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Representing an Ordered Set as the Intersection of Super Greedy Linear Extensions

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Abstract. A linear extension $[x_1 < x_2 < \cdots < x_t]$ of a finite ordered set $\mathscr{P} = (P, <)$ is super greedy if it can be obtained using the following procedure: Choose x_1 to be a minimal element of $\mathscr{P}$; suppose $x_1, \ldots, x_i$ have been chosen; define $p(x)$ to be the largest $j \leqslant i$ such that $x_j < x$ if such a j exists and 0 otherwise; choose x_{i+1} to be a minimal element of $P - \{x_1, \ldots, x_i\}$ which maximizes p. Every finite ordered set $\mathscr{P}$ can be represented as the intersection of a family of super greedy linear extensions, called a super greedy realizer of $\mathscr{P}$. The super greedy dimension of $\mathscr{P}$ is the minimum cardinality of a super greedy realizer of $\mathscr{P}$. Best possible upper bounds for the super greedy dimension of $\mathscr{P}$ are derived in terms of $|P - A|$ and width $(P - A)$, where A is a maximal antichain.

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1. Introduction

It is well known that every ordered set $\mathscr{P}$ can be represented as the intersection of some family of linear orders, each of which is a linear extension of $\mathscr{P}$. Such a family is called a *realizer* of $\mathscr{P}$. The *dimension* of the ordered set $\mathscr{P}$, denoted by $\dim(\mathscr{P})$, is the minimum size of a realizer of $\mathscr{P}$. The concept of dimension has proved to be a useful invariant in the study of ordered sets. The reader is referred to Kelly and Trotter [5] for a survey of this subject. A natural extension of the concept of dimension arises from requiring the linear extensions that form a realizer to have certain additional properties. Let C be a class of linear extensions of $\mathscr{P}$. A family Σ of linear extensions of $\mathscr{P}$ is a C *realizer* of $\mathscr{P}$ if Σ is a

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realizer of $\mathscr{P}$ and every linear extension in Σ comes from C. The *C dimension* of $\mathscr{P}$, denoted $\dim_C(\mathscr{P})$ is defined to be the minimum size of a C realizer of $\mathscr{P}$ if one exists and is undefined otherwise. A comparison between the dimension and the C dimension of an ordered set can be used as a measure of the richness of the class C. Several authors, including Bouchitté *et al.* [2], Kierstead and Trotter [7], and Rival and Zaguia [11], have studied the G, or greedy, dimension of ordered sets $\mathscr{P}$, where G is the class of greedy linear extensions of $\mathscr{P}$. In this article we investigate the SG, or super greedy, dimension of ordered sets $\mathscr{P}$, where SG is the class of super greedy linear extensions of $\mathscr{P}$. We pause now to define the notions of greedy and super greedy linear extension.

Let $\mathscr{P} = (P, <)$ be a finite ordered set on t elements. For a subset S of P we denote the set of minimal elements of P restricted to S by MIN(S). The open upset of S is $U(S) = \{x \in P : s < x \text{ for some } s \in S\}$. If $S = \{s\}$ we may write $U(s)$ for $U(S)$. Consider the following (non-deterministic) algorithm for constructing a linear ordering $[x_1, \ldots, x_t]$ on P.

ALGORITHM LIN
SET $R = P$, $M = \text{MIN}(R)$
FOR $i = 0, \ldots, t-1$
  CHOOSE $x_{i+1} \in M$
  SET $R = R - \{x_{i+1}\}$, $M = \text{MIN}(R)$
END

For any sequence of choices of the points x_{i+1}, algorithm LIN produces a linear extension of $\mathscr{P}$; and every linear extension of $\mathscr{P}$ is obtained from LIN by a suitable sequence of choices of the points x_{i+1}. We can obtain a more restrictive class of linear extensions by further restricting the choice of the x_{i+1}. For example a *greedy* linear extension of $\mathscr{P}$ is a linear extension of $\mathscr{P}$ which is obtained from algorithm LIN together with the additional tie breaking rule:

T_1: Prefer elements which cover x_{i+1-1} for x_{i+1}.

More precisely, a linear extension of $\mathscr{P}$ is *greedy* iff it is obtained from the following algorithm GREEDY by a suitable sequence of choices of the points x_{i+1}.

ALGORITHM GREEDY
SET $R = P$, $M = \text{MIN}(R)$, $G = M$
FOR $i = 0, \ldots, t-1$
  CHOOSE $x_{i+1} \in G$
  SET $R = R - \{x_{i+1}\}$, $M = \text{MIN}(R)$
  IF $M \cap U(x_{i+1}) \neq \emptyset$
    THEN SET $G = M \cap U(x_{i+1})$
    ELSE SET $G = M$
END

Note that since G is always a nonempty subset of M, we do indeed get a linear extension of $\mathscr{P}$.

In this paper we are concerned with the even more restrictive class of super greedy linear extensions of $\mathscr{P}$. A greedy linear extension of $\mathscr{P}$ is *super greedy* if it is obtained from algorithm GREEDY together with the additional tie breaking rules:

T_2: Prefer elements which cover x_{i+1-2} for x_{i+1}.

⋮

T_t: Prefer elements which cover x_{i+1-t} for x_{i+1}.

Such a tie breaking scheme is implemented according to its order, i.e., T_i has higher priority than T_j if $i<j$. More precisely, a linear extension of $\mathscr{P}$ is super greedy iff it is obtained from the following algorithm SUPER GREEDY by a suitable sequence of choices of the points x_{i+1}.

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ALGORITHM SUPER GREEDY
SET R = P,   M = MIN(R),   SG = M
FOR i = 0, ..., t − 1
   CHOOSE x_{i+1} ∈ SG
   SET R = R − {x_{i+1}},   M = MIN(R),   j = i
   WHILE M ∩ U(x_j) = ∅   AND   j ≠ 0   DO SET j = j − 1
   IF j ≠ 0
      THEN SET SG = M ∩ U(x_j)
      ELSE SET SG = M
END
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These definitions are illustrated on the ordered set $\mathscr{P}$ and three of its linear extensions shown in Figure 1. The linear extension $\mathscr{L}_1$ is not greedy because $c=x_3$ is not an element of $G=\{d, e\}$ at stage $i=2$. The linear extension $\mathscr{L}_2$ is greedy, but is not super greedy. Note that $c=x_4$ is an element of $G=M=\{e, c\}$, but is not an element of $SG=\{e\}$, at stage $i=3$. The linear extension $\mathscr{L}_3$ is super greedy.

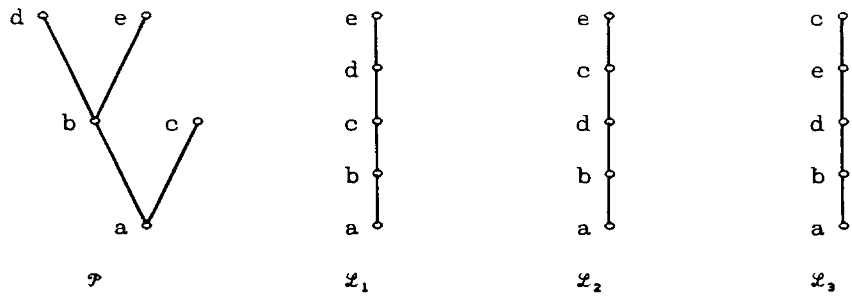

Fig. 1. Of the linear extensions $\mathscr{L}_1$, $\mathscr{L}_2$, and $\mathscr{L}_3$ of $\mathscr{P}$, $\mathscr{L}_1$ is not greedy, $\mathscr{L}_2$ is greedy but not super greedy, and $\mathscr{L}_3$ is super greedy.

