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A RAMSEY THEORETIC PROBLEM FOR FINITE ORDERED SETS

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1. Introduction

In this paper, we consider the following Ramsey theoretic problem for finite ordered sets:

> For each $n \geq 1$, what is the least integer $f(n)$ so that for every ordered set P of width n, there exists an ordered set Q of width $f(n)$ such that every 2-coloring of the points of Q produces a monochromatic copy of P?

Before presenting our results on this problem, we pause to introduce some basic notation and terminology and to make some observations concerning the background of this problem. Throughout the paper, we consider only finite ordered sets. If P is an ordered set and $x, y \in P$, we write $x \parallel y$ when x and y are incomparable. For a positive integer r, we let $\boldsymbol{r} = \{1, 2, \ldots, r\}$.

An *r-coloring* of an ordered set Q is a mapping $\phi : Q \to \boldsymbol{r}$ of the points of Q to a set of r elements. In this setting, the elements of $\boldsymbol{r}$ are called *colors*. When ϕ is an r-coloring of Q and $\alpha \in \boldsymbol{r}$, a subordered set P of Q such that $\phi(x) = \alpha$ for every $x \in P$ is called a monochromatic subordered set (of color α). In this paper, we are primarily interested in the case $r = 2$.

Accordingly, we write $Q \to P$ when every 2-coloring of Q produces a monochromatic copy of P, i.e., for every 2-coloring $\phi : Q \to \boldsymbol{2}$, there exists an $\alpha \in \boldsymbol{2}$ and a monochromatic subordered set P' of color α so that P' is isomorphic to P. To indicate that the statement that $Q \to P$ is false, we will write $Q \not\to P$.

Lemma 1. *For every ordered set P, there exists an ordered set Q so that $Q \to P$.*

Proof. Given an arbitrary ordered set P, consider the ordered set Q whose point

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set is $\{(x, y): x, y \in P\}$ and whose order is given by $(x_1, y_1) < (x_2, y_2)$ in Q if and only if ($x_1 < x_2$ in P) or ($x_1 = x_2$ and $y_1 < y_2$ in P). In other words, Q is obtained by replacing each point of P by a copy of P. Now consider any 2-coloring ϕ of Q. For each $x \in P$, the subordered set $P_x = \{(x, y): y \in P\}$ is isomorphic to P. If all points of any P_x are mapped by ϕ to color 1, then we have a monochromatic copy of P of color 1. So we may assume that for each $x \in P$, there exists $y_x \in P$ so that $\phi(x, y_x) = 2$. But this implies that $\{(x, y_x): x \in P\}$ is a monochromatic copy of P of color 2. □

With the existence question settled so easily, we can turn our attention to more delicate questions regarding specific combinatorial parameters. Three such parameters of interest are cardinality, length, and width. So that we can formulate all three problems simultaneously, for $i = 1, 2, 3$, we let $g_i(P)$ denote respectively the cardinality, length, and width of the ordered set P. Then for each $i = 1, 2, 3$, let $f_i(n)$ be the least positive integer so that for every ordered set P with $g_i(P) = n$, there exists an ordered set Q with $g_i(Q) = f_i(n)$ so that $Q \to P$.

Lemma 2. *For $i = 1, 2, 3$ and all $n \geq 1$, $2n - 1 \leq f_i(n) \leq n^2$.*

Proof. The substitution construction of Lemma 1 establishes the upper bound. For $i = 1$, the lower bound follows from the observation that if Q is an ordered set containing m points and $m < 2n - 1$, then we observe that $Q \not\to P$. To see this, consider a 2-coloring ϕ of Q which maps any subset of $\lceil \frac{1}{2}m \rceil$ points from P to color 1 and the subset consisting of the remaining $\lfloor \frac{1}{2}m \rfloor$ points to color 2.

For $i = 2$, given an ordered set Q of length $m < 2n - 1$, we can partition Q into m antichains $Q = A_1 \cup A_2 \cup \cdots \cup A_m$ and map the points in the first $\lceil \frac{1}{2}m \rceil$ antichains to color 1 and the remaining points to color 2. This 2-coloring does not produce a monochromatic chain of length n. The argument when $i = 3$ is dual. □

Given the similarity of the problems to this point, it is somewhat surprising that they suddenly diverge. First, we observe that $f_1(n)$ is on the order of n^2. To see this, consider the following example produced in collaboration with Saks and West.

