}{X}_3,f_3,k_3)$ so that for every r-coloring ψ of the copies of $(\underset{\sim}{X}_1,f_1,k_1)$ contained in (X_3,f_3,k_3), there exists a monochromatic copy of $(\underset{\sim}{X}_2,f_2,k_2)$.

Although the best possible theorem is not known, we can at least prove the following result.

Theorem 22 [34]: For every LP-lattice (X_1,f_1,k_1) and every $r \ge 2$, there exists an LP-lattice $(\underset{\sim}{X}_2,f_2,k_2)$ so that if ψ is an r-coloring of the points in $\underset{\sim}{X}_2$, then there exists a monochromatic copy of $\underset{\sim}{X}_1$ contained in $\underset{\sim}{X}_2$.

Before proceeding to the proof, we need some well known results from chromatic theory for hypergraphs. Recall that a t-uniform hypergraph is a pair (X,E) where X is a set and E is a family of subsets of $\underset{\sim}{X}$, called edges, with each $|E| = t$ for every edge $E \in E$. The chromatic number of the hypergraph $H = (X,E)$, denoted $\chi(H)$, is the least r for which there exists an r-coloring $\psi: X \to \{1,2,\ldots,r\}$ so that there is no monochromatic edge $E \in E$. We refer the reader to [3] for extensive background material concerning hypergraphs. Here, we will need one special result due to Erdös and Hajnal [8]. For every pair r,t of integers with $t \ge 2$, there exists a t-uniform hypergraph $H = (X,E)$ with $\chi(H) > r$ so that $|E_1 \cap E_2| \le 1$ for every $E_1,E_2 \in E$ with $E_1 \ne E_2$.

We now proceed to prove our theorem in two stages. First, we determine an LP-lattice $(\underset{\sim}{X}_3,f_3,k_3)$ with $k_3 = r(k_1-1) + 1$ which satisfies the following condition: If ψ_0 is any r-coloring of the points in $(\underset{\sim}{X}_3,f_3,k_3)$ so that $\psi_0(x) = \psi_0(y)$ whenever $f_3(x) = f_3(y)$, i.e., points at the same level receive the same color, then there exists a monochromatic copy of $(\underset{\sim}{X}_1,f_1,k_1)$ contained in $(\underset{\sim}{X}_3,f_3,k_3)$. This is easy to accomplish. We simply set $k_3 = r(k_1 - 1) + 1$ and take $(\underset{\sim}{X}_3, f_3,k_3)$ as the free sum of $\binom{k_3}{k_1}$ copies of $\underset{\sim}{X}_1$ with the leveling f_3 defined so that for each k_1 element subset $S \subseteq \{1,2,\ldots,k_3\}$, there exists a copy of $\underset{\sim}{X}_1$ whose levels are the k_1 levels in S. That $(\underset{\sim}{X}_3,f_3,k_3)$ has the desired property then follows immediately from the pigeon-hole principle.

The second part of the proof involves a more complicated construction. Given integers r,k_3,k_4 with $r \ge 2$ and $1 \le k_4 < k_3$ and an arbitrary LP-lattice $(\underset{\sim}{X}_3, f_3, k_3)$ we show that there exists an LP-lattice $(\underset{\sim}{X}_2,f_2,k_2)$ with $k_2 = k_3$ so that if ψ is an r-coloring of the points $\underset{\sim}{X}_2$ and in addition each of the first k_4 levels is monochromatic, then there exists a copy of $(\underset{\sim}{X}_3,f_3,k_3)$ contained in $(\underset{\sim}{X}_2,f_2,k_2)$ so that each of the first $k_4 + 1$ levels of this copy of $(\underset{\sim}{X}_3,f_3,k_3)$ is monochromatic. In this part of the proof, we need not be concerned about the origin of the LP-lattice $(\underset{\sim}{X}_3,f_3,k_3)$: it is arbitrary.

Without loss of generality we may assume that the $k_4 + 1^{st}$

level has $t \geq 2$ points; otherwise the conclusion follows immediately. Label the $k_4 + 1^{st}$ level as $\{a_1, a_2, \ldots, a_s\}$. Next, choose a t-uniform hypergraph $H = (V,E)$ with $\chi(H) > r^s$ and $|E_1 \cap E_2| \leq 1$ for every $E_1, E_2 \in E$ with $E_1 \neq E_2$. Let $V = \{u_1, u_2, \ldots, u_n\}$. We form the LP-lattice (X_2, f_2, k_2) as follows:

1. The $k_4 + 1^{st}$ level is an antichain of n points labelled by the vertices $u_1, u_2, \ldots, u_n$ of the hypergraph H.
2. For each $i = 1, 2, \ldots, k_4$ with $i \neq k_4$, the i^{th} level consists of the points $\{(y,E): y \in A_i, E \in E\}$.
3. The ordering on X_2 is defined by:
   a. For every $E \in E$, if $E = \{u_{i_1}, u_{i_2}, \ldots, u_{i_t}\}$ with $i_1 < i_2 < \ldots < i_t$, then $(y,E) < u_{i_j}$ in $X_2 \iff y < a_j$ in X_3 and $(y,E) > u_{i_j}$ in $X_2 \iff y > a_j$ in X_3.
   b. For every $E \in E$, $(y,E) \leq (z,E)$ in $X_2 \iff y \leq z$ in X_3.
   c. If $E_1, E_2 \in E$ and $E_1 \neq E_2$, then $(y,E_1) < (z,E_2) \iff$ there is vertex $x = u_i \in E_1 \cap E_2$ so that $(y,E_1) < x$ and $x < (z,E_2)$ are required by a.

From this definition, it follows easily that (X_2, f_2, k_2) is an LP-lattice. The desired coloring property follows easily from the fact that the chromatic number of H exceeds r, and thus any r-coloring of the points in X_2 will produce a monochromatic edge $E \in E$.

By combining these two results and inducting on the second, the proof of our theorem is easily obtained. □

It is no small surprise to see that the number of levels appearing in the preceeding theorem is the absolute minimum required by the pigeon-hole principle. Specifically, if (X_1, f_1, k_1) and (X_2, f_2, k_2) are LP-lattices and $k_2 < r(k_1 - 1) + 1$, then it is easy to produce an r-coloring of the points in X_2 so that there will not be a monochromatic copy of (X_1, f_1, k_1) contained in (X_2, f_2, k_2). Simply color all points in the first $k_1 - 1$ levels of X_2 with color 1, all points in the next $k_1 - 1$ levels of X_2 with color 2, etc. The surprise here is that the required number of levels is so small.

Let us pause to consider the details of the proof of Theorem 22 to see what happens if we restrict our attention to leveled lattices. The only problem is a trivial one, namely modifying the constructions so as to insure that we always have a leveled lattice. However, this difficulty is easy to overcome since we need only add a new 0 and a new 1 during each step of the construction. (Of course, this destroys the simple bound on the number of levels.) We then have the following surprising result.

Theorem 23 [34]: For every $r \geq 2$ and every lattice L_1, there exists a lattice L_2 so that for every r-coloring of the elements of L_2, there exists a cover preserving embedding of L_1 into a monochromatic sublattice of L_2. □

Applying the preceding theorem to a lattice $L_1 = 2$ yields the following curious corollary (also proven by Bollobás) answering a question posed by Rival and Sands (see [34]).

Theorem 24 [34]: For every r, there exists a lattice L so that the Hasse diagram of L has chromatic number exceeding r. □

6. Open Problems in Ramsey Theory for Posets

We close the first part of this paper with a brief list of open problems in Ramsey theory for posets.

1. Does there exist a function $f(n,r)$ so that if L_1 is a distributive lattice of length n, then there exists a distributive lattice L_2 of length $f(n,r)$ so that for every r-coloring of the elements of L_2, there exists a cover preserving embedding of L_1 into a monochromatic sublattice of L_2?
2. Let $g(n,r)$ be the least integer so that for every poset $X_1 = (X_1, P_1)$ with $|X_1| = n$, there exists a poset $X_2 = (X_2, P_2)$ with $|X_2| = g(n,r)$ so that for every r-coloring of the points of X_2, there exists an embedding of X_1 into a monochromatic subposet of X_2. It is easy to see that $r(n-1) + 1 \leq g(n,r) \leq n^r$. Determine $g(n,r)$.
3. Let $h(n,r)$ be the least integer so that for every poset $X_1 = (X_1, P_1)$ of width n, there exists a poset $X_2 = (X_2, P_2)$ of width $h(n,r)$ so that for every r-coloring of the points of X_2, there is an embedding of X_1 into a monochromatic subposet X_2. It is easy to see that $r(n-1) + 1 \leq h(n,r) \leq n^r$. Determine $h(n,r)$.

PART II: IRREDUCIBLE POSETS AND CHROMATIC GRAPH THEORY

1. Irreducible Posets - Some Historical Perspectives.

It follows immediately from the definition of dimension for a poset (X,P) that if (Y,Q) is a subposet, then $\dim(Y,Q) \leq \dim(X,P)$. We then say (X,P) is irreducible if there is an integer $t \geq 2$ so that $\dim(X,P) = t$ but $\dim(Y,Q) < t$ for every proper nonempty subposet (Y,Q). In order to specify the value of t, we will also say (X,P) is t-irreducible.

For a poset (X,P) and a point $x \in X$, an elementary inequality of Hiraguchi [19] asserts that $\dim(X,P) \leq 1 + \dim(X - \{x\}, P(X - \{x\}))$. Thus a poset (X,P) is t-irreducible if $\dim(X,P) = t$ and $\dim(X - \{x\}, P(X - \{x\})) = t-1$ for every $x \in X$. Trivially, the only 2-irreducible poset is a two element antichain. In sharp contrast, the problem of determining all 3-irreducible posets is quite difficult as the list [21], [52] contains nine infinite families and 18 "odd" examples.