The original motivation for studying greedy linear extensions stemmed from their connection with the jump number problem. However, Bouchitté *et al.* [2] observed that every ordered set has a greedy realizer and proceeded to identify several classes of ordered sets for which the greedy dimension was the same as the dimension. So it was natural to further investigate greedy dimension. Kierstead and Trotter [7] obtained best possible bounds for the greedy dimension of $\mathscr{P}$ in terms of both the cardinality and the width of $P - A$, where A is a maximal antichain. Super greedy linear extensions were introduced by Pretzel [10] as a natural restriction of the notion of greedy linear extension. He observed that every ordered set has a super greedy realizer and in fact $\dim_{SG}(\mathscr{P}) \leqslant \text{width}(\mathscr{P})$. Pretzel asked whether Kierstead and Trotter's work on greedy dimension could be extended to super greedy dimension. This paper answers Pretzel's question. It was later observed by Golumbic [3] and others that super greedy linear extensions are closely related to depth first search. If one adds an artificial least element to $\mathscr{P}$ and then does a depth first search starting at this element, and records each element of P the last time it is visited during this search, then one obtains a super greedy linear extension of $\mathscr{P}$ in reverse order; moreover all super greedy linear extensions of $\mathscr{P}$ can be obtained in this way. The reader is referred to Kierstead and Trotter [8] and Bouchitte *et al.* [1] for a general introduction to super greedy linear extensions. Some problems involving super greedy linear extensions with constraints are shown to be NP-complete in Kierstead [6].

In the remainder of this section we shall review our notation. The principal theorems are introduced in Section 2. The rest of the paper is devoted to their proofs.

Let $\mathscr{P} = (P, <)$ be a fixed finite ordered set and let S be a subset of P. Then the set of maximal elements of $\mathscr{P}$ restricted to S is denoted by $\text{MAX}(S)$. If S is a chain $\max(S)$ denotes the largest element of S and $\min(S)$ denotes the smallest element of S. The *closed upset* of S is

$$U[S] = \{x \in P : s \leqslant x \text{ for some } s \in S\};$$

the *closed downset* of S is

$$D[S] = \{x \in P : x \leqslant s \text{ for some } s \in S\};$$

and the *open upset* of S is

$$D(S) = \{x \in P : x < s \text{ for some } s \in S\}.$$

In the case that $S = \{s\}$ we may write s instead of S in each of these notations. We write $x < S$ if $x < s$ for all $s \in S$ and $S < x$ if $s < x$ for all $s \in S$. We denote the cover relations by $<:$, i.e., $x <: y$ iff $x < y$ and there does not exist $z \in P$ such that $x < z < y$. We write $x \parallel y$ if x is incomparable to y and $x \mid y$ if x is comparable to y. If $\mathscr{L} = (P, <)$ is a linear extension of $\mathscr{P}$, we write $x < y$ in $\mathscr{L}$ for $x <_{\mathscr{L}} y$. We write $\mathscr{L}$-$\max(S)$ for the greatest element of S in the linear order $\mathscr{L}$ and $\mathscr{L}$-

min(S) has a similar meaning. For T a subset of P we say that T is over S in $\mathscr{L}$, denoted T/S, if whenever $t \in T$, $s \in S$, and $t \parallel s$, then $s < t$ in $\mathscr{L}$.

2. The Principal Theorems

The primary purpose of this paper is to develop inequalities for the super greedy dimension of an ordered set $\mathscr{P}$ in terms of the cardinality and width of $P - A$, where A is a maximal antichain in $\mathscr{P}$. This is a continuation of work begun by Hiraguchi [4] and Trotter [12] for ordinary dimension and continued by Kierstead and Trotter [7] for greedy dimension. For the rest of this paper A will denote a maximal antichain in the ordered set under consideration, usually $\mathscr{P}$, U will be the open upset of A, and D will be the open downset of A.

The next theorem extends a result of Trotter [12] and Kimble [9] for ordinary dimension.

THEOREM A (Kierstead and Trotter [7]). *Let A be a maximal antichain in an ordered set $\mathscr{P} = (P, <)$ such that $|P - A| \geqslant 2$. Then $\dim_G(\mathscr{P}) \leqslant |P - A|$.* □

The situation is considerably more complicated for super greedy dimension. The following theorem details the possibilities.

THEOREM 1. *Let A be a maximal antichain in an ordered set $\mathscr{P} = (P, <)$ such that $|P - A| \geqslant 2$. Let $D = D(A)$, $U = U(A)$, $m = |D|$, and $n = |U|$. Then each of the following inequalities* (i) *is true and* (ii) *is best possible.*

(a) *If $D \neq \emptyset$ and $U \neq \emptyset$, then $\dim_{SG}(\mathscr{P}) \leqslant m \cdot n + 1$.*
(b) *If $A = MIN(P)$, then $\dim_{SG}(\mathscr{P}) \leqslant n$.*
(c) *If $A = MAX(P)$, then $\dim_{SG}(\mathscr{P}) \leqslant m$.* □

Next we consider the effect of the parameter width$(P - A)$. For ordinary dimension we have

THEOREM B (Trotter [12]). *Let A be a maximal antichain in an ordered set $\mathscr{P} = (P, <)$ such that $width(P - A) = n \geqslant 1$. Then each of the following inequalities* (i) *is true and* (ii) *is best possible.*

(a) *$\dim(\mathscr{P}) \leqslant 2n + 1$.*
(b) *If $A = MIN(P)$, then $\dim(\mathscr{P}) \leqslant n + 1$.*
(c) *If $A = MAX(P)$, then $\dim(\mathscr{P}) \leqslant n + 1$.* □

For greedy dimension the situation is:

THEOREM C (Kierstead and Trotter [7]). *Let A be a maximal antichain in an ordered set $\mathscr{P} = (P, <)$ such that $width(P - A) = n \geqslant 1$. Then each of the following inequalities* (i) *is true and* (ii) *is best possible.*

(a) *$\dim_G(\mathscr{P}) \leqslant \max\{3, n^2 + n\}$.*
(b) *If $A = MIN(P)$ then $\dim_G(\mathscr{P}) \leqslant \max\{2, 2n - 1\}$.*

(c) *If* $A = MAX(P)$ *then* $dim_G(\mathscr{P}) \leqslant n+1$. □

The next theorem fully describes the situation for super greedy dimension.

THEOREM 2. (a) *For every positive integer* m *there exists an ordered set* $\mathscr{P} = (P, <)$ *with a maximal antichain* A *such that* $width(P-A) = 1$, *but* $dim_{SG}(\mathscr{P}) = m$.

Let A *be a maximal antichain in an ordered set* $\mathscr{P} = (P, <)$ *such that* $width(P - A) = n \geqslant 1$. *Then each of the following inequalities* (i) *is true and* (ii) *is best possible.*

(b) *If* $A = MIN(P)$ *then* $dim_{SG}(\mathscr{P}) \leqslant 2n$.
(c) *If* $A = MAX(P)$ *then* $dim_{SG}(\mathscr{P}) \leqslant n+1$. □

In the next section we will provide the examples which show the upper bounds of Theorems 1 and 2 are best possible. This will enable the reader to gain some facility in working with the notion of super greedy realizer. In Section 4 we lay the groundwork for proving the upper bounds of Theorems 1 and 2. In particular, we establish some technical lemmas which will provide us with super greedy linear extensions possessing special properties. In Section 5 these super greedy linear extensions will be used to construct realizers witnessing the upper bounds of Theorems 1 and 2.