Lemma 3. *Let P_m be the ordered set consisting of the disjoint sum of an m-element chain and an $m - 1$-element antichain. If $Q \to P_m$, then Q contains at least $m^2 + 2m - 2$ points.*

Proof. Let C_1 be a chain of maximum size in Q. Since the length of P is m and $Q \to P$, we know that $|C_1| \geq 2m - 1$. For each $i = 2, 3, \ldots, m - 1$, let C_i be a chain of maximum size in $Q - (C_1 \cup C_2 \cup \cdots \cup C_{i-1})$. Consider the 2-coloring of Q which maps all points in $C_1 \cup C_2 \cup \cdots \cup C_{m-1}$ to color 1 and the remaining points to color 2. The monochromatic copy of P_m produced by this 2-coloring must be of color 2. So the subordered set of Q mapped to color 2 has at least $2m-1$ points and length at least m. This implies that $|C_i| \geqslant m$ for $i = 1, 2, 3, \ldots, m-1$. Thus $|Q| \geqslant (2m-1) + (2m-1) + m(m-2)$ and $|Q| \geqslant m^2 + 2m - 2$. □

Theorem 1. *For each* $n \geqslant 2$, $\frac{1}{4}n^2 \leqslant f_1(n) \leqslant n^2 - n + 1$.

Proof. The lower bound follows from Lemma 3. The upper bound follows from the observation that in the substitution argument given in Lemma 1, we need only replace all points of P except one by a copy of P. □

It is a matter of mathematical taste as to whether Theorem 1 constitutes a complete solution to the problem of determining $f_1(n)$. However, there is no ambiguity concerning $f_2(n)$. The following theorem is due to Nešetřil and Rödl [1].

Theorem 2. *For each* $n \geqslant 1$, $f_2(n) = 2n - 1$, *i.e., for every ordered set P of length n, there exists an ordered set Q of length $2n-1$ so that every 2-coloring of Q produces a monochromatic copy of P.* □

The results of this paper will reveal the subtlety of the problem of determining $f(n) = f_3(n)$. As is the case in seveal other combinatorial problems involving antichains and chains in ordered sets, the problem involving width and partitions into chains is more complex than its dual counterpart.

2. The principal theorem

In this section, we will prove the following result.

Theorem 3. *For each* $n \geqslant 2$, $f(n) \geqslant 2n$, *i.e., there exists an ordred set P of width n so that for every ordered set Q of width $2n-1$, there exists a 2-coloring of Q which does not produce a monochromatic copy of P.*

Proof. For each $i \geqslant 1$ and each $n \geqslant 2$, let $P(i, n)$ denote the ordered set whose point set is the set of ordered pairs in the cartesian product $\boldsymbol{i} \times \boldsymbol{n}$. The ordering is defined by $(a, b) < (c, d)$ in $P(i, n)$ if and only if $(a+1 < c)$ or $(a < c$ and $b = d)$.

We then define a second ordered set $T(i, n)$ whose point set is also $\boldsymbol{i} \times \boldsymbol{n}$ by the rule $(a, b) < (c, d)$ in $T(i, n)$ if and only if $a < c$.

In the remainder of the argument, we take $P = P(31, n)$. For each $i = 1, 2, \ldots, 31$, the n-element antichain $A_i = \{(i, j) \colon 1 \leqslant j \leqslant n\}$ is called the ith *rank* of P.

Now let Q be an arbitrary ordered set of width $2n-1$. We will now proceed to construct a 2-coloring ϕ of Q which does not produce a monochromatic copy of P. We begin by choosing an arbitrary chain partition $Q = C_1 \cup C_2 \cup \cdots \cup C_{2n-1}$.

We next partition Q into subsets called *layers*. This partition will be denoted $Q = Q_1 \cup Q_2 \cup \cdots \cup Q_t$ and the integer t will count the number of layers in Q. Q_1 will be a down set in Q, and for each $i = 2, 3, \ldots, t$, Q_i will be a down set in $Q - (Q_1 \cup Q_2 \cup \cdots \cup Q_{i-1})$. (Recall that a subset S is a down set in P when $x \in S$ and $y < x$ imply $y \in S$.) The formal definition of this partitioning of Q into layers is given recursively:

S_1: Set $P_0 = Q$.