The study of "critical" combinatorial structures is a common theme in combinatorial mathematics. It is a natural topic to be investigated whenever there is an integer valued parameter defined on a class of structures satisfying the property that it is monotonically decreasing on substructures. Among the most popular instances of this phenomena in graph theory involve the parameters: chromatic number and genus. For example, N. Robertson and P. Seymour have just announced that for each $k \geq 1$, there are only finitely many graphs (up to homeomorphs) which have genus k and are critical with respect to this property. This result is a sweeping generalization of the well known theorem of Kuratowski characterizing planar graphs in terms of two forbidden subgraphs K_5 and $K_{3,3}$.

Irreducible posets have played an important role in the development of dimension theory for partially ordered sets. At times this role has been enlightening but curiously, in some instances, some misleading trails have emerged from properties of irreducible posets discovered in preliminary work. In retrospect, it may have been only the lack of knowledge about the more subtle aspects of posets which is to blame. Here are some examples.

In 1950, T. Hiraguchi [20] proved that if $|X| \geq 4$, then $\dim(X,P) \leq [|X|/2]$. On the other hand, this inequality is sharp since for each $n \geq 1$, the poset $\underset{\sim}{S}_n$ determined by the family of all 1-element and n-1-element subsets of an n-element set ordered by inclusion is an n-irreducible poset having 2n points. $\underset{\sim}{S}_n$ is called the "standard" example of an n-irreducible poset. For integers n,k with $n \geq 3$ and $k \geq 0$, Trotter [45] defined the (generalized) <u>crown</u> $\underset{\sim}{S}^k_n$ as a poset of height one with maximal elements $a_1, a_2, \ldots, a_{n+k}$, minimal elements $b_1, b_2, \ldots, b_{n+k}$ and ordering defined by $b_i I \{a_i, a_{i+1}, \ldots, a_{i+k}\}$ (cyclically) and $b_i < a_j$ for all other a_j. When $k = 0$, the crown $\underset{\sim}{S}^0_n$ is isomorphic to the standard example $\underset{\sim}{S}_n$. The family $\{\underset{\sim}{S}^k_3 : k \geq 0\}$ was introduced by Baker, Fishburn and Roberts [2] who first used the term "crown" to describe this family of 3-irreducible posets. In a certain sense, the posets in $\{\underset{\sim}{S}^k_3 : k \geq 0\}$ play a role in dimension theory for posets which is somewhat analogous to the role played by odd cycles in chromatic graph theory.

Among a group of researchers investigating questions in dimension theory at the 1971 summer conference at Bowdoin College, we recall that several (including the second author of this paper) naively believed that $\underset{\sim}{S}^k_n$ was always an n-dimensional poset, and it came as a small surprise to discover that $\dim \underset{\sim}{S}^2_4$ was in fact three. Motivated by this small example, Trotter [45] developed the general formula $\dim(\underset{\sim}{S}^k_n) = \lceil 2(n+k)/(k+2) \rceil$ which shows that for fixed n, the value of $\dim \underset{\sim}{S}^k_n$ decreases monotonically to 3 as k increases.

In 1971, only a few examples of 3-irreducible posets which were not crowns in $\{\underset{\sim}{S}^k_3 : k \geq 0\}$ were known. One was the six point "chevron" appearing in Hiraguchi's paper [20]. A seven point 3-irreducible poset appeared in Baker, Fishburn, and Roberts [2]. During this summer conference, several researchers began to investigate the problem of determining a forbidden subposet characterization of Hiraguchi's inequality. In fact, Bogart and Trotter [4] completed

the even case during the Bowdoin conference by showing that if $n \geq 3$ and $|X| = 2n$, then $\dim(X,P) < n$ unless (X,P) is the standard example $\underset{\sim}{S}^0_n$, or $n = 3$ and (X,P) is Hiraguchi's chevron or its dual. The preliminary work on the odd case produced a large number of 3-irreducible posets on seven points. Using ad-hoc methods, Trotter determined all 3-irreducible posets on at most seven points and circulated a chart showing the examples.

In 1973, R. Kimble [25] completed the characterization of Hiraguchi's inequality by proving that if $n \geq 4$ and $|X| \leq 2n+1$, then $\dim(X,P) < n$ unless (X,P) contains the standard example $\underset{\sim}{S}^0_n$. The statement for $n = 3$ is much more complicated because of the presence of a large number of 3-irreducible posets on 6 or 7 points. The pattern beginning to emerge from these results was that questions involving 3-irreducible posets were complicated by the occurence of pathology, but for $t \geq 4$, questions involving t-irreducible posets yielded cleaner and more elegant solutions.

This viewpoint was subsequently reinforced by other developments. In 1973, Trotter [47] and Kimble [25] independently discovered a short proof of Hiraguchi's inequality based on the following result.

<u>Theorem 1</u>: Let A be an antichain in a poset (X,P). If $|X - A| = n \geq 2$, then $\dim(X,P) \leq n$. □

In 1975, Trotter [48] published a forbidden subposet characterization of the inequality in Theorem 1. The case $n = 2$ is trivial. Once again, the case $n = 3$ is complicated since the chevron and several of the seven point 3-irreducible posets consist of an antichain and three other points. Some of these posets have the property that some of the points in $X - A$ are on one side of A and other points are on the opposite side of A. However, for $n \geq 4$, the characterization becomes elegant as there is a family F_n containing $2n+1$ posets, each one being an n-irreducible poset containing a maximum antichain A so that the poset obtained by deleting A is a maximal antichain B of n points all on the same side of A. The posets in F_n all have the same basic structure and are describable by a very simple rule. This is not the case when $n = 3$.

Hiraguchi's inequality continues to serve as a source of interesting research problems in dimension theory. Most of this research is centered on attempts to answer the following conjecture, which if true would yield a stronger result than the original inequality, (see [24]).

<u>Conjecture</u>: If (X,P) is a poset and $|X| \geq 3$, then there exists a distinct pair $\{x,y\}$ of points in X so that $\dim(X,P) \leq 1 + \dim(X - \{x,y\}, P(X - \{x,y\}))$ □

In [49] Trotter constructed several infinite families of irreducible posets. These constructions explain in part the difficulty of the conjecture since they possess properties which if excluded yield conditions under which the conjecture is known to hold. One such condition is given by the following elementary result.

Theorem 2 [19]: If (X,P) is a poset, x is a maximal element, y is a minimal element, and x is incomparable with y in P, then $\dim(X,P) \leq 1 + \dim(X - \{x,y\}, P - \{x,y\})$. □

It is therefore natural to ask whether for every $n \geq 3$, there exists an n-irreducible poset in which every maximal element is greater than every minimal element. The answer is yes for $n = 3$ as evidenced by the chevron. But the constructions in [49] show that the answer when $n \geq 4$ is also yes.

Despite the variety of the examples constructed in [49], it is interesting to note that the explicit construction of irreducible posets containing large chains was an elusive task. In fact, three distinct existence proofs were given before actual constructions were found. Trotter's 1974 paper [46] was titled "Irreducible Posets with Large Height Exist" because, at the time, he viewed that particular statement as the most important consequence of the paper. From the current vantage point, it is now clear that the major result was a complicated proof that the following inequality is best possible. (No simple proof has been found to date.)

Theorem 3: Let A be an antichain in a poset (X,P) with $X-A \neq \emptyset$. If $n = \text{width}(X - A, P(X - A))$, then $\dim(X,P) \leq 2n + 1$. □

A second existence proof is given in [5] where it is shown that interval orders with large dimension exist (The proof involves an elementary application of Ramsey's Theorem). It is also shown that there exists a function $f: N \to N$ so that if (X,P) is an interval order in which the longest chain has size n, then $\dim(X,P) \leq f(n)$. The best known upper bound [5] for $f(n) \leq c \log n$. However, regardless of the precise form of $f(n)$, it is clear that among the class of interval orders, there exist irreducible posets of arbitrarily large height.

The first major breakthrough in research on irreducible posets came in 1975 with the pioneering work [23] of Kelly and Rival on planar lattices. This paper was important in three ways. First, it represents a signal contribution to an area of research which might loosely be described as combinatorial lattice theory. Secondly, it underscored the fundamental importance of diagrams for posets and their connections to central areas of research in graph theory. Thirdly, it produced infinite families of 3-irreducible posets some of arbitrarily large width and some of arbitrarily large height (none with both!). Shortly after this advancement, Kelly [21] and Trotter and Moore [52] independently completed the determination of all 3-irreducible posets. The combinatorial techniques required to complete this determination were quite complicated. The difficulty stems from the fact that the list of 3-irreducible posets included 9 infinite families and 18 "odd" examples. Without careful organization, arguments could easily get bogged down in a maze of seemingly unending cases. Using the list of 3-irreducible posets, it was then possible to prove that for each $t \geq 3$, there exist 5-irreducible posets of arbitrarily large height. For $t \geq 4$, this proof was not, however, an explicit construction.

After this work was completed, it seemed reasonable that a complete characterization of all irreducible posets of arbitrary dimension was possible. If the pattern held true, then the most difficult case, that of determining all 3-irreducible posets, had already been completed.

As is often the case in mathematical research, nothing of real significance happened in the subject of irreducible posets for a period of several years. Then in less than one year, the real answers came quickly, and they constituted a genuine surprise. We mention first the complexity Theorems. The major result is due to Yannakakis [57].

Theorem 4: For every $t \geq 3$, the problem of determining whether a poset (X,P) satisfies $\dim(X,P) \leq t$ is NP-complete. □

The preceding result does not hold for $t = 2$, since to test whether a poset (X,P) has dimension at most two, it suffices to test the complement of the comparability graph of (X,P) to see whether it is a comparability graph. This is a polynomial size problem (see Golumbic [10]).

This complexity theorem suggests that the structure of irreducible posets is not as previously envisioned. Despite the pathology inherent in the odd examples, the 3-irreducible posets are in a sense relatively simple. It is the t-irreducible posets for $t \geq 4$ which are the unmanageable beasts.

In 1981, Kelly [22] developed a technique for constructing irreducible posets which he called "dimension products". He was motivated in part by a technique, also referred to as a dimension product, introduced by Trotter in [49]. Trotter's product was applied only to posets of height one while Kelly's could be applied to posets of arbitrary height. Kelly's method accomplished for the first time an explicit construction of t-irreducible posets of arbitrarily large height.