3. Examples

Before beginning our constructions we note that for any ordered set $\mathscr{P}$, $\dim(\mathscr{P}) \leqslant \dim_{SG}(\mathscr{P})$. An ordered set of the type shown in Figure 2 is called the standard example of width n. It is well known to have dimension n. Thus we have:

Proof of Theorems 1.b.ii, 1.c.ii, and 2.c.ii. By the above observation these results follow immediately from the standard example and Theorem B.c.ii. □

Proof of Theorem 1.a.ii. For any positive integers m and n we must construct an ordered set $\mathscr{P} = (P, <)$ with a maximal antichain A such that

$$|D| = m, \qquad |U| = n \quad \text{and} \quad \dim_{SG}(\mathscr{P}) \geqslant m \cdot n + 1.$$

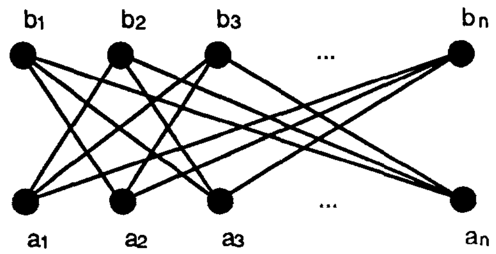

Fig. 2. The standard example of width n.

Let $P = D \cup A \cup U$, where

$$D = \{d_1, \ldots, d_m\}, \qquad A = \{x, y\} \cup \{a_{i,j} : i = 1, \ldots, m \text{ and } j = 1, \ldots, n\},$$
$$U = \{u_1, \ldots, u_n\}$$

are all antichains of $\mathscr{P}$. Let $\mathscr{P}$ be the transitive closure of the following cover relations, where $i = 1, \ldots, m$ and $j, k = 1, \ldots, n$:

(i) $d_i <: a_{i,j} <: u_k$ iff $k \neq j$;
(ii) $x <: u_j$; and
(iii) $d_i <: y$.

Figure 3 shows $\mathscr{P}$ if $m = 2$ and $n = 3$.

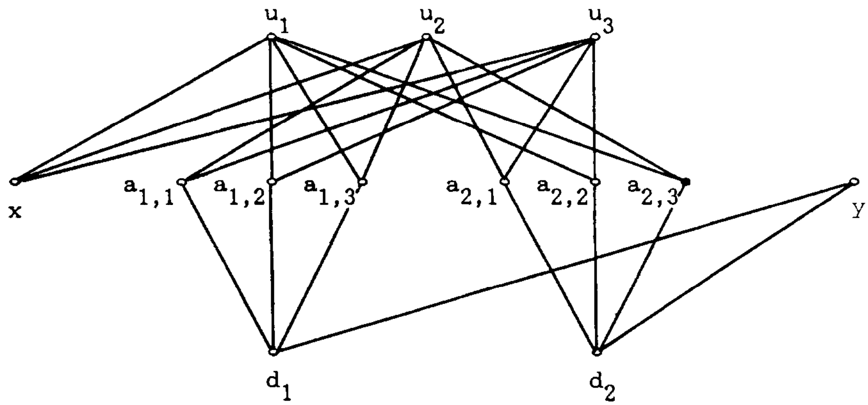

Fig. 3. An ordered set $\mathscr{P}$ such that $|D| = 2$, $|U| = 3$, and $\dim_{SG}(\mathscr{P}) \geqslant 2 \cdot 3 + 1$.

Next we show that $\dim_{SG}(\mathscr{P}) \geqslant m \cdot n + 1$. Each of the following conditions must be met by some linear extension $\mathscr{L}$ in any realizer of $\mathscr{P}$:

$$(C_{i,j})\colon u_j < a_{i,j} \text{ in } \mathscr{L} \text{ for } i = 1, \ldots, m \text{ and } j = 1, \ldots, n; \text{ and } (\overline{C})\colon y < x \text{ in } \mathscr{L}.$$

Thus it suffices to show that at most one of these conditions can be met by any super greedy linear extension $\mathscr{L}$ of $\mathscr{P}$. Fix such an $\mathscr{L}$. First notice that for all $i = 1, \ldots, m$ and $k = 1, \ldots, n$:

(*) if $d_i < d_j$ in $\mathscr{L}$ then $a_{i,k} < d_j$ in $\mathscr{L}$; and
(**) if $d_i < x$ in $\mathscr{L}$ then $a_{i,k} < x$ in $\mathscr{L}$.

This is because once d_i is chosen during algorithm SUPER GREEDY each $a_{i,k}$ is in M and has higher priority under the tie breaking scheme than either d_j or x.

If $\mathscr{L}$ satisfies $(\overline{C})$ then $D < y < x < U$ in $\mathscr{L}$. Thus, by (**) $A - \{x, y\} < x$ in $\mathscr{L}$ and no condition $(C_{i,j})$ is satisfied by $\mathscr{L}$. Now suppose $\mathscr{L}$ satisfies $(C_{i,j})$ and consider another condition $(C_{h,k})$. If $i \neq h$ we have $\{d_h, d_i\} < u_j < a_{i,j}$ in $\mathscr{L}$ and $d_i < u_k$. By (*), $d_h < d_i$ in $\mathscr{L}$. Thus, using (*) again $a_{h,k} < d_i < u_k$ in $\mathscr{L}$ and $(C_{h,k})$

fails. Otherwise $i=h$ and $k \neq j$. But then $a_{h,k} < u_j < a_{i,j} < u_k$ in $\mathscr{L}$ and again $(C_{h,k})$ fails. We conclude that $\dim_{SG}(\mathscr{P}) > m \cdot n + 1$. □

Proof of Theorem 2.a. Fix a positive integer m. We must construct an ordered set $\mathscr{P} = (P, <)$ with an antichain A such that width$(P - A) = 1$ and $\dim_{SG}(\mathscr{P}) \geqslant m$. Let $P = C \cup A$, where $C = \{d_1 < d_2 < \cdots d_m < u_1 < u_2 < \cdots < u_m\}$ is a chain in $\mathscr{P}$ and $A = \{a_1, a_2, \ldots, a_m, u_1\}$ is a maximal antichain in $\mathscr{P}$ such that $d_i <: a_i <: u_{i+1}$ for $i = 1, \ldots, m-1$ and $d_m <: a_m$. Figure 4 shows $\mathscr{P}$ if $m = 3$.

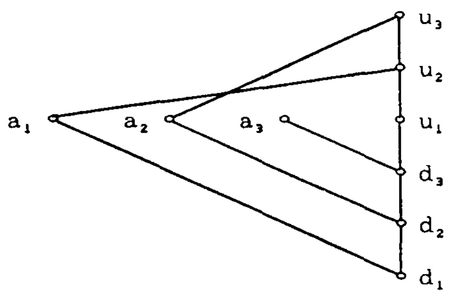

Fig. 4. An ordered set $\mathscr{P}$ with maximal antichain A such that width$(P - A) = 1$ and $\dim_{SG}(\mathscr{P}) = 3$.

We must show that $\dim_{SG}(\mathscr{P}) \leqslant m$. Each of the following conditions must be met by some linear extension $\mathscr{L}$ in any realizer of $\mathscr{P}$:

(C_i) $u_i < a_i$ in $\mathscr{L}$ for $i = 1, \ldots, m$.

Thus it suffices to show that at most one of these conditions can be met by any super greedy linear extension $\mathscr{L}$ of $\mathscr{P}$. Fix such an $\mathscr{L}$. We will show that (C_i) implies not (C_j) for $1 \leqslant i < j \leqslant m$. First notice that for all $i = 1, \ldots, m-1$:

$(*)$ if $d_{i+1} < a_i$ in $\mathscr{L}$ then $a_j < a_i$ in $\mathscr{L}$.