S_2: If $P_\beta \cap C_j = \emptyset$ for some $j \in \mathbf{2n-1}$, set $Q_{\beta+1} = P_\beta$; otherwise let $R_\beta = \{\min(P_\beta \cap C_j): j \in \mathbf{2n-1}\}$ and set $Q_{\beta+1} = \{x \in P_\beta$: there exists $y \in R_\beta$ with $x \not> y$ in $Q\}$.

S_3: Set $P_{\beta+1} = P_\beta - Q_{\beta+1}$ and return to S_2 if $P_{\beta+1} \neq \emptyset$. Else stop and set $t = \beta + 1$.

In the remainder of the argument, we will assume that the number t of layers is even. If the algorithm described above results in an odd value of t, we simply add to Q a $2n-1$-element antichain A with $a < x$ for every $x \in Q$ and every $a \in A$. We note some essential properties of this partitioning into layers in a sequence of claims.

Claim 1. If $x \in Q_\beta$, $y \in Q_\gamma$ and $x < y$ in Q, then $\beta \leqslant \gamma$.

Proof. The result follows immediately from the defining property S_2. □

Claim 2. If $x \in Q_\beta$, $y \in Q_\gamma$ and $\beta + 2 \leqslant \gamma$, then $x < y$ in Q.

Proof. From S_2, we observe that $y > z$ for every $z \in R_\beta$, else $y \in Q_{\beta+1}$. Among the $2n-1$ points in R_β, there is one, say z_j, which comes from the same chain as x. We then have $y > z_j > x$ in Q. □

Claim 3. If $1 \leqslant \beta < t$ and S is an isomorphic image of $T(4, n)$ in $Q_\beta \cup Q_{\beta+1}$, then one of $S \cap Q_\beta$ and $S \cap Q_{\beta+1}$ contains an isomorphic image of $T(2, n)$.

Proof. Let $h: T(4, n) \rightarrow S$ be an embedding. If $h(2, j) \in Q_{\beta+1}$ for some $j \in \mathbf{n}$, then $\{h(i, j): 3 \leqslant i \leqslant 4,\ j \in \mathbf{n}\}$ is isomorphic to $T(2, n)$ and is contained in $Q_{\beta+1}$; otherwise $\{h(i, j): 1 \leqslant i \leqslant 2, j \in \mathbf{n}\}$ is isomorphic to $T(2, n)$ and is contained in Q_β. □

Next, for each $\beta = 1, 2, \ldots, t$, we define a graph G_β whose vertex set is the set $\mathbf{2n-1}$. In G_β, a pair $\{j, k\}$ of distinct integers from $\mathbf{2n-1}$ is an edge if and only if $(C_j \cup C_k) \cap Q_\beta$ contains an isomorphic image of $T(2, 2)$.

Claim 4. For each $\beta \in \boldsymbol{t}$ and each $j \in \mathbf{2n-1}$, there exists $k \in \mathbf{2n-1}$ with $k \neq j$ so that $\{j, k\}$ is not an edge in G_β.

Proof. Suppose to the contrary that there exists $\beta \in \boldsymbol{t}$ and $j \in \mathbf{2n-1}$ so that $\{j, k\}$ is an edge in G_β for every $k \in \mathbf{2n-1}$ with $k \neq j$. For each such k, choose a copy of $T(2, 2)$ contained in $(C_j \cup C_k) \cap Q_\beta$. Of the four points in this copy of $T(2, 2)$, two belong to $C_j \cap Q_\beta$ and the other two belong to $C_k \cap Q_\beta$. Let z_k be the larger of the two points in $C_j \cap Q_\beta$. Then let $z_0 = \max\{z_k : k \neq j\}$. It follows that $z_0 > x$ for every $x \in R_\beta$ and thus by S_2, we would conclude that $z_0 \in Q_{\beta+1}$. □

We now group the layers into consecutive pairs. For each $\beta = 1, 2, \ldots, \frac{1}{2}t$, we define the βth *section* of Q, denoted S_β, by $S_\beta = Q_{2\beta-1} \cup Q_{2\beta}$.