Shortly after learning of Kelly's results, Trotter and Ross [53], [54] came up with a more general construction scheme using techniques from chromatic graph theory combined with the dimension product concept to prove the following result.

Theorem 5: Let $t \geq 3$ and let (X,P) be an arbitrary t-dimensional poset. Then there exists a $t + 1$-irreducible poset containing (X,P) as a subposet. □

The theorem completely destroyed the mythical belief that t-irreducible posets were somehow more elegant structures than 3-irreducible posets when $t \geq 4$. To the contrary, it shows that an arbitrary degree of pathology can be built into t-irreducible posets. In particular, note that Theorem 5 does not hold when $t = 2$ since no 3-irreducible poset has both large width and large height, say both exceeding 4. In fact, there are numerous examples of 2-dimensional posets which are not subposets of 3-irreducible posets. This follows from the fact that in each $n \geq 6$, there are at most 27 3-irreducible posets on n points. However, the constructions used to prove Theorem 5 also yield the following inequality.

Theorem 6: Let $t \geq 4$. Then there exist positive constants c_1 and c_2 with $c_1 > 1$ so that for every $n \geq 2t$, there are

at least $c_1 c_2 n^2$ t-irreducible posets on n points. □

We will now proceed to develop the construction techniques required to prove Theorem 5 and 6. We begin by developing a link between posets and hypergraphs.

2. Posets, Hypergraphs and Consistent Linear Extensions

A set $\mathcal{P} = \{L_1, L_2, \ldots, L_t\}$ of t (not necessarily distinct) linear extensions of a poset (X,P) is called a realizer of P if $x_1 \leq x_2$ in P if and only if $x_1 \leq x_2$ in L_i for each $i = 1,2,\ldots,t$. The dimension of (X,P) is then the minimum size of a realizer of P. Let I_P denote the set of all incomparable pairs of (X,P). Then it is easy to see that a collection $\mathcal{P} = \{L_1, L_2, \ldots, L_t\}$ of linear extensions of P is a realizer of P if and only if for each $(x_1,x_2) \in I_P$, there exists some $L_i \in \mathcal{P}$ so that $x_2 < x_1$ in L_i. If $I \in I_P$ and $\mathcal{P} = \{L_1, L_2, \ldots, L_t\}$ is a collection of linear extensions of P, then we say $\mathcal{P}$ reverses I if for every $(x_1,x_2) \in I$, there exists some $L_i \in \mathcal{P}$ with $x_2 < x_1$ in L_i. The dimension of a poset (X,P) with $I_P \neq \emptyset$ is then the minimum number of linear extensions of P required to reverse all incomparable pairs in I_P.

For a binary relation R, we let $\hat{R} = \{(r_1,r_2): (r_2,r_1) \in R\}$. $\hat{R}$ is called the dual or reverse of R. When (X,P) is not a chain, it follows that $\dim(X,P)$ is the least t for which there exists a partition $\hat{I}_P = I_1 \cup I_2 \cup \ldots \cup I_t$ so that for each $i = 1,2,\ldots,t$, there exists a linear entension L_i of P with $x_1 < x_2$ in L_i for every $(x_1,x_2) \in I_i$. It is therefore natural to consider the following question:

If $I \subseteq I_P$, under what conditions does there exist a linear extension L of P with $x_1 < x_2$ in L for every $(x_1,x_2) \in I$?

The answer is simple. A set $\{c_i,d_i): 1 \leq i \leq m\} \subseteq I_P$ is called an alternating cycle of length m when $d_i \leq c_{i+1}$ in P for $i = 1,2,\ldots,m-1$ and $d_m \leq c_1$ in P. It is easy to show that these sets provide an answer to this question (see [51], for example).

Lemma 7: Let (X,P) be a poset and let $I \subseteq I_P$. Then there exists a linear extension L of P with $x_1 < x_2$ in P_L for every $(x_1,x_2) \in I$ if and only if I does not contain an alternating cycle. □

An alternating cycle $\{(c_i,d_i): 1 \leq i \leq m\}$ is said to be strong if $d_i \leq c_j$ if and only if $j = i+1$ for $1 \leq i \leq m-1$, and $d_m \leq c_j$ if and only if $j = 1$. It is straightforward to verify that if $I \subseteq I_P$ and I contains an alternating cycle, then I also contains a strong alternating cycle. Furthermore, if I is a strong alternating cycle, then no proper subset of I contains an alternating cycle.

Lemma 8: Let (X,P) be a poset and let $I \subseteq I_P$. Then there exists a linear extension L of P with $x_1 < x_2$ in P_P for every $(x_1,x_2) \in I$ if and only if I does not contain a strong alternating cycle. □

With a poset $\underset{\sim}{X}$, we associate a hypergraph $G_{\underset{\sim}{X}}$ so that the dimension of $\underset{\sim}{X}$ is the same as the chromatic number of $G_{\underset{\sim}{X}}$. As in Part I, we use the definition of the chromatic number of $G_{\underset{\sim}{X}}$, $\chi(G_{\underset{\sim}{X}})$, as the least number of colors required to assign colors to the vertices of $G_{\underset{\sim}{X}}$ so that no edge of $G_{\underset{\sim}{X}}$ has all of its vertices assigned to the same color. The scheme for defining $G_{\underset{\sim}{X}}$ is immediate. The vertex set of $G_{\underset{\sim}{X}}$ is the set I_p of incomparable pairs and a subset $E \subseteq I_{\underset{\sim}{X}}$ is an edge if its reverse $\hat{E}$ is a strong alternating cycle.

For computational purposes, the hypergraph $G_{\underset{\sim}{X}}$ contains too many vertices to be of real value in determining the dimension of $\underset{\sim}{X}$. However there is a natural way to determine a subhypergraph $H_{\underset{\sim}{X}}$ of $G_{\underset{\sim}{X}}$ so that $H_{\underset{\sim}{X}}$ and $G_{\underset{\sim}{X}}$ have the same chromatic number, and (in many cases) the structure of $H_{\underset{\sim}{X}}$ is more readily analyzed.

An incomparable pair $(x_1,x_2) \in I_P$ is called a nonforced pair if $x_3 < x_1$ implies $x_3 < x_2$ for all $x_3 \in X$ and $x_2 < x_4$ implies $x_1 < x_4$ for all $x_4 \in X$. We let N_P denote the set of all nonforced pairs. It is customary to treat P_{N_P} as both a binary relation and a directed graph. The graph theoretic properties of the digraph N_P are central to the theory of rank for partially ordered sets, and we refer the reader to [27], [28], [29], and [30] for additional material on this topic. Here we will require some elementary properties of N_P which we state without proof and refer the reader to [28] for details.

Lemma 9: Let (X,P) be a poset and let $\mathcal{P} = \{L_1, L_2, \ldots, L_t\}$ be a set of linear extensions of P. Then $\mathcal{P}$ is a realizer of P if and only if for each $(x_1,x_2) \in N_P$, there exists some $L_i \in \mathcal{P}$ so that $x_2 < x_1$ in L_i. □

It follows immediately from the preceding lemma that if (X,P) is not a chain, then the dimension of (X,P) is the minimum number of linear extensions of P required to reverse the nonforced pairs in N_P. This observation allows us to determine a subhypergraph $H_{\underset{\sim}{X}}$ of $G_{\underset{\sim}{X}}$ which has the same chromatic number as $G_{\underset{\sim}{X}}$. The vertex set of $H_{\underset{\sim}{X}}$ is the set N_P of nonforced paris of $(\underset{\sim}{X},P)$. A subset $E \subseteq N_P$ is an edge in $H_{\underset{\sim}{X}}$ if and only if $\hat{E}$ is a strong alternating cycle. It is often the case that the hypergraph $H_{\underset{\sim}{X}}$ has relatively simple structure; in particular, it is frequently a simple graph whose coloring properties can be easily determined.

To illustrate, when $\underset{\sim}{X} = (X,P) = \underset{\sim}{S}^0_n$, the standard example, it is easy to verify that $H_{\underset{\sim}{X}}$ is K_n - the complete graph on n vertices. Also when $\underset{\sim}{X} = (\underset{\sim}{X},P)$ is Hiraguchi's chevron, $H_{\underset{\sim}{X}}$ is a 5-cycle [50].

Lemma 10: If (X,P) is a poset and N_P is its set of nonforced pairs, then the binary relation $P \cup N_P$ is transitive. □

A subposet $\underset{\sim}{Y} = (Y,Q)$ of a poset $\underset{\sim}{X} = (X,P)$ is said to be partitive (in $\underset{\sim}{X}$) if the following two conditions are satisfied:

i. If $x \in X - Y$ and $x > y$ for some $y \in Y$, then $x > y$ for all $y \in Y$.

ii. If $x \in X - Y$ and $x < y$ for some $y \in Y$, then $x < y$ for all $y \in Y$.

The partitive subposet $\underset{\sim}{Y}$ in a poset $\underset{\sim}{X}$ is nontrivial when $2 \le |Y| < |X|$. The following result is a special case of the formula for the dimension of an ordinal sum (see [19] or [24].

Lemma 11: If $\underset{\sim}{Y} = (Y,Q)$ is a nontrivial partitive subposet of a poset $\underset{\sim}{X} = (X,P)$ and $y_0 \in Y$, then $\dim(\underset{\sim}{X}) = \max\{\dim(\underset{\sim}{X} - (\underset{\sim}{Y} - \{y_0\})), \dim(\underset{\sim}{Y})\}$. □

In particular, it follows that if $t \ge 2$ and $\underset{\sim}{X}$ is t-irreducible, then $\underset{\sim}{X}$ contains no nontrivial partitive subposets.

Lemma 12: Let $\underset{\sim}{X} = (X,P)$ be a poset and N_P the set of nonforced pairs. If the binary relation $P \cup N_P$ contains a directed cycle $\{(x_i,x_{i+1}): 1 \le i < m\} \cup \{(x_m,x_1)\}$ where $m \ge 2$, then the subposet $A = \{x_1,x_2,\ldots,x_m\}$ is a partitive subposet of $\underset{\sim}{X}$. Furthermore, A is an antichain. □

The only 2-irreducible poset is a two element antichain. For this poset $I_P = N_P = P \cup N_P$ and each of these binary relations is a directed cycle of length two. But for $t \ge 3$, no such pathology can occur.