This is because once d_{i+1} is chosen a_{i+1} and d_{i+2} are in M and have higher priority in the tie breaking scheme than a_i. So $\{a_{i+1}, d_{i+2}\} < a_i$ in $\mathscr{L}$. But after d_{i+2} is chosen a_{i+2} and d_{i+3} are in M and have higher priority than a_i, etc. Now suppose $\mathscr{L}$ satisfies (C_i). Then $d_{i+1} < u_i < a_i$ in $\mathscr{L}$. Thus by $(*)$ $a_j < a_i < u_j$ in $\mathscr{L}$ and (C_j) is false for $j > i$. This clearly implies that at most one of the conditions is satisfied by $\mathscr{L}$. We conclude that $\dim_{SG}(\mathscr{P}) \geqslant m$. □

Proof of Theorem 2.b.ii. Fix a positive integer n. We must construct an ordered set $\mathscr{P} = (P, <)$ such that width$(P - \text{MIN}(P)) = n$ and $\dim_{SG}(\mathscr{P}) \geqslant 2n$. The ordered set $\mathscr{P}$ consists of four layers:

$$\text{MIN}(P) = \{a_1, a_1', a_2, a_2', \ldots, a_n, a_n'\}, \qquad X = \{x_1, \ldots, x_n\},$$
$$Y = \{y_1, \ldots, y_n\} \quad \text{and} \quad \text{MAX}(P) = \{z_1, \ldots, z_n\}.$$

$\mathscr{P}$ is the transitive closure of the following cover relations, where $i, j = 1, \ldots, n$:

(i) $a_i' <: x_i <: y_i <: z_i$;
(ii) $a_i <: z_i$;
(iii) $a_i <: y_j$ if $i \neq j$; and
(iv) $a_i' <: y_j$ if $i \neq j$.

Figure 5 shows $\mathscr{P}$ in the case $n = 3$.

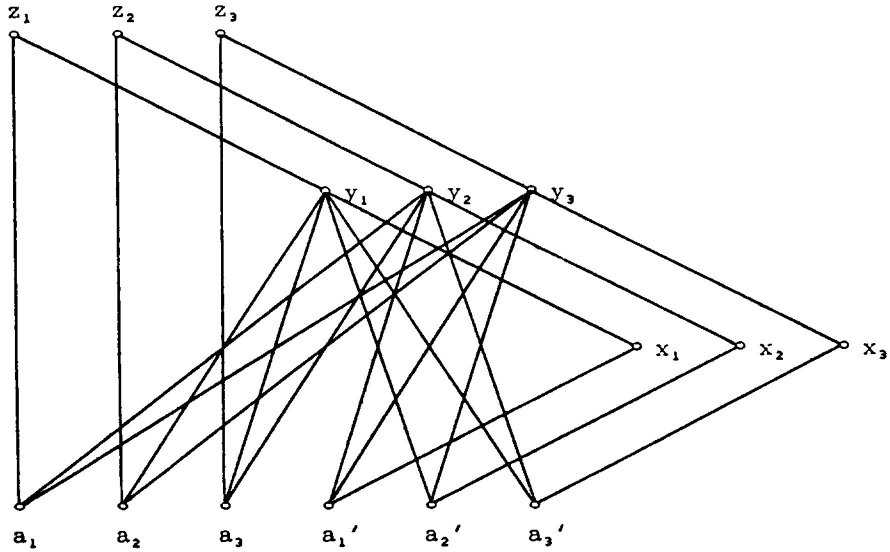

Fig. 5. An ordered set $\mathscr{P}$ such that width$(P - \text{MIN}(P)) = 3$ and $\dim_{SG}(\mathscr{P}) \geqslant 2 \cdot 3$.

Now we show that $\dim_{SG}(\mathscr{P}) \geqslant 2n$. This is surely the case if $n = 1$ so assume $n > 1$. Each of the following conditions must be met by some linear extension $\mathscr{L}$ in any realizer of $\mathscr{P}$, where $i = 1, \ldots, n$ and $\oplus$ is addition modulo n:

(C_i) $y_i < a_i$ in $\mathscr{L}$; and
(D_i) $z_{i \oplus 1} < x_i$ in $\mathscr{L}$.

Thus it suffices to show that at most one of these conditions is satisfied by any super greedy linear extension $\mathscr{L}$ of $\mathscr{P}$. In fact it is clear that any linear extension that satisfies (C_i) cannot satisfy any of the other conditions. Now let $\mathscr{L}$ be a super greedy linear extension. Then for $i, j = 1, \ldots, n$:

$(*)$ if $a_i' < a_j'$ in $\mathscr{L}$ then $x_i < a_j'$ in $\mathscr{L}$.

This is because once a_i' is chosen $x_i \in M$ and has higher priority in the tie breaking scheme than a_j'. Thus if $\mathscr{L}$ satisfies (D_i) then $a_j' < a_i'$ for all $j \neq i$, since otherwise $x_i < a_j' < z_{i \oplus 1}$ in $\mathscr{L}$. It follows that if $\mathscr{L}$ satisfies (D_i) then $\mathscr{L}$ does not satisfy any of the other conditions. We conclude that $\dim_{SG}(\mathscr{P}) \geqslant 2n$.

4. Super Greedy Linear Extensions

We begin this section by introducing some concepts that will simplify the construction of super greedy realizers in Section 5. Later we will prove the existence of super greedy linear extensions with special properties in which these concepts are incorporated.

Let $\mathscr{P}=(P,<)$ be an ordered set and let $\mathscr{F}=\{\mathscr{Q}_x : x\in P\}$ be a family of ordered sets indexed by the points of P, where $\mathscr{Q}_x=(Q_x,<_x)$. The *ordinal sum* of $\mathscr{F}$ over $\mathscr{P}$, denoted $\Sigma_{\mathscr{P}}\mathscr{F}$, is the ordered set $\mathscr{R}=(R,\prec)$, where $R=\{(x,y): x\in P \text{ and } y\in Q_x\}$ and $(x,y)\prec(x',y')$ iff $x<x'$ or both $x=x'$ and $y<_x y'$. We will make repeated use of the consequences of the following elementary lemma.

LEMMA D (Kierstead and Trotter [8]). *Let $\mathscr{P}$ be an ordered set and let $\mathscr{F}$ be a family of ordered sets indexed by the points of $\mathscr{P}$. Then $dim_{SG}(\Sigma_{\mathscr{P}}\mathscr{F})\leqslant max\{dim_{SG}(\mathscr{P}), dim_{SG}(\mathscr{Q}) : \mathscr{Q}\in\mathscr{F}\}$.* □

An ordered set is said to be *indecomposable* if it cannot be expressed as a nontrivial ordinal sum. In light of Lemma D and the fact that all our upper bounds are at least 2 we may assume, without loss of generality, for the rest of the paper that $\mathscr{P}$ is indecomposable. In particular, we assume

(A1) Either $D(x)\neq D(y)$ or $U(x)\neq U(y)$ for all $x, y\in P$, $x\neq y$.
(A2) There exists $y\in P$ such that x is comparable to y, for all $x\in P$.

We define the set $N_{\mathscr{P}}$ of *nonforced pairs* of $\mathscr{P}$ by $N_{\mathscr{P}}=\{(x,y)\in P\times P: x\parallel y \text{ and } D(x)\subset D(y) \text{ and } U(y)\subset U(x)\}$. The set of nonforced pairs $N_{\mathscr{P}}$ has two useful elementary properties. Firstly, if Σ is a collection of linear extensions of $\mathscr{P}$, then Σ is a realizer of $\mathscr{P}$ iff for every $(x,y)\in N_{\mathscr{P}}$ there exists $\mathscr{L}\in\Sigma$ such that $y<x$ in $\mathscr{L}$. Secondly, if we define $\mathscr{P}^*=(P,<^*)$ by $x<^* y$ iff $x<y$ or $(x,y)\in N_{\mathscr{P}}$, then, using (A1), it follows that $\mathscr{P}^*$ is an ordered set that extends $\mathscr{P}$.

When B is a subset of the distinguished maximal antichain A of $\mathscr{P}$, we define a quasi-order $\mathscr{Q}_B=(U,<_B)$ on the open upset of A by: $x<_B y$ iff $D(x)\cap B\subset D(y)\cap B$. We say that a subset W of U is a *B-chain* if $x\mid y$ in Q_B for all $x, y\in W$, and a subset S of U is a *B-antichain* if $x\parallel y$ in Q_B for all $x, y\in S$. A linear order $[w_1<w_2<\cdots<w_k]$ of a B-chain W is consistent if $w_i<_B w_j$ for all $i<j$. Note that a B-chain always has a consistent linear order, but not necessarily a unique one.