Claim 5. Let β satisfy $1 \leq \beta \leq \frac{1}{2}t$. Then there exists a subset $W_\beta \subset \mathbf{2n-1}$ so that:

(1) $|W_\beta| = n$;

(2) $T_\beta = \{x \in S_\beta : x \in C_j \text{ and } j \in W_\beta\}$ does not contain an isomorphic copy of $T(4, n)$.

Proof. We first assume $n \geq 3$. In this case, we apply Claim 4 twice to choose integers k_1, k_2 (not necessarily distinct) with $1 \neq k_1$ and $1 \neq k_2$ so that $\{1, k_1\}$ is not an edge in $G_{2\beta-1}$ and $\{1, k_2\}$ is not an edge in $G_{2\beta}$. We then let W_β be any n-element subset of $\mathbf{2n-1}$ containing 1, k_1, and k_2, and let $T_\beta = \{x \in S_\beta : x \in C_j$ and $j \in W_\beta\}$.

To show that (2) is valid, we observe that if (2) is violated, then by Claim 3 either $S \cap Q_{2\beta-1}$ or $S \cap Q_{2\beta}$ contains a copy of $T(2, n)$. This would imply that either W_β is a complete subgraph of $G_{2\beta-1}$ or W_β is a complete subgraph of $G_{2\beta}$. But neither of these statements is true.

Now consider the case $n = 2$. It follows from Claim 4 that $G_{2\beta-1}$ contains a vertex of degree zero so we may choose $j_0 \in \mathbf{3}$ so that $\{j_0, k\}$ is not an edge in $G_{2\beta-1}$ for all $k \in \mathbf{3}$. Then choose $k_0 \in \mathbf{3}$ with $k_0 \neq j_0$ so that $\{j_0, k_0\}$ is not an edge in $G_{2\beta}$ and set $W_\beta = \{j_0, k_0\}$. The remainder of the argument is the same as before. □

We are now ready to define the 2-coloring ϕ of Q. First, we apply Claim 5 to define the n-element sets W_β for $\beta = 1, 2, \ldots, \frac{1}{2}t$. We then set $T_\beta = \{x \in S_\beta : x \in C_j$ and $j \in W_\beta\}$ and $T'_\beta = S_\beta - T_\beta$ for $\beta = 1, 2, \ldots, \frac{1}{2}t$. Finally, define:

$$\phi(x) = \begin{cases} 1 & \text{if } x \in T_\beta \text{ and } \beta \text{ is odd,} \\ 1 & \text{if } x \in T'_\beta \text{ and } \beta \text{ is even,} \\ 2 & \text{if } x \in T_\beta \text{ and } \beta \text{ is even,} \\ 2 & \text{if } x \in T'_\beta \text{ and } \beta \text{ is odd.} \end{cases}$$

To complete the proof, we must show that this 2-coloring of Q does not produce a monochromatic copy of P. Suppose to the contrary that $\alpha \in \{1, 2\}$ and that $h: P \to Q$ is an embedding of P onto a monochromatic copy of P of color α.

Claim 6. For each $\beta = 1, 2, \ldots, \frac{1}{2}t$, there is no monochromatic copy of $T(4, n)$ contained in S_β.

Proof. This is an immediate consequence of Claim 5 and the definition of ϕ. □

Claim 7. Let i, j satisfy $1 \leq i < j \leq 31$ and $i + 10 \leq j$. Then let β be an integer with $1 \leq \beta \leq \frac{1}{2}t$. If $h(A_i) \cap S_\beta \neq \emptyset$, then $h(A_j) \cap S_\beta = \emptyset$.