Lemma 13: Let $t \ge 3$, let (X,P) be a t-irreducible poset, and let N_P be the set of nonforced pairs. Then the binary relation $P \cup N_P$ is acyclic - that is, it contains no directed cycles. □

For any poset (X,P) for which $P \cup N_P$ is acyclic, we can consider $P \cup N_P$ as a partial order on X. With this interpretation, $(X,P \cup N_P)$ is an extension of the poset (X,P). For such posets, a linear extension $L: [x_1 < x_2 < x_3 < \ldots < x_n]$ of $P \cup N_P$ is said to be consistent if $i < j$ whenever $x_i < x_j$ in P and $x_i\, I\, x_j$ in P. A maximal element x of the poset $(X,P \cup N_P)$ is called a strongly maximal element of (X,P).

Lemma 14: Let (X,P) be a poset and let N_P be the set of nonforced pairs. If $P \cup N_P$ is acyclic and x is a strongly maximal element of (X,P), then there exists a consistent linear extension $L: [x_1 < x_2 < \ldots < x_n]$ of $P \cup N_P$ with $x = x_n$. □

If (X,P) is a t-dimensional poset and L is a consistent linear extension of $P \cup N_P$, then L cannot belong to any realizer of size t of P since L reverses no nonforced pairs. On the other hand, we will frequently make minor modifications in a consistent linear extension to obtain one which does belong to a realizer of size t. Here is one such instance; others will be discussed later.

In the remainder of this paper, we will adopt the convention of using symbols such as $X_0,X_1,\ldots,X_t$ to denote linear extensions of a poset $\underset{\sim}{X} = (X,P)$. This device will allow us to simultaneously treat linear extensions of several different posets. We will use the symbol $N_{\underset{\sim}{X}}$ to denote the set of nonforced pairs in the poset $\underset{\sim}{X} = (X,P)$.

For a poset (X,P) and an element $x \in X$, we let $D(x) = \{y: y < x \text{ in } P\}$, $I(x) = \{y: y\, I\, x \text{ in } P\}$ and $U(x) = \{y: y > x \text{ in } P\}$. If $X_0: [x_1 < x_2 < \ldots < x_n]$ is a consistent linear extension of $P \cup N_P$, where $D(x_n) = \{x_1,x_2,\ldots,x_s\}$, and $I(x_n) = \{x_{s+1},x_{s+2},\ldots,x_{n-1}\}$, then the linear order $X_0^*: [x_1 < x_2 < x_3 < \ldots < x_s < x_n < x_{s+2} < \ldots < x_{n-1}]$ is called the reverse of X_0. Note that X_0^* is a linear extension of P but that X_0^* is not, in general, a linear extension of P but that X_0^* is not, in general, a linear extension of $P \cup N_P$. The following lemma shows that X_0^* belongs to a realizer of size t for P when X is t-irreducible. The result is a special case of the theorem due to Hiraguchi [19] which states that the removal of a point from a poset decreases the dimension by at most one.

Lemma 15: Let $t \ge 3$ and let $\underset{\sim}{X} = (X,P)$ be a t-irreducible poset. Also let X_0 be a consistent linear extension of $P \cup N_P$. Then let x be the strongly maximal element of $\underset{\sim}{X}$ which is the greatest element in X_0. If $\{X_1',X_2',\ldots,X_{t-1}'\}$ is a realizer of $\underset{\sim}{X} - \{x\}$ and for each $i = 1,2,\ldots,t-1$, we form a linear extension X_i of $\underset{\sim}{X}$ by adding x to X_i' as the largest element, then $\{X_0^*,X_1,X_2,\ldots,X_{t-1}\}$ is a realizer of X. □

3. The Embedding Poset when $t = 3$.

In this section, we present an infinite family of 3-irreducible posets $\{A(n,3): n \ge 1\}$ whose special properties will be particularly useful in the proof of Theorem 5 in the case $t = 3$. It is of secondary importance that these posets are irreducible. What actually matters is that they each have a consistent linear extension (which of course cannot belong to any realizer of size 3) such that if any one of a large number of minor modifications is made, the resulting linear extension belongs to a realizer of size 3.

For each $n \geq 1$, the poset $A(n,3)$ is a 3-irreducible poset for which the linear extension A_0: $[a_1 < a_2 < a_3 < \ldots < a_{2n+5}]$ is consistent. In Figure 1, we show a diagram for $\underset{\sim}{A}(n,3)$; for clarity, only the subscripts are shown

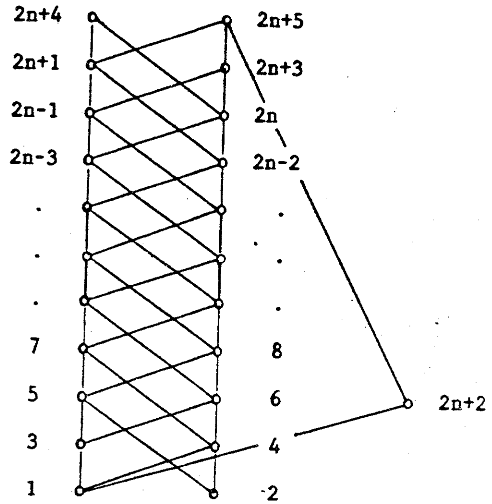


Figure 1

Let $\underset{\sim}{A} = \underset{\sim}{A}(n,3)$; then it is straightforward to verify that the set $N_{\underset{\sim}{A}}$ of nonforced pairs is the union of the two sets $N'_{\underset{\sim}{A}}$ and $N''_{\underset{\sim}{A}}$ where

$$N'_{\underset{\sim}{A}} = \{(a_i, a_{i+1}): 1 \leq i \leq 2n,\ 2n+2 \leq i \leq 2n+4\},$$

and $N''_{\underset{\sim}{A}} = \{(a_2, a_{2n+2}), (a_3, a_{2n+2}), (a_{2n+2}, a_{2n+4})\}$.

The following lemma gives important information on the structure of the hypergraph $H_{\underset{\sim}{A}}$.

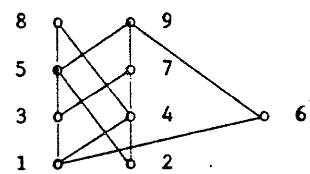
Lemma 16: If $n \geq 2$ and $\underset{\sim}{A} = \underset{\sim}{A}(n,3)$, then the hypergraph $H_{\underset{\sim}{A}}$ is a simple graph.

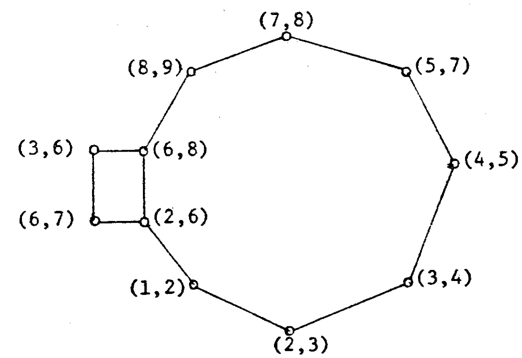
Proof: Let E be an edge in the hypergraph $H_{\underset{\sim}{A}}$ and suppose that

E contains at least three vertices of $H_{\underset{\sim}{A}}$. After relabelling, we may assume that $\hat{E} = \{(c_i,d_i): 1 \leq i \leq m\}$ is a strong alternating cycle in $\underset{\sim}{A}$ with $m = |E| \geq 3$. Since $C = \{c_1,c_2,\ldots,c_m\}$ and $D = \{d_1,d_2,\ldots,d_m\}$ are antichains in $\underset{\sim}{A}$, we may conclude that $m = 3$ and that $a_{2n+2} \in C \cap D$. Without loss of generality, we may assume that $a_{2n+2} = c_1$ and $a_{2n+2} = d_3$. Then since $(d_1,c_1) = (d_1,a_{2n+2}) \in E$, we know that either $d_1 = a_2$ or $d_1 = a_3$. Similarly, either $c_3 = a_{2n+3}$ or $c_3 = a_{2n+4}$. In any case, we would conclude that $d_1 < c_3$ which contradicts the assumption that $\hat{E}$ is a strong alternating cycle. □

Lemma 17: If $n \geq 2$ and $\underset{\sim}{A} = \underset{\sim}{A}(n,3)$, then the graph $H_{\underset{\sim}{A}}$ is a 3-colorable graph. Furthermore, the removal from $H_{\underset{\sim}{A}}$ of any one of the $2n$ vertices in the set $\{(a_i,a_{i+1}): 1 \leq i \leq 2n\}$ leaves a 2-colorable graph. □

We illustrate the preceding lemma when $n = 2$. For clarity only the subscripts are shown.





Lemma 17 has not been stated in the strongest possible form, but we are only interested in certain vertices in the graph $H_{\underset{\sim}{A}}$. Now the consistent linear extension A_0 reverses no nonforced pairs so there is no realizer of size 3 to which A_0 belongs. Now let i be any integer with $1 \leq i \leq 2n$ and let A_0^i be the linear extension of $\underset{\sim}{A}$ obtained by interchanging a_i and a_{i+1} in A_0 – that is,

A_0^i: $[a_1 < a_2 < \dots < a_{i-1} < a_{i+1} < a_i < a_{i+2} < a_{i+3} < \dots < a_{2n+5}]$.

We now show that each A_0^i belongs to a realizer of size 3 of $\underset{\sim}{A}$.

<u>Lemma 18</u>: Let $n \geq 2$ and let $\underset{\sim}{A} = \underset{\sim}{A}(n,3)$. Then for each $i = 1,2,\dots,2n$, there exist linear extensions A_1^i, A_2^i so that $\{A_0^i, A_1^i, A_2^i\}$ is a realizer of $\underset{\sim}{A}$.