An antichain B is said to be *rooted* in a chain E if every element of B either covers an element of E or is minimal; B is said to be *strongly* rooted in E if every element of B either covers the maximal element of E or is minimal.

We now present the technical lemmas which assert the existence of super greedy linear extensions with special properties needed for the proofs of Theorems 1 and 2. Only Theorem 2.c uses Lemmas 6 and 7. The reader may choose to see how these lemmas are used in Section 5 before reading their proofs.

LEMMA 3. *Let E be a chain contained in D and B be an antichain contained in A such that* (*) *either $E=\emptyset$ and $B \subset MIN(P)$ or $E \neq \emptyset$ and every element of B covers $\hat{e}=max(E)$. Let W be a B-chain contained in U. Then there exists a super greedy linear extension $\mathscr{L}=\mathscr{L}(B, W, E)$ of $\mathscr{P}$ satisfying each of the following properties:*

(3.1) *$E/(P-U[B_1])$ in $\mathscr{L}$, where $B_1=\{b \in B: b \parallel w$ for some $w \in W\}$;*
(3.2) *B/W in $\mathscr{L}$;*
(3.3) *$B/(A-B)$ in $\mathscr{L}$;*
(3.4) *if $x, y \in A-B$, $(x, y) \in N_{\mathscr{P}}$, and either $x, y \in \mathrm{MIN}(P)$ or $\mathscr{L}$-$max\{d: d <: x\} = \mathscr{L}$-$max\{d: d <: y\}$, then $y < x$ in $\mathscr{L}$;*
(3.5) *if $x, y \in B$, $(x, y) \in N_{\mathscr{P}}$ and $U(x) \cap W = U(y) \cap W$, then $y < x$ in $\mathscr{L}$.*

Proof. If $E \neq \emptyset$ let $\bar{E}$ be an extension of E to a maximal chain in $D[\hat{e}]$; otherwise let $\bar{E}=\emptyset$. Let $[w_1 < w_2 < \cdots < w_k]$ be a consistent extension of W. For each $x \in B_1$ we define weight(x) to be the least j such that $x \leqslant w_j$; if not $x \leqslant w_j$ for all j, we define weight(x) to be infinite; if $x \notin B_1$ we define weight(x) to be 0. We let $\mathscr{L}$ be a super greedy linear extension constructed by adding the following tie breaking scheme to the algorithm SUPER GREEDY.

T_1: Prefer elements of $(A \cup D)-(B-\bar{E})$.
T_2: Minimize weight(x_{i+1}).
T_3: Avoid elements of $\bar{E}$.
T_4: $\mathscr{P}^*$-maximize x_{i+1}.

In order to show that $\mathscr{L}$ satisfies property (3.1), we prove the stronger result:

(3.1′) $\bar{E}/(P-U[B_1])$ in $\mathscr{L}$.

Suppose (3.1′) is false and let stage i be the first time that (3.1′) is violated. Then $x_{i+1} \in \bar{E}$ and there exists $y \in R-U[B_1]$ such that $x_{i+1} \parallel y$. (At stage i, $R=P-\{x_1, \ldots, x_i\}$.) We can, and do, choose $y \in M$, where $M=\mathrm{MIN}(R)$. Since $x_{i+1} \in \bar{E}$, x_{i+1} is not preferred to y by T_1. Since $y \notin B_1$, x_{i+1} is not preferred to y by T_2. On the other hand, $y \notin \bar{E}$, so y is preferred to x_{i+1} by T_3. We conclude that $y \notin SG$. Thus there exists $j \leqslant i$ such that $x_j < x_{i+1}$, $x_j \parallel y$, and, for $j < k \leqslant i$, $x_k \parallel x_{i+1}$ and $x_k \parallel y$. By the choice of i, $x_j \notin \bar{E}$. Let x_h be the greatest element of $\bar{E}$ that is less than x_{i+1}. Since x_j witnesses that $x_{i+1} \notin \mathrm{MIN}(P)$, such an element x_h exists by the maximality of $\bar{E}$. By the choice of i, $h<j$. By the maximality of $\bar{E}$, not $x_h < x_j$, and thus $x_h \parallel x_j$. Since $x_j < x_{i+1}$, $x_j \notin U[B_1]$. But then $h < i+1$ contradicts the choice of i and property (3.1′) must hold.

Now suppose property (3.2) is false at some stage i. Then $x_{i+1} \in B_1$ and there exists $w_r \in R \cap W$ such that $x_{i+1} \parallel w_r$. Choose $y \in M$ so that $y \leqslant w_r$. Since $x_{i+1} \in B$, x_{i+1} is not preferred to y by T_1. However, weight(y) $\leqslant r$, while weight(x_{i+1}) $> r$, and thus y is preferred to x_{i+1} by T_2. We conclude that $y \notin SG$. Thus there exists $j \leqslant i$ such that $x_j < x_{i+1}$, $x_j \parallel y$, and, for $j < k \leqslant i$, $x_k \parallel x_{i+1}$ and $x_k \parallel y$. Since

$x_{i+1} \notin \mathrm{MIN}(P)$ and B is strongly rooted in E, $\hat{e} <: x_{i+1}$. Thus, by property (3.1), $x_j = \hat{e}$ and $y \in U[B_1]$. Say $b \in B_1$ such that $b \leqslant y \leqslant w_r$ and $b \parallel w_s$. Then $s < r$ and $|W| \geqslant 2$. (If, as will be the case in Lemma 4, $|W| = 1$, then there is already a contradiction.) By (*), $\hat{e} <: b$. But this contradicts $y \parallel x_j = \hat{e}$ and so property (3.2) holds.

Next suppose that property (3.3) is violated at stage i. Then $x_{i+1} \in B$ and there exists $a \in R \cap (A - B)$ such that $x_{i+1} \parallel a$. Let $y \in M$ such that $y \leqslant a$. Notice that $y \notin \bar{E}$ by (*), because if $E \neq \emptyset$ then $\hat{e} <: x_{i+1}$. Thus $y \in (A \cup D) - (B \cup \bar{E})$ and is preferred to x_{i+1} by T_1. We conclude that $y \notin SG$. Thus there exists $j \leqslant i$ such that $x_j < x_{i+1}$, $x_j \parallel y$, and, for $j < k \leqslant i$, $x_k \parallel x_{i+1}$ and $x_k \parallel y$. Since $x_{i+1} \notin \mathrm{MIN}(P)$, $\hat{e} <: x_{i+1}$. By property (3.1), $x_j = \hat{e}$. But then $x_j \parallel y$ contradicts property (3.1), so property (3.3) must be true.

Now suppose property (3.4) is false at stage i. Then there exists $y \in R$ such that x_{i+1}, $y \in A - B$, $(x_{i+1}, y) \in N_{\mathscr{P}}$, and (**) either x_{i+1}, $y \in \mathrm{MIN}(P)$ or $\mathscr{L}$-$\max\{d : d <: x_{i+1}\} = \mathscr{L}$-$\max\{d : d <: x_{i+1}\}$. By (**) $y \in SG$. Since $y \notin B \cup E$, x_{i+1} is preferred to y by neither T_1, T_2, nor T_3. But y is preferred to x_{i+1} by T_4. This contradiction shows that property (3.4) is true.