Proof. Choose $x_1 \in A_i$ so that $h(x_1) \in S_\beta$. Suppose there exists a point $x_2 \in A_j$ so that $h(x_2) \in S_\beta$. Then consider the subset T of P consisting of all points from ranks A_{i+2}, A_{i+4}, A_{i+6}, A_{i+8}. It is clear that $h(T)$ is a monochromatic copy of $T(4, n)$ contained in S_β. The contradiction completes the proof. □

Claim 8. Let i satisfy $1 \leq i < 31$ and let $x \in A_i$. Then let $y \in A_{i+1}$ satisfy $x < y$ in P. If $h(x) \in Q_\beta$ and $h(y) \in Q_\gamma$, then:

(1) $\gamma \leq \beta + 2$;

(2) If β is odd and the width of the subordered set in $Q_\beta \cup Q_{\beta+1}$ determined by the points of color α is $n - 1$, then $\gamma \leq \beta + 1$.

Proof. We first establish (1). Suppose to the contrary that $\gamma > \beta + 2$ and choose $z \in A_i$ with $z \neq x$. Then $x \parallel z$ and $y \parallel z$ in P. Choose δ so that $h(z) \in Q_\delta$. Then $\delta \leq \beta + 1$, else $h(x) < h(z)$ by Claim 2. However, Claim 2 now implies $h(z) < h(y)$. The contradiction establishes (1).

We now prove (2). Suppose to the contrary that the conditions of the hypothesis of (2) are satisfied but the conclusion is not, i.e., assume that $\gamma = \beta + 2$. The argument above shows that $h(z) \in Q_{\beta+1}$ for every $z \in A_i$ with $z \neq x$. However, this implies that $h(A_i) \subset Q_\beta \cup Q_{\beta+1}$, and thus the points of color α in $Q_\beta \cup Q_{\beta+1}$ contain an n-element antichain. The contradiction completes the proof of the claim. □

Claim 9. There exists an integer β with $1 \leq \beta \leq \frac{1}{2}t$, an integer i with $11 \leq i \leq 31$, an an element $x \in A_i$ so that:

(1) $h(x) \in Q_{2\beta}$;

(2) The width of the subordered set of S_β determined by the points of color α is $n - 1$.

Proof. Choose γ so that $h(11, 1) \in S_\gamma$. Suppose first that the width of the subordered set of S_γ determined by the points of color α is $n - 1$. If $h(11, 1) \in Q_{2\gamma}$, we are done, so we can assume that $h(11, 1) \in Q_{2\gamma-1}$. By Claim 7,

$h(21, 1) \notin Q_{2\gamma-1}$. Choose the least i so that $11 < i \leq 21$ and $h(i, 1) \notin Q_{2\gamma-1}$. By Claim 8, $h(i, 1) \in Q_{2\gamma}$.

So we can now assume that the width of the subordered set of S_γ determined by the points of color α is n. Choose the least i with $11 < i < 21$ so that $h(i, 1) \notin S_\gamma$. By Claim 8, $h(i, 1) \in S_{\gamma+1}$. If $h(i, 1) \in Q_{2\gamma+2}$, we are done. So we may assume $h(i, 1) \in Q_{2\gamma+1}$. Choose the least j so that $i < j \leq 31$ and $h(j, 1) \notin Q_{2\gamma+1}$. As before, Claim 8 implies that $h(j, 1) \in Q_{2\gamma+2}$. □

We are now ready to obtain the final contradiction. Let β and i be the least positive integers satisfying the conclusion of Claim 9. Then choose the least j so that $h(A_j) \cap Q_{2\beta} \neq \emptyset$. By Claim 7, we know that $j > 1$. Then choose $x = (j, k) \in A_j$ so that $h(x) \in Q_{2\beta}$. It follows that $h(z) \in Q_{2\beta-1}$ for every $z \in A_{j-1}$ with $z \neq (j-1, k)$. However, this implies that the image of $\{x\} \cup (A_{j-1} - \{(j-1, k)\})$ is a monochromatic n-element antichain of color α in S_β. The contradiction completes the proof of our theorem. □

3. Concluding remarks

The reader may note that there appears to be hope for improving the lower bound on $f(n)$ given in Theorem 3. In particular, for large values of n, there appears to be great freedom in the selection of the set W_β in Claim 5. However, we have been unsuccessful to date in our efforts to take advantage of this freedom, and we hesitate to conjecture the correct order of magnitude of $f(n)$.

Reference

[1] J. Nešetřil and V. Rödl, Combinatorial partitions of finite posets and lattices—Ramsey lattices, Algebra Universalis 19 (1984) 106–119.