<u>Proof</u>: From Lemma 17, we note that for each $i = 1,2,\dots,2n$, the graph $H_{\underset{\sim}{A}} - \{(a_i,a_{i+1})\}$ can be 2-colored using the colors $\{1,2\}$. Then for each $j = 1,2$, let A_j^i be a linear extension of $\underset{\sim}{A}$ which reverses all nonforced pairs which have been assigned color j. Since A_0^i reverses the nonforced pair (a_i,a_{i+1}), it follows that $\{A_0^i, A_1^i, A_2^i\}$ reverses $N_{\underset{\sim}{A}}$ and these linear extensions are a realizer of $\underset{\sim}{A}$. □

4. <u>The Embedding Theorem when $t = 3$</u>

Suppose that $\underset{\sim}{X} = (X,P)$ and $\underset{\sim}{Y} = (Y,P)$ are disjoint subposets of a poset $\underset{\sim}{Z} = (Z,R)$. We say that $\underset{\sim}{Y}$ is an <u>upper filter</u> of $\underset{\sim}{X}$ if the following two conditions are satisfied:

i. For every $(x_1,x_2) \in N_{\underset{\sim}{X}}$ there exists $y \in Y$ with $x_1 < Y$ and x_2 I y in $\underset{\sim}{Z}$.

ii. For every $x \in X$ and every $y \in Y$, $y \not< x$ in $\underset{\sim}{Z}$.

A linear extension Z_0 of $\underset{\sim}{Z}$ is called an <u>injection of</u> $\underset{\sim}{X}$ <u>over</u> $\underset{\sim}{Y}$ when $x \in X$, $y \in \underset{\sim}{Y}$, and x I y in $\underset{\sim}{Z}$ imply that $x > y$ in Z_0. The concepts of upper filters and injections are related by the following elementary result.

<u>Lemma 19</u>: Let $\underset{\sim}{X} = (X,P)$ and $\underset{\sim}{Y} = (Y,Q)$ be disjoint subposets in a poset $\underset{\sim}{Z} = (Z,R)$, and let Y be an upper filter of $\underset{\sim}{X}$. If $\dim(\underset{\sim}{X}) = t$ and Z_0 is a linear extension of $\underset{\sim}{Z}$ which is an injection of $\underset{\sim}{X}$ over $\underset{\sim}{Y}$, then Z_0 cannot belong to a realizer of size t for $\underset{\sim}{Z}$.

<u>Proof</u>: Any realizer of $\underset{\sim}{Z}$ must reverse all incomparable pairs in $\underset{\sim}{Z}$ and must therefore reverse the incomparable pairs in $N_{\underset{\sim}{X}}$. Now let $(x_1,x_2) \in N_{\underset{\sim}{X}}$; choose an element $y \in \underset{\sim}{Y}$ so that $x_1 < y$ and x_2Iy in $\underset{\sim}{Z}$. Since Z_0 is an injection of $\underset{\sim}{X}$ over $\underset{\sim}{Y}$, we must have $x_1 < y < x_2$ in Z_0. Thus Z_0 reverses no pairs in $N_{\underset{\sim}{X}}$. If $\mathcal{R}$ is a realizer of size t for $\underset{\sim}{Z}$ and $Z_0 \in \mathcal{R}$, then the restrictions of the other $t-1$ linear extensions to $\underset{\sim}{X}$ must reverse $N_{\underset{\sim}{X}}$. Since $\dim \underset{\sim}{X} = t$, this is impossible. □

If $\underset{\sim}{X} = (X,P)$ and $\underset{\sim}{Y} = (Y,Q)$ are disjoint subposets of a poset $\underset{\sim}{Z}$, we write $\underset{\sim}{X} < \underset{\sim}{Y}$ when $x < y$ for every $x \in X$ and $y \in Y$. Similarly, we write $\underset{\sim}{X}$ I $\underset{\sim}{Y}$ when x I y for every $x \in X$ and $y \in Y$.

We are now ready to present the proof of Theorem 5 for the case $t = 3$. The reader is encouraged to compare this construction with the graph construction used by B. Toft [44] to show the existence of color critical graphs having a relatively large percentage of edges.

<u>Theorem 20</u>: Let $\underset{\sim}{X} = (X,P)$ be a poset with $\dim(X,P) \leq 3$. Then there exists a 4-irreducible poset containing (X,P) as a subposet.

<u>Proof</u>: We first construct a 4-dimensional poset $\underset{\sim}{W} = (W,X)$ containing (X,P) as a subposet. In general (W,S) will not be irreducible, but we will prove that (W,S) contains a 4-irreducible subposet which also contains (X,P) as a subposet. The poset (W,S) is the union of five disjoint subposets $\underset{\sim}{X}$, $\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{U}$, and $\underset{\sim}{V}$, with $\underset{\sim}{A}$ I $\underset{\sim}{U}$, $\underset{\sim}{A} < \underset{\sim}{V}$, $\underset{\sim}{U} < (\underset{\sim}{X} \cup \underset{\sim}{B})$, and $\underset{\sim}{V}$ I$(\underset{\sim}{X} \cup \underset{\sim}{B})$. Furthermore, $\underset{\sim}{X} \cup \underset{\sim}{B}$ will be an upper filter of $\underset{\sim}{A}$ and $\underset{\sim}{V}$ will be an upper filter of $\underset{\sim}{U}$. The posets $\underset{\sim}{A}$ and $\underset{\sim}{U}$ will be 3-irreducible posets.

Since $\dim \underset{\sim}{X} \leq 3$, there exists linear extensions X_1, X_2, and X_3 of $\underset{\sim}{X}$ so that $x \leq x'$ in $\underset{\sim}{X}$ if and only if $x \leq x'$ in X_i for $i = 1,2,3$. These linear extensions need not be distinct. We suppose that $|X| = m$ and let X_1: $[x_1 < x_2 < x_3 < \dots < x_m]$. Next choose an integer n so that $n \geq 2$ and $2n \geq m$. Then the subposet $\underset{\sim}{A}$ is $\underset{\sim}{A}(n,3)$.

The subposet $\underset{\sim}{B}$ is a chain containing $2n+4-m$ points $\{b_{m+1} < b_{m+2} < \dots < b_{2n+4}\}$. For each $j = 1,2,\dots,m$, $a_i < x_j$ in S if and only if $i \leq j$. Also, for each $j = m+1, m+2, \dots, 2n+4$, $a_i < b_j$ in S if and only if $i \leq j$. To see that $\underset{\sim}{X} \cup \underset{\sim}{B}$ is an upper filter of $\underset{\sim}{A}$, we consider an arbitrary nonforced pair $(a_i,a_j) \in N_{\underset{\sim}{A}}$. Since A_0 is consistent, we know that $i < j$. If $1 \leq i \leq m$, then $x_i < a_i$ and x_i I a_j; if $m+1 \leq i \leq 2n+4$, then $b_i > a_i$ and b_i I a_j. Therefore, $\underset{\sim}{X} \cup \underset{\sim}{B}$ is an upper filter of $\underset{\sim}{A}$ as claimed.

The subposet $\underset{\sim}{U}$ is the standard example of a 3-irreducible poset with the points labelled so that the linear extension U_0: $[u_1 < u_2 < u_3 < u_4 < u_5 < u_6]$ is consistent, the subposets $\{u_1,u_2,u_3\}$ and $\{u_4,u_5,u_6\}$ are antichains, and $u_i < u_{3+j}$ if and only if $i \neq j$ for each $i, j = 1,2,3$.

The subposet $\mathbf{V}$ consists of a single point $\{v\}$ with $u_i < v$ if and only if $i = 1,2,3$. Note that $N_{\mathbf{U}} = \{(u_i,u_{3+i}) : i = 1,2,3\}$, so that $\mathbf{V}$ is an upper filter of $\mathbf{U}$. This completes the definition of the poset (W,S).

We now show that $\dim (W,S) \geq 4$. Suppose to the contrary that $\dim (W,S) \leq 3$. Since (W,S) contains the 3-dimensional poset $\mathbf{A}$, we conclude that $\dim (W,S) = 3$. Then let $\mathcal{S} = \{S_1,S_2,S_3\}$ be a realizer of S. We know that these three linear extensions reverse all incomparable pairs in (W,S), but we are particularly interested in how they reverse the incomparable pairs in the following two subsets of I_S:

$$N_1 = \{(a,z) \in I_S : \ a \in \mathbf{A}, \ \ z \in \mathbf{X} \cup B\}$$

$$N_2 = \{(u_i,v) \in I_S : \ i = 4,5,6\}$$

It is easy to see that no linear extension of S can reverse a pair from N_1 and a pair from N_2. For if $(a,z) \in N_1$, and $z < a$ in S_i, then $u < z < a < v$ in S_i for every $u \in \mathbf{U}$ and $v \in \mathbf{V}$. Similarly, if $(u,v) \in N_2$ and $v < u$ in S_i, then $a < v < u < z$ for every $a \in \mathbf{A}$ and $z \in \mathbf{X} \cup \mathbf{B}$. On the other hand, no linear extension in $\mathcal{S}$ can reverse N_1 for this would imply that it is an injection of $\mathbf{A}$ over $\mathbf{X} \cup \mathbf{B}$ in violation of Lemma 19. Similarly, no linear extension in $\mathcal{S}$ can reverse N_2. We conclude that at least two linear extensions in $\mathcal{S}$ reverse pairs in N_2. But this is clearly impossible. The contradiction shows that $\dim(W,S) \geq 4$.

We next show that $\dim (\mathbf{W} - \{x_i\}) = 3$ for every $x_i \in X$. To accomplish this, we choose an arbitrary point $x_i \in X$ and construct a realizer $\{S_1',S_2',S_3'\}$ of the subposet $\mathbf{W} - \{x_i\}$. First, let $\{U_0^*,U_1,U_2\}$ be a realizer of $\mathbf{U}$ where U_0^* is the reverse of the consistent linear extension U_0 and U_i is formed from U_i' by adding u_6 as the largest element. Then let $(U \cup V)_0^*$ be the linear extension of $\mathbf{U} \cup \mathbf{V}$ defined by $(U \cup V)_0^* : [u_1 < u_2 < u_6 < v < u_4 < u_5]$. The restriction of $(U \cup V)_0^*$ to $\mathbf{U}$ is U_0^*; furthermore $(U \cup V)_0^*$ reverses (u_4,v) and (u_5,v) but not (u_6,v).