Finally, suppose that property (3.5) is false at stage i. Then $x_{i+1} \in B$ and there exists $y \in R \cap B$ such that $(x_{i+1}, y) \in N_{\mathscr{P}}$ and $U(x_{i+1}) \cap W = U(y) \cap W$. By (*) and property (3.1) $y \in SG$. Since x_{i+1} is preferred to y by neither T_1, T_2, nor T_3, but y is preferred to x_{i+1} by T_4 this is a contradiction. Thus property (3.5) is true. □

LEMMA 4. *Let E be a chain contained in D, B a subset of A strongly rooted in E, and v and w elements of U such that $D(v) \cap B \not\subset D(w) \cap B$. Then there exists a super greedy linear extension $\mathscr{M} = \mathscr{M}(B, v, w, E)$ of $\mathscr{P}$ such that:*

(4.1) *$E/(P - U[B_1])$ in $\mathscr{M}$, where $B_1 = \{b \in B : b \parallel w\}$;*
(4.2) *B/w in $\mathscr{M}$;*
(4.3) *v/P in $\mathscr{M}$;*
(4.4) *if x, $y \in B$, $(x, y) \in N_{\mathscr{P}}$, $x \in MIN(P)$ iff $y \in MIN(P)$, and $x < w$ iff $y < w$, then $y < x$ in $\mathscr{M}$.*

Proof. Let $W = \{w\}$ and define $\hat{e}$, $\bar{E}$, and weight as in the proof of Lemma 3. Let $\hat{b} \in B$ such that $\hat{b} < v$, $\hat{b} \parallel w$, and $\hat{b}$ is $\mathscr{P}^*$-minimal among all such elements. Choose a chain F which is maximal among all chains with minimal element $\hat{b}$ and maximal element v. Let $\mathscr{M}$ be a super greedy linear extension constructed by adding the following tie breaking scheme to algorithm SUPER GREEDY.

T_1: Minimize weight (x_{i+1}).
T_2: Avoid elements of $\bar{E}$.
T_3: Avoid elements of F.
T_4: $\mathscr{P}^*$-maximize x_{i+1}.

Properties (4.1) and (4.2) are proved just as properties (3.1) and (3.2) were in the previous lemma with the exception that $|W| = 1$ is used instead of (*) at the end of the proof of (4.2).

To prove (4.3) we actually prove the stronger result:

(4.3′) F/P in $\mathcal{M}$.

Suppose that (4.3′) is false and let stage i be the first time that (4.3′) is violated. Then $x_{i+1} \in F$ and there exists $y \in R$ such that $y \parallel x_{i+1}$. We can, and do, choose $y \in M$. First suppose $x_{i+1} = \hat{b}$. Since weight $(x_{i+1}) = \infty$, x_{i+1} is not preferred to y by T_1. If x_{i+1} is preferred to y by T_2 then $y \in \bar{E}$ and weight$(y) = 0$; thus y is preferred to x_{i+1} by T_1. Certainly y is preferred to x_{i+1} by T_3. We conclude that $y \notin SG$. Thus there exists $j \leqslant i$ such that $x_j < x_{i+1}$, $x_j \parallel y$, and, for $j < k \leqslant i$, $x_k \parallel x_{i+1}$ and $x_k \parallel y$. Thus $x_{i+1} \notin \text{MIN}(P)$. Since B is strongly rooted in E, $\hat{e} <: x_{i+1} = \hat{b}$, and it follows from (4.1) that $x_j = \hat{e}$. But then $y \parallel x_j$ contradicts (4.1).

Now suppose $x_{i+1} \neq \hat{b}$. By the choice of i, $\hat{b} < y$. Thus weight$(y) = 0$ and $y \notin \bar{E}$. It follows that x_{i+1} is preferred to y by neither T_1 nor T_2. Certainly T_3 prefers y to x_{i+1}. Thus $y \notin SG$. So there exists $j \leqslant i$ such that $x_j < x_{i+1}$, $x_j \parallel y$, and, for $j < k \leqslant i$, $x_k \parallel x_{i+1}$ and $x_k \parallel y$. By the choice of i, $x_j \notin F$. Let x_k be the largest element of F less than x_{i+1}. Then $k < j$. If $x_k \parallel x_j$, k contradicts the choice of i; if $x_k < x_j$, then x_j contradicts the maximality of F. We conclude that property (4.3) is true.

Finally, suppose (4.4) is false. Say $x_{i+1} \in B$ and there exists $y \in R \cap B$ such that $(x_{i+1}, y) \in N_{\mathcal{P}}$, $x_{i+1} \in \text{MIN}(P)$ iff $y \in \text{MIN}(P)$, and $x_{i+1} < w$ iff $y < w$. We claim that $y \in SG$. If $x_{i+1} \in \text{MIN}(P)$ then $y \in \text{MIN}(P)$ also and it follows that $y \in SG$. If $x_{i+1} \notin \text{MIN}(P)$ then neither is y, and since B is strongly rooted in E, both x_{i+1} and y cover $\hat{e}$. Thus $\hat{e} = x_j$ for some $j \leqslant i$. By property (4.1) $\hat{e}$ is the $\mathcal{M}$-largest lower cover of x and y. Thus $y \in M$ and for $j < k \leqslant i$, $x_k \parallel x_{i+1}$ and $x_k \parallel y$. We conclude $y \in SG$. Since $x_{i+1} < w$ iff $y < w$, weight$(x_{i+1}) =$ weight(y) and x_{i+1} is not preferred to y by T_1. Clearly x_{i+1} is not preferred to y by T_2. By the choice of $\hat{b}$, x_{i+1} is not preferred to y by T_3. However T_4 prefers y to x_{i+1}, which contradicts the choice of x_{i+1}. We conclude that property (4.4) is true.

LEMMA 5. *Let B be a subset of MIN(P) and let E be a chain in $P -$ MIN(P). Then there exists a super greedy linear extension $\mathcal{N} = \mathcal{N}(B, E)$ of $\mathcal{P}$ such that:*

(5.1) *E/P in $\mathcal{N}$; and*

(5.2) *if $x, y \in B$ and $(x, y) \in N_{\mathcal{P}}$, then $y < x$ in $\mathcal{N}$.*

Proof. Choose a $\mathcal{P}^*$-minimal element $\hat{e}$ of MIN(P) such that $E \cup \{\hat{e}\}$ is a chain. Let $\bar{E}$ be a maximal chain containing $E \cup \{\hat{e}\}$. We let $\mathcal{N}$ be a super greedy linear extension constructed by adding the following tie breaking scheme to the algorithm SUPER GREEDY.

T_1: Avoid elements of $\bar{E}$.
T_2: $\mathscr{P}^*$-maximize x_{i+1}.

To show that $\mathscr{N}$ satisfies property (5.1) we actually prove the stronger result:

(5.1′) $\bar{E}/P$ in $\mathscr{N}$.

The argument is essentially the same as the proof of (3.1′), but simpler, and we leave it to the reader.

Now suppose property (5.2) is false at stage i. Say $x_{i+1} \in B$ and there exists $y \in R \cap B$ such that $(x_{i+1}, y) \in N_{\mathscr{P}}$. Since $x_{i+1}, y \in \text{MIN}(P)$ and $x_{i+1} \in SG$, $y \in SG$. By the choice of $\hat{e}$, x_{i+1} is not preferred to y by T_1, while y is preferred to x_{i+1} by T_2, contradicting the choice of x_{i+1}. We conclude that property (5.2) is true. □

LEMMA 6. *Let B be a subset of A. Then there exists a super greedy linear extension $\mathscr{S} = \mathscr{S}(B)$ of $\mathscr{P}$ such that:*

(6.1) *for all $x \in D$ and $y \in B$ with $x \| y$, if there exists $e \in D$ such that $e < x$ and $y \in A_e = \{b \in B : e = \mathscr{S}\text{-max}\{d \in D : d <: b\}$, then $y < x$ in $\mathscr{S}$.*

Proof. Let $\mathscr{S}$ be a super greedy linear extension constructed by adding the following tie breaking rule to algorithm SUPER GREEDY.

T_1: Prefer elements of B.