Since $1 \leq i \leq m \leq 2n$, there exists by Lemma 12 a realizer $\{A_0^i,A_1^i,A_2^i\}$ of the 3-irreducible poset $\mathbf{A}$ where A_0^i is obtained from A_0 by interchanging a_i and a_{i+1}. We let $(A \cup X \cup B)_0^i$ denote the linear extension of $\mathbf{A} \cup \mathbf{X} \cup (\mathbf{B} - \{x_i\})$ defined by:

$$(A \cup X \cup B)_0^i : \ [a_1 < x_1 < a_2 < x_2 < a_3 < x_3 < \ldots$$
$$< x_{i-2} < a_{i-1} < a_{i+1} < a_i < x_{i+1} < a_{i+2} < x_{i+2} < \ldots$$
$$< a_m < x_m < a_{m+1} < b_{m+1} < a_{m+2} < b_{m+2} < \ldots$$
$$< a_{2n+4} < b_{2n+4} < a_{2n+5}].$$

Then it follows that the restriction of $(A \cup X \cup B)_0^i$ to $\mathbf{A}$ is A_0^i, and the restriction of $(A \cup X \cup B)_0^i$ to $\mathbf{X} - \{x_i\}$ is X_0^*. Furthermore, we note that $(A \cup X \cup B)_0^i$ is an injection of $\mathbf{A}$ over $\mathbf{X} \cup (\mathbf{B} - \{x_i\})$. It also reverses the nonforced pair $(a_i,a_{i+1}) \in N_{\mathbf{X}}$. We then define:

$$S_1' = V_1 < (A \cup X \cup B)_0^i < V,$$

$$S_2' = A_1^i < U_2' < V < u_6 < X_2 - \{x_i\} < B, \ \text{and}$$

$$S_3' = A_2^i < (U \cup V)_0^* < X_3 - \{x_i\} < B.$$

To see that these linear orders form a realizer of $\mathbf{W} - \{x_i\}$, we make the following observations:

1. Each S_j' is clearly a linear extension of $\mathbf{W} - \{x_i\}$.
2. $\mathbf{U} < \mathbf{A}$ and $\mathbf{X} \cup (\mathbf{B} - \{x_i\}) < \mathbf{V}$ in S_1'.
3. $\mathbf{B} < \mathbf{U}$ and $\mathbf{V} < \mathbf{X} \cup (\mathbf{B} - \{x_i\})$ in S_2'.
4. S_1' is an injection of $\mathbf{A}$ over $\mathbf{X} \cup (\mathbf{B} - \{x_i\})$.
5. S_2' and S_3' reverse N_2.
6. The restriction of $\{S_1',S_2',S_3'\}$ to $\mathbf{A}$ is $\{A_0^i,A_1^i,A_2^i\}$.
7. The restriction of S_j' to $\mathbf{X} - \{x_i\}$ is $X_j - \{x_i\}$ for $j = 1,2,3$.
8. The restriction of S_j' to U is U_j for $j = 1,2$ and the restriction of S_3' to U is U_0^*.

At this point, the proof of Theorem 20 is essentially complete. We have shown that $\dim (W,S) \geq 4$ but that the removal of any point from the subposet $\mathbf{X} \subseteq \mathbf{W}$ leaves a three dimensional poset. Now for every $t \geq 2$, a t-dimensional poset contains a t-irreducible subposet, and any 4-irreducible subposet of (W,S) must obviously contain $\mathbf{X}$ as a subposet. □

Although the poset $\mathbf{W} = (W,S)$ is not 4-irreducible, it is nearly so. There are exactly two 4-irreducible subposets of $\mathbf{W}$; they are $\mathbf{W} - \{b_{2n+1}\}$ and $\{\mathbf{W} - b_{2n+2}\}$.

5. The Embedding Posets when $t > 4$.

In this section, we construct for each $t \geq 4$ an infinite family $\{\underset{\sim}{A}(n,t): n \geq 1\}$ of t-dimensional posets each of which possesses a consistent linear extension in which any one of a large number of minor modifications allows the resulting extension to belong to a realizer of $\underset{\sim}{A}(n,t)$ of size t. The construction will utilize the concept of a dimension product as introduced in [22] by D. Kelly.

If $\underset{\sim}{X} = (X,P)$ and $\underset{\sim}{Y} = (W,Q)$ are posets, then the cartesian product $\underset{\sim}{X} \times \underset{\sim}{Y}$ is the poset whose point set is the set of all pairs (x,y) where $x \in X$ and $y \in Y$ with $(x_1,y_1) \leq (x_2,y_2)$ in $\underset{\sim}{X} \times \underset{\sim}{Y}$ if and only if $x_1 \leq x_2$ in P and $y_1 \leq y_2$ in Q. The cartesian product of n copies of $\underset{\sim}{X}$ is denoted $\underset{\sim}{X}^n$. It is easy to see that $\dim(\underset{\sim}{X} \times \underset{\sim}{Y}) \leq \dim \underset{\sim}{X} + \dim \underset{\sim}{Y}$. The following result of Baker [1] gives a sufficient condition for equality to hold.

Theorem 21: Let $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ be posets. If $\underset{\sim}{X}$ contains distinct points x_1, x_2 and $\underset{\sim}{Y}$ contains distinct points y_1, y_2 so that $x_1 \leq x \leq x_2$ in $\underset{\sim}{X}$ for every $x \in \underset{\sim}{X}$ and $y_1 \leq y \leq y_2$ in $\underset{\sim}{Y}$ for every $y \in \underset{\sim}{Y}$, then $\dim(\underset{\sim}{X} \times \underset{\sim}{Y}) = \dim \underset{\sim}{X} + \dim \underset{\sim}{Y}$. □

For a poset $\underset{\sim}{X}$, Let $\overline{\underset{\sim}{X}}$ denote the poset obtained from $\underset{\sim}{X}$ by adding two new points, one larger than every point in $\underset{\sim}{X}$ and the other less than every point in $\underset{\sim}{X}$. Then $\dim(\overline{\underset{\sim}{X}}) = \dim \underset{\sim}{X}$ for every poset $\underset{\sim}{X}$. Furthermore, $\dim(\overline{\underset{\sim}{X}} \times \overline{\underset{\sim}{Y}}) = \dim \underset{\sim}{X} + \dim \underset{\sim}{Y}$. Kelly's dimension product identifies a (relatively small) subposet of $\overline{\underset{\sim}{X}} \times \overline{\underset{\sim}{Y}}$ whose dimension is $\dim \underset{\sim}{X} + \dim \underset{\sim}{Y}$, and in many cases, Kelly's construction precisely determines an irreducible subposet. Such will be the case in the construction we now present.

For each $n \geq 1$ and each $t \geq 4$, $\underset{\sim}{A}(n,t)$ will be a t-dimensional subposet of the cartesian product $\underset{\sim}{A}(n,3) \times \underline{2}^{t-3}$. We will find it convenient to use notation similar to that employed in Section 3 and 4 rather than the notation of Kelly.

The poset $\underset{\sim}{A}(n,t)$ is the union of four subposets $\underset{\sim}{A}'(n)$, $\underset{\sim}{A}''(n)$, $\underset{\sim}{A}_{t-3}$, and $\underset{\sim}{B}_{t-3}$. The subposets $\underset{\sim}{A}'(n)$ and $\underset{\sim}{A}''(n)$ are both copies of $\underset{\sim}{A}(n,3)$ labelled $\{a_i': 1 \leq i \leq 2n+5\}$ and $\{a_i'': 1 \leq i \leq 2n+5\}$ respectively. Furthermore $a_i'' \leq a_j'$, $a_i' \leq a_j'$ and $a_j'' < a_j'$ if and only if $a_i \leq a_j$ in $\underset{\sim}{X}(n,3)$.

The subposets $\underset{\sim}{A}_{t-3}$ and $\underset{\sim}{B}_{t-3}$ are both $t-3$ element antichains labelled $\{a_1, a_2, \ldots, a_{t-3}\}$ and $\{b_1, b_2, \ldots, b_{t-3}\}$ respectively. Furthermore, $b_i < a_j$ if and only if $i \neq j$ for all $i, u = 1,2,\ldots, t-3$. In addition $\underset{\sim}{A}'(n) > \underset{\sim}{B}_{t-3}$ and $\underset{\sim}{A}''(n) < \underset{\sim}{A}_{t-3}$. This completes the definition of the poset $\underset{\sim}{A}(n,t)$.

For graphs G_1 and G_2, the join of G_1 and G_2, denoted by $G_1 + G_2$, is the graph obtained by taking disjoint copies of G_1 and G_2 and adding an edge between x_1 and x_2 for every vertex $x_1 \in G_1$ and $x_2 \in G_2$. Clearly, the chromatic number of $G_1 + G_2$ is the sum of the chromatic numbers of G_1 and G_2. For an integer $k \geq 1$, let K_k denote the complete graph on k vertices. It is straightforward to verify the following results concerning the hypergraph associated with $\underset{\sim}{A}(n,t)$.