Suppose property (6.1) is false and let stage i be the first time that property (6.1) is violated. Then $x_{i+1} \in D$ and for some $e \in D$ with $e < x_{i+1}$ there exists $y \in R \cap A_e$ such that $x_{i+1} \| y$. Let $e = x_j$. Since $y \in A_e$, $y \in M$ at each stage k for $j \leqslant k$. Thus $x_j < x_k$ for $j < k \leqslant i$. By the choice of i, $e <: x_{i+1}$ and so $x_{i+1} \| x_k$ for $j < k \leqslant i$. Thus $y \in SG$ at stage i. But y is preferred to x_{i+1} by T_1, contradicting the choice of x_{i+1}. Thus property (6.1) is true. □

LEMMA 7. *Let E be a chain in P-MAX(P) and let B be a subset of MAX(P) rooted in E. Then there exists a super greedy linear extension $\mathscr{T} = \mathscr{T}(B, E)$ such that:*

(7.1) *$E/(P - B)$ in $\mathscr{T}$;*
(7.2) *$B/(P - B)$ in $\mathscr{T}$; and*
(7.3) *if $x, y \in B$ and $(x, y) \in N_{\mathscr{P}}$, then $y < x$ in $\mathscr{T}$.*

Proof. Let $\bar{E}$ be a maximal chain in $P - \text{MAX}(P)$ which contains E. We let $\mathscr{T}$ be a super greedy linear extension constructed by adding the following tie breaking scheme to algorithm SUPER GREEDY.

T_1: Avoid elements of B.
T_2: Avoid elements of $\bar{E}$.
T_3: $\mathscr{P}^*$-maximize x_{i+1}.

Again the proof that property (7.1) holds is essentially the same as the proof that property (3.1) holds and is left to the reader.

Suppose property (7.2) is false at stage i. Then $x_{i+1} \in B$ and there exists $y \in R - B$ such that $x_{i+1} \parallel y$. We can, and do, choose $y \in M$. Since y is preferred to x_{i+1} by T_1, $y \notin SG$. So there exists $j < i$ such that $x_j < x_{i+1}$, $x_j \parallel y$, and, for $j < k \leqslant i$, $x_k \parallel x_{i+1}$ and $x_k \parallel y$. Thus $x_j <: x_{i+1}$. Since B is rooted in E and $x_{i+1} \notin \mathrm{MIN}(P)$, x_{i+1} covers some element $e \in E$. By property (7.1) $e = x_j$; but then $e \parallel y$ contradicts property (7.1). We conclude that $\mathscr{T}$ satisfies property (7.2).

Suppose property (7.3) is false at stage i. Then $x_{i+1} \in B$ and there exists $y \in R \cap B$ such that $(x_{i+1}, y) \in N_{\mathscr{P}}$ and $x_{i+1} \parallel y$. Note that $y \in M$ since otherwise there exists $d \in D$ such that $d < y$ and $d \parallel x_{i+1}$, which contradicts property (7.2). Since $y \in M$ and $D(x) \subset D(y)$, $y \in SG$. But x is preferred to y by neither T_1 nor T_2, while y is preferred to x by T_3. This contradiction shows that $\mathscr{T}$ satisfies property (7.3). □

5. Upper Bounds

In this section we use Lemmas 3 through 7 to construct various super greedy realizers of $\mathscr{P}$. Recall that we may assume that $\mathscr{P}$ has properties (A1) and (A2). When referring to a super greedy linear extension constructed by Lemma 3, for example, we will simply write $\mathscr{L}(B, W, E)$. If one of the parameter sets is a singleton we may write the element instead of the set.

Proof of Theorem 1.a.i. We must show that $\mathscr{P}$ has a super greedy realizer Σ of cardinality at most $m \cdot n + 1$, where $m = |D|$ and $n = |U|$. We begin by introducing some necessary and rather lengthy notation. Fix elements $\hat{d} \in \mathrm{MAX}(D)$ and $\hat{u} \in U$ such that $D(\hat{u})$ is maximal. Let $A_0 = A \cap \mathrm{MIN}(P)$, $A_1 = A - A_0$, $P_0 = U[A_0]$, and $P_1 = P - P_0$. Let $\mathscr{P}_0$ be $\mathscr{P}$ restricted to P_0 and $\mathscr{P}_1$ be $\mathscr{P}$ restricted to P_1. Notice that if $\mathscr{L}_0$ is a super greedy linear extension of $\mathscr{P}_0$ and $\mathscr{L}_1$ is a super greedy linear extension of $\mathscr{P}_1$, then $\mathscr{L}_1 + \mathscr{L}_0$ is a super greedy linear extension of $\mathscr{P}$.

Let $A_{\hat{d}} = \{a \in A_1 : \hat{d} <: a\}$. Partition $U - \{\hat{u}\}$ into two element $A_{\hat{d}} \cup A_0$-antichains and if necessary one $A_{\hat{d}} \cup A_0$-chain W'. Let G' be the union of these $A_{\hat{d}} \cup A_0$-antichains. For each $g \in G'$ let g' be the other element of the $A_{\hat{d}} \cup A_0$-antichain to which g belongs. If $W' \cup \{\hat{u}\}$ is an $A_{\hat{d}} \cup A_0$-chain, let $W = W' \cup \{\hat{u}\}$ and $G = G'$; otherwise let $W = W'$ and $G = G' \cup \{\hat{u}\}$. Let $\hat{w} = \hat{u}$ if $\hat{u} \in W$; otherwise let $\hat{w}$ be some maximal element of W in the quasi order $\mathscr{Q}_{A_{\hat{d}} \cup A_0}$. If $\hat{u} \in G$, then $\{\hat{u}, \hat{w}\}$ is an $A_{\hat{d}} \cup A_0$-antichain. In this case let $\hat{u}' = \hat{w}$ and $\hat{w}' = \hat{u}$.

Now we begin the construction of Σ. Let: $\mathscr{R}_0 = \mathscr{L}_1 + \mathscr{L}_0$, where $\mathscr{L}_0 = \mathscr{L}(A_0, W, \emptyset)$ calculated in $\mathscr{P}_0$ and $\mathscr{L}_1 = \mathscr{L}(A_{\hat{d}}, \emptyset, \hat{d})$ calculated in $\mathscr{P}_1$.

Note that the parameter sets of $\mathscr{L}_0$ and $\mathscr{L}_1$ satisfy the hypothesis of Lemma 3. In particular, if $B' \subset B$ then any B-chain is a B'-chain. Partition A_1 into $\{A_d : d \in D\}$, where $A_d = \{a \in A_1 : d = \mathscr{L}_1\}\text{-max}\{d \in D : d <: a\}$. Note that this definition of $A_{\hat{d}}$ is consistent with the previous definition by property (3.1). For each $u \in U$, we define $\mathscr{R}_{\hat{d}}, u$ by:

$$\mathscr{R}_{\hat{d},u} = \begin{cases} \mathscr{M}(A_{\hat{d}} \cup A_0, u', u, \hat{d}) & \text{if } u \in G \\ \mathscr{N}(\emptyset, u) & \text{if } u \in W - \{\hat{w}\} \\ \mathscr{L}(A_{\hat{d}}, W, \hat{d}) & \text{if } u = \hat{w}. \end{cases}$$

Notice that if $u \in G$ then $D(u') \cap (A_{\hat{d}} \cup A_0) \not\subset D(u) \cup (A_{\hat{d}} \cup A_0)$. With this remark we leave it to the reader to check that the above parameter sets satisfy the hypothesis of the appropriate lemmas.

Finally, for $d \in D - \{\hat{d}\}$ and $u \in U$, let $\mathscr{R}_{d,u} = \mathscr{L}(A_d, u, d)$.

We claim that $\Sigma = \{\mathscr{R}_0\} \cup \{\mathscr{R}_{d,u} : d \in D \text{ and } u \in U\}$ is a super greedy realizer of $\mathscr{P}$ of cardinality $m \cdot n + 1$. Clearly every linear extension in Σ is super greedy and the cardinality is as claimed. Thus it suffices to show that for every non-forced pair (x, y) there exists a linear extension in Σ which puts x over y. We break the argument into four main cases: $x \in D$, $x \in U$, $x \in A$ and $y \in U$, and $x, y \in A$. Note that it is not possible to have $x \in A$ and $y \in D$.

Case 1: $x \in D$. Then $y \notin U[A_x]$. Thus $y < x$ in

$$\mathscr{R}_{x,\hat{w}} = \begin{cases} \mathscr{L}(A_x, W, x) & \text{if } x = \hat{d} \\ \mathscr{L}(A_x, \hat{w}, x) & \text{if } x \in D - \{\hat{d}\} \end{cases}$$

by property (3.1).

Case 2: $x \in U$. By the choice of $\hat{u}$ and assumption (A1) $x \neq \hat{u}$. Thus either $x' \in G$ or $x \in W - \{\hat{w}\}$.

Case 2.1: $x' \in G$. Then $y < x$ in $\mathscr{R}_{\hat{d},x'} = \mathscr{M}(A_{\hat{d}} \cup A_0, x, x', \hat{d})$ by property (4.3).

Case 2.2: $x \in W - \{\hat{w}\}$. Then $y < x$ in $\mathscr{R}_{\hat{d},x} = \mathscr{N}(\emptyset, x)$ by property (5.1).

Case 3: $x \in A$ and $y \in U$. Then either $x \in A_d$, where $d \neq \hat{d}$, $x \in A_{\hat{d}} \cup A_0$ and $y \in G$, $x \in A_{\hat{d}}$ and $y \in W$, or $x \in A_0$ and $y \in W$.

Case 3.1: $x \in A_d$, where $d \neq \hat{d}$. Then $y < x$ in $\mathscr{R}_{d,y} = \mathscr{L}(A_d, y, d)$ by property (3.2).

Case 3.2: $x \in A_{\hat{d}} \cup A_0$ and $y \in G$. Then $y < x$ in $\mathscr{R}_{\hat{d},y} = \mathscr{M}(A_{\hat{d}} \cup A_0, y', y, \hat{d})$ by property (4.2).

Case 3.3: $x \in A_{\hat{d}}$ and $y \in W$. Then $y < x$ in $R_{\hat{d},\hat{w}} = \mathscr{L}(A_{\hat{d}}, W, \hat{d})$ by property (3.2).

Case 3.4: $x \in A_0$ and $y \in W$. Then $y < x$ in $\mathscr{R}_0 = \mathscr{L}_1 + \mathscr{L}_0$, where $\mathscr{L}_0 = \mathscr{L}(A_0, W, \emptyset)$ calculated in $\mathscr{P}_0$, by property (3.2).

Case 4: $x, y \in A$. Then either $x, y \in A_0$, $x, y \in A_d$ for some $d \in D$, $x \in A_d$ and $y \in A_e$, where $d \neq e$, or $x \in A_0$ and $y \in A_1$. It is not possible to have $x \in A_1$ and $y \in A_0$.

Case 4.1: $x, y \in A_0$. Then $y < x$ in $\mathscr{R}_{\hat{d},\hat{w}} = \mathscr{L}(A_{\hat{d}}, W, \hat{d})$ by property (3.4).

Case 4.2: $x, y \in A_d$. Then $y < x$ in $\mathscr{R}_0 = \mathscr{L}_1 + \mathscr{L}_0$, where $\mathscr{L}_1 = (A_{\hat{d}}, \emptyset, \hat{d})$ calculated in $\mathscr{P}_1$, by property (3.5) or (3.4) depending on whether or not $d = \hat{d}$.

Case 4.3: $x \in A_d$ and $y \in A_e$, where $d \neq e$. Then $y < x$ in

$$\mathscr{R}_{d,\hat{w}} = \begin{cases} \mathscr{L}(A_d, W, d) & \text{if } d = \hat{d} \\ \mathscr{L}(A_d, \hat{w}, d) & \text{if } d \neq \hat{d} \end{cases}$$

by property (3.3).

Case 4.4: $x \in A_0$ and $y \in A_1$. Then $y < x$ in $\mathscr{R}_0 = \mathscr{L}_1 + \mathscr{L}_0$. □

Proof of Theorem 1.b.i. We must show that $\mathscr{P}$ has a super greedy realizer Σ of cardinality at most $n = |U|$. Define $\hat{u}$, g', G, W, $\hat{w}$, $\hat{w}'$, and $\hat{u}'$ just as in the last proof, but with respect to the quasi order $\mathscr{C}_A$ rather than $\mathscr{C}_{A_0 \cup A_{\hat{d}}}$. For each $u \in U$, define

$$\mathscr{R}_u = \begin{cases} \mathscr{M}(A, u', u, \emptyset) & \text{if } u \in G \\ \mathscr{N}(A, u) & \text{if } u \in W - \{\hat{w}\} \\ \mathscr{L}(A, W, \emptyset) & \text{if } u = \hat{w}. \end{cases}$$

We claim that $\Sigma = \{\mathscr{R}_u : u \in U\}$ is a super greedy realizer of $\mathscr{P}$ of cardinality n. Clearly every linear extension in Σ is super greedy and the cardinality is correct. Thus it suffices to show that for every nonforced pair (x, y) there exists a linear extension in Σ which puts x over y. We break the argument into three cases: $x \in U$, $x \in A$ and $y \in U$, and $x, y \in A$.

Case 1: $x \in U$. By the choice of $\hat{u}$ and assumption (A1) $x \neq \hat{u}$. Thus either $x' \in G$ or $x \in W - \{\hat{w}\}$.

Case 1.1: $x' \in G$. Then $y < x$ in $\mathscr{R}_{x'} = \mathscr{M}(A, x, x', \emptyset)$ by property (4.3).

Case 1.2: $x \in W - \{\hat{w}\}$. Then $y < x$ in $\mathscr{R}_x = \mathscr{N}(A, x)$ by property (5.1).

Case 2: $x \in A$ and $y \in U$. Then $y \in G$ or $y \in W$.

Case 2.1: $y \in G$. Then $y < x$ in $\mathscr{R}_y = \mathscr{M}(A, y', y, \emptyset)$ by property (4.2).

Case 2.2: $y \in W$. Then $y < x$ in $\mathscr{R}_{\hat{w}} = \mathscr{L}(A, W, \emptyset)$ by property (3.2).

Case 3: $x, y \in A$. By assumption (A2) $U(y) \neq \emptyset$. Say $y < u$. Then $x < u$ also. Either $u \in G$, $U(x) \cap W = U(y) \cap W$ or $u \in W$ and $U(y) \cap W \subsetneq U(x) \cap W$.

Case 3.1: $u \in G$. Then $y < x$ in $\mathscr{R}_u = \mathscr{M}(A, u', u, \emptyset)$ by property (4.4).

Case 3.2: $U(x) \cap W = U(y) \cap W$. Then $y < x$ in $\mathscr{R}_{\hat{w}} = \mathscr{L}(A, W, \emptyset)$ by property (3.5).

Case 3.3: $u \in W$ and $U(y) \cap W \subsetneq U(x) \cap W$. Let $w \in (U(x) - U(y)) \cap W$. Then $w \neq \hat{w}$. So $y < x$ in $\mathscr{R}_w = \mathscr{N}(A, w)$ by property (5.2).

Proof of Theorem 1.c.i. We must show that $\mathscr{P}$ has a super greedy realizer Σ of cardinality at most $m = |D|$. For each $d \in D$ let $B_d = \{a \in A : d <: a\}$. By assumption (A2) $\{B_d : d \in D\}$ covers A. For each $d \in D$ define:

$$\mathscr{R}_d = \mathscr{L}(b_d, \emptyset, d).$$

We claim that $\Sigma = \{\mathscr{R}_d : d \in D\}$ is a super greedy realizer of $\mathscr{P}$ of cardinality m