Lemma 22: Let $n \geq 1$ and $t \geq 4$. Also let $\underset{\sim}{A}_1 = \underset{\sim}{A}(n,3)$ and $\underset{\sim}{A}_2 = \underset{\sim}{A}(n,t)$. Then:

1. The set $N_{\underset{\sim}{A}_2}$ of nonforced pairs of $\underset{\sim}{A}_2$ is the union of two sets $N'_{\underset{\sim}{A}_2}$ and $N''_{\underset{\sim}{A}_2}$ where $N'_{\underset{\sim}{A}_1} = \{(a_i'', a_j'): (a_i, a_j) \in N_A\}$ and $N''_{\underset{\sim}{A}} = \{(b_i, a_i): 1 \leq i \leq t-3\}$.
2. If $n \geq 2$, the hypergraph $H_{\underset{\sim}{A}_2}$ is a simple graph.
3. The induced subgraph G_1 of $H_{\underset{\sim}{A}_2}$ whose vertex set is $N'_{\underset{\sim}{A}_2}$ is isomorphic to $H_{\underset{\sim}{A}_1}$.
4. The induced subgraph G_2 of $H_{\underset{\sim}{A}_2}$ whose vertex set is $N''_{\underset{\sim}{A}_2}$ is isomorphic to K_{t-3}.
5. $H_{\underset{\sim}{A}_2} = G_1 + G_2$.
6. $\chi(H_{\underset{\sim}{A}_2}) = \dim(\underset{\sim}{A}_2) = t$.
7. For each $i = 1,2,\ldots,2n$, the removal of the vertex (a_i'', a_{i+1}') from $H_{\underset{\sim}{A}_2}$ leaves a graph with chromatic number $t-1$. □

Note that $\underset{\sim}{A}(n,t)$ is not t-irreducible but that the removal of a_1' and a_{2n+5}'' leaves a t-irreducible subposet. For symmetry we include these points in the definition of the embedding posets $\underset{\sim}{A}(n,t)$. The following linear order is easily seen to be a consistent linear extension of $\underset{\sim}{A}(n,t)$:

$$A_0: \ [b_1 < b_2 < b_3 < \ldots < b_{t-3} <$$
$$a_1'' < a_1' < a_2'' < a_2' < a_3'' < a_3' < a_4'' < \ldots <$$

$a''_{2n+3} < a'_{2n+3} < a''_{2n+4} < a'_{2n+4} < a''_{2n+5} < a'_{2n+5} <$

$a_1 < a_2 < a_3 < \ldots < a_{t-3}]$.

For each $i = 1,2,\ldots,2n$, we let A_0^i denote the linear extension of $\underset{\sim}{A}$ obtained by interchanging $a''_i < a'_i$ with $a''_{i+1} < a'_{i+1}$. Note that exactly four entries in A_0 change places in this modification. Note further that A_0^i reverses the nonforced pair (a''_i, a'_{i+1}) and thus by Statement 7 in Lemma 22, there exist linear extensions $A_1^i, A_2^i, \ldots, A_{t-1}^i$ so that $\{A_0^i, A_1^i, A_2^i, \ldots, A_{t-1}^i\}$ is a realizer of $\underset{\sim}{A}$.

6. The Embedding Theorem when $t > 4$

The proof of the following theorem is quite similar to the proof of Theorem 20 so we will present only the key steps.

Theorem 23: If $t \geq 4$ and $\underset{\sim}{X} = (X,P)$ is a poset whose dimension is at most t, then there exists a $t+1$-irreducible poset containing (X,P) as a subposet.

Proof: We begin by constructing a $t+1$-dimensional poset $\underset{\sim}{W} = (W,S)$ containing (X,P) as a subposet. Suppose X has m elements. Let n be an integer with $2n \geq m$. Let $\{X_1, X_2, \ldots X_t\}$ be a realizer of $\underset{\sim}{X}$ with $X_1: [x_1 \;\; x_2 \;\; \ldots \;\; x_m]$. Then the poset $\underset{\sim}{W}$ is the union of following disjoint subposets $\underset{\sim}{X}, \underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{U}, \underset{\sim}{V}, \underset{\sim}{C}$, and $\underset{\sim}{D}$ which are defined as follows:

1. $\underset{\sim}{A} = \underset{\sim}{A}(n,t)$.
2. $\underset{\sim}{B}$ is a $2n+4-m$ element chain labelled $\{b_{m+1} < b_{m+2} < \ldots < b_{2n+4}\}$.
3. $\underset{\sim}{U}$ is the standard t-dimensional poset labeled $\{u_1, u_2, \ldots, u_{2t}\}$ with $u_i < u_{t+j}$ if and only if $i \neq j$ for all i, $j = 1,2,\ldots,t$.
4. $\underset{\sim}{V} = \{v\}$ is a one point poset.
5. The subposets $\underset{\sim}{C} = \{c_1, c_2, \ldots, c_{t-3}\}$ and $\underset{\sim}{D} = \{d_1, d_2, \ldots, d_{t-3}\}$ are $t-3$ point antichains.

To complete the definition of $\underset{\sim}{W}$, we describe the comparabilities between these subposets:

6. $\underset{\sim}{X} < \underset{\sim}{B}$, $(\underset{\sim}{X} \cup \underset{\sim}{B})\; I\; \underset{\sim}{C}$ and $(\underset{\sim}{X} \cup \underset{\sim}{B}) > \underset{\sim}{D}$
7. For each $j = 1,2,\ldots,m$, $a''_i < a'_i < x_j$ if and only if $i \leq j$.

8. For each $j = m+1, m+2, \ldots, 2n+4$, $a''_i < a'_i < b_j$ if and only if $i \leq j$.
9. $\underset{\sim}{V} > \{u_1, u_2, \ldots, u_t\}$ and $\underset{\sim}{V}\; I\; \{u_{t+1}, u_{t+2}, \ldots, u_{2t}\}$.
10. For each $i, j = 1,2,\ldots,t-3$, $d_i < c_j$ if and only if $i \neq j$.
11. $(\underset{\sim}{A} \cup \underset{\sim}{U}) < \underset{\sim}{C}$, $(\underset{\sim}{A} \cup \underset{\sim}{D}) < \underset{\sim}{V}$, and $(\underset{\sim}{U} \cup \underset{\sim}{D}) < (\underset{\sim}{X} \cup \underset{\sim}{B})$.
12. $(\underset{\sim}{X} \cup \underset{\sim}{B})\; I\; (\underset{\sim}{V} \cup \underset{\sim}{C})$, $\underset{\sim}{V}\; I\; \underset{\sim}{C}$, $\underset{\sim}{A}\; I\; (\underset{\sim}{U} \cup \underset{\sim}{D})$, and $\underset{\sim}{U}\; I\; \underset{\sim}{D}$.

Note that $\underset{\sim}{X} \cup \underset{\sim}{B}$ is an upper filter of $\underset{\sim}{A}$ and $\underset{\sim}{V}$ is an upper filter of $\underset{\sim}{U}$. We now show that $\dim (W,S) \geq t+1$. To the contrary, suppose that $\{S_1, S_2, \ldots, S_t\}$ is a realizer of S. Then these extensions reverse all nonforced pairs of I_S. In particular they reverse all nonforced pairs in the following three sets:

$N_1 = \{(a,z)\colon\; a \in \underset{\sim}{A},\;\; z \in \underset{\sim}{X} \cup \underset{\sim}{B},\; a\; I\; z\}$

$N_2 = \{(u,v)\colon\; u \in \underset{\sim}{U},\;\; v \in \underset{\sim}{V},\; u\; I\; v\}$

$N_3 = \{(d_i, c_i)\colon\; i = 1,2,\ldots,t-3\}$

We then note that no linear extension of S can reverse pairs from any two of these three sets. Furthermore, a linear extension can reverse at most one pair from N_3. This implies that one of the S'_is is an injection of $\underset{\sim}{A}$ over $\underset{\sim}{X} \cup \underset{\sim}{B}$ or an injection of $\underset{\sim}{U}$ over $\underset{\sim}{V}$. Neither of these statements can be true, so we conclude that $\dim (",S) \geq t+1$.

We next show that $\dim \underset{\sim}{W} - \{x_i\} = t$ for $i = 1,2,\ldots,m$. To accomplish this we make the following observations. There is a linear extension $(A \cup X \cup B)_0^i$ of $\underset{\sim}{A} \cup \underset{\sim}{X} \cup \underset{\sim}{B}$ such that:

a. The restriction to $\underset{\sim}{A}$ is A_0^i.
b. The restriction to $(\underset{\sim}{X} \cup \underset{\sim}{B}) - \{x_i\}$ is $X_1 - \{x_i\} < \underset{\sim}{B}$.
c. It is an injection of $\underset{\sim}{A}$ over $(\underset{\sim}{X} \cup \underset{\sim}{B}) - \{x_i\}$.

Furthermore, there is a realizer $\{U_0^*, U_1, U_2, \ldots, U_{t-1}\}$ of $\underset{\sim}{U}$ where U_0^* is the reverse of the consistent linear extension U_0: $[u_1 < u_2 < \ldots < u_{2t}]$, and U_i is formed from U'_i by adding u_{2t} as the largest element. It follows that there is a linear extension $(U \cup V)_0^*$ such that:

a. The restriction to U is U_0^*.
b. It reverses all nonforced pairs in N_2 except (u_{2t}, v).

Let C_0 and D_0 be arbitrary linear orders on C and D respectively. Then define:

$$S_1' = D_0 < U_1 < (A \cup X \cup B)_0^i < V < C_0$$

$$S_2' = U_2 < A_1^i < D_0 - \{d_1\} < c_1 < d_1 < C_0 - \{c_1\} < X_2 - \{x_i\} < B < V$$

$$S_3' = U_3 < A_2^i < D_0 - \{d_2\} < c_2 < d_2 < C_0 - \{c_2\} < X_3 - \{x_i\} < B < V$$

$$\vdots$$

$$S_{t-2}' = U_{t-2} < A_{t-3}^i < D_0 - \{d_{t-3}\} < c_{t-3} < d_{t-3} < C_0 - \{c_{t-3}\} < X_{t-2} - \{x_i\} < B < V$$

$$S_{t-1}' = D_0 < A_{t-2}^i < U_{t-1}' < V < U_{2t} < X_{t-1} - \{x_i\} < B < C_0$$

$$S_t' = D_0 < A_{t-1}^i < (U \cup V)_0^* < X_t - \{x_i\} < B < C_0.$$

These t linear extensions of $\underset{\sim}{W} - \{x_i\}$ form a realizer so we conclude that $\dim(\underset{\sim}{W} - \{x_i\}) = t$ for each $i = 1,2,\ldots,m$. It follows that any $t+1$-irreducible subposet of $\underset{\sim}{W}$ contains $\underset{\sim}{X}$ as a subposet and the proof of our theorem is complete. □

7. Open Problems in Dimension Theory

We close with five open problems in dimension theory for ordered sets.

1. Does every poset (X,P) with $|X| \geq 3$ contain a distinct pair $x,y \in X$ so that $\dim(X,P) \leq 1 + \dim(X - \{x,y\}, P(X - \{x,y\}))$?
2. For every pair m,n of integers with $1 \leq m \leq n$, do there exist posets $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ with $\dim \underset{\sim}{X} = m$ and $\dim \underset{\sim}{Y} = \dim(\underset{\sim}{X} \times \underset{\sim}{Y}) = n$?
3. For every $t \geq 2$, does there exist a t-irreducible poset $\underset{\sim}{X}$ having a planar order diagram?
4. For each $t \geq 1$, construct a $2t+1$-irreducible poset (X,P) containing an antichain A so that the width of the subposet $(X - A, P(X - A))$ is t.
5. What is the maximum value $f(n)$ of the dimension of an interval order in which the maximum size of a chain is n?

For additional open problems in dimension theory, we encourage the reader to consult the listing provided in [48].

References

1. K.A. Baker, "Dimension, Join-independence, and Breath in Partially Ordered Sets," (1961) (unpublished).
2. K.A. Baker, P.C. Fishburn and F.S. Roberts, "Partial Orders of Dimension 2," Networks 2 (1971) 11-28.
3. C. Berge, "Graphes et Hypergraphes," Dunod, Paris, 1970.
4. K.P. Bogart and W.T. Trotter, "Maximal Dimensional Partially Ordered Sets II. Characterization of 2n-element posets with dimension n," Discrete Math. 5 (1973) 33-45.
5. K.P. Bogart, I. Rabinovitch and W.T. Trotter, "A Bound on the Dimension of Interval Orders," J. Combinatorial Theory (Ser. A) 21 (1976) 319-328.
6. R.P. Dilworth, "A Decomposition Theorem for Partially Ordered Sets," Ann. Math. 51 (1950) 161-166.
7. B. Dushnik and E.W. Miller, "Partially Ordered Sets," Amer. J. Math. 63 (1941) 600-610.
8. P. Erdös and A. Hajnal, "On Chromatic Number of Graphs and Set-systems," Acta Math. Acad. Sci. Hung. 17, (1966) 61-99.
9. J. Folkman, Personal Communication.
10. M.C. Golumbic, "The Complexity of Comparability Graph Recognition and Coloring," Computing 18, (1977) 199-203.
11. R.L. Graham, "Euclidean Ramsey Theorems on the n-sphere," J. Graph Theory 7 (1983) 105-114.
12. R.L. Graham, "Rudiments of Ramsey Theory," CBMS Regional Conference Series in Math.," Amer. Math. Soc. Vol. 45 (1981)
13. R.L. Graham, "Recent Developments in Ramsey Theory," To Appear.
14. R.L. Graham and B.L. Rothschild, "Ramsey's Theorem for n-parameter Sets," Trans. Amer. Math. Soc. 159 (1971) 257-292.
15. R.L. Graham, B.L. Rothschild and J.H. Spencer, "Ramsey Theory," John Wiley & Sons, Inc. (1980).
16. A.W. Hales and R.I. Jewett, "Regularity and Positional Games," Trans. Amer. Math. Soc. 106 (1963) 222-229.
17. E. Harzheim, "Combinatorial Theorems on Contractive Mappings in Power Sets," Discrete Math 40 (1982) 193-202.
18. E. Harzheim, "A Constructive Proof for a Theorem on Contractive Mappings in Power Sets," Discrete Math. 45 (1983) 99-106.

19. T. Hiraguchi, "On the Dimension of Partially Ordered Sets," Sci. Rep. Kanazawa Univ. 4 (1951) 77-94.

20. T. Hiraguchi, "On the dimension of Orders," Sci. Rep. Kanazawa Univ. 4 (1955) 1-20.

21. D. Kelly, "The 3-irreducible Partially Ordered Sets," Canad. J. Math. 29 (1977) 367-383.

22. D. Kelly, "On the Dimension of Partially Ordered Sets," Discrete Math. 35 (1981) 135-156.

23. D. Kelly and I. Rival, "Planar Lattices", Canad. J. Math. 27 (1975a) 636-665.

24. D. Kelly and W.T. Trotter, "Dimension Theory for Ordered Sets," in Ordered Sets, I. Rival (ed.), D. Reidel (1982) 171-211.

25. R.J. Kimble, "Extermal Problems in Dimension Theory for Partially Ordered Sets," Ph.D. thesis, M.I.T. (1973).

26. K. Leeb, Personal Communication.

27. S.B. Maurer and I. Rabinovitch, "Large Minimal Realizers of a Partial Order," Proc. Amer. Math. Soc. 66 (1978) 211-216.

28. S.B. Maurer, I. Rabinovitch and W.T. Trotter, "Large Minimal Realizers of a Partial Order II," Discrete Math. 31 (1980a) 297-314.

29. S.B. Maurer, I. Rabinovitch and W.T. Trotter, "A Generalization of Turán's Theorem," Discrete Math. 32 (1980b) 167-189.

30. S.B. Maurer, I. Rabinovitch and W.T. Trotter, "Partially Ordered Sets with Equal Rank and Dimension," Congressus Numerantium 29 (1980c) 627-637.

31. J. Nesetril and V. Rödl, "Partitions of Relational and Set Systems," J. Comb. Th. A 22 (1977) 289-312.

32. J. Nesetril and V. Rödl, "Partition (Ramsey) Theory - a Survey, in: Coll. Math. Soc. Janos Bolyai," Combinatorics 18 (1978) 759-792.

33. J. Nesetril and V. Rödl, "Partition Theory and Its Applications, in B. Bollobas. ed.," Surveys of Combinatorics 38 (1979) 96-156.

34. J. Nesetril and V. Rödl, "Combinatorial Partitions of Finite Posets and Lattices - Ramsey Lattices," Algebra Universalis (1983).

35. O. Ore, "Theory of Graphs," Colloquium Publication, Amer. Math. Soc. 38 (1962)

36. R. Rado, "Note on Combinatorial Analysis," Proc. London Math. Soc. 48 (1943) 122-160.

37. R. Rado, "A Theorem on Chains of Finite Sets, II," Acta Arith. 43 (1971) 257-261.

38. F.P. Ramsey, "On a Problem of Formal Logic," Proc. London Math. Soc. 30 (1930) 264-286.

39. I. Rival and B. Sands, "On the Adjacency of Vertices to the Vertices of an Infinite Subgraph," J. London Math. Soc. 21 (1980) 393-400.

40. B.L. Rothschild, "A Generalization of Ramsey's Theorem and a Conjecture of Rota," Doctoral Dissertation, Yale University, (1967).

41. I. Schur, "Uber die Kongruenz $x^m + y^m \equiv z^m$ (mod p)," Jahresbericht der D. Math. Verein 25 (1916) 114-117.

42. D. Scott and P. Suppes, "Foundational Aspects of Theories of Measurement," J. Symbolic Logic 23 (1958) 113-128.

43. E. Szpilrajn, "Sur l'entension de l'ordre Partiel," Fund. Math. 16 (1930) 386-389.

44. B. Toft, "On the Maximal Number of Edges of Critical k-chromatic Graphs," Studia Sci. Math. Hungar, 5 (1970) 461-470.

45. W.T. Trotter, "Dimension of the crown S_n^k," Discrete Math. 8 (1974a) 85-103.

46. W.T. Trotter, "Irreducible Posets with Large Height Exist, J. Combinatorial Theory (Ser. A) 17 (1974) 337-344.

47. W.T. Trotter, "Inequalities in Dimension Theory for Posets," Proc. Amer. Math. Soc. 47 (1975b) 311-316.

48. W.T. Trotter, "A Forbidden Subposet Characterization of an Order-dimension Inequality," Math. Systems Theory 10 (1976a) 91-96.

49. W.T. Trotter, "On the Construction of Irreducible Partially Ordered Sets," U.S.C. Technical Memo (1974).

50. W.T. Trotter, "Graphs and Partially Ordered Sets," in Selected Topics in Graph Theory II, L. Beineke and R. Wilson (eds.), Academic Press, (1983) 237-268.

51. W.T. Trotter and J.I. Moore, "The Dimension of Planar Posets," J. Combinatorial Theory (B) 22 (1976) 54-67.

52. W.T. Trotter and J.I. Moore, "Characterization Problems for Graphs, Partially Ordered Sets, Lattices, and Families of Sets," Discrete Math. 4 (1976a) 361-368.

53. W.T. Trotter and J.A. Ross, "Every t-irreducible Partial Order is a Suborder of a t+1-irreducible Partial Order," Annals of Discrete Math 17 (1983) 613-621.

54. W.T. Trotter and J.A. Ross, "For $t \geqq 3$, Every t-dimensional Partial Order is a Suborder of a t+1-irreducible Partial Order" Proceedings of the 1981 Hungarian Conference on Combinatorics and Graph Theory held at Eger, Hungary. To Appear.

55. B.L. van der Waerden, "Beweis einer Baudet'schen Vermutung," Nieuw Arch. Wisk. 15 (1927) 212-216.

56. D. West, W.T. Trotter, G.W. Peck, and P. Schor, "Regressions and Monotone Chains: A Ramsey-type Extremal Problem for Partial Orders," Combinatorics 4 (1984) 117-119.

57. M. Yannakakis, "On the Complexity of the Partial Order Dimension Problems," SIAM J. of Computing (1983).

ORDERED RANKED POSETS, REPRESENTATIONS OF INTEGERS AND INEQUALITIES FROM EXTREMAL POSET PROBLEMS

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ABSTRACT

Let L be the set of subsets of a finite set ordered by inclusion. Sperner found the maximum size of an antichain in L. Later Kruskal proved a theorem stronger than Sperner's result. It has many applications and yields representations of integers in terms of *cascades*. There are *cascade inequalities*, many of which can be proved using Daykin's algorithm, but the proofs of others are based on manipulation of binomial coefficients.

I will describe some applications of the above and also describe how both Leeb and Clement have proved a version of Kruskal's theorem for different generalizations of L. In each generalization we wish to solve the problems already solved in L, and this leads to an axiomatic approach.

I. Rival (ed.), Graphs and Orders, 395–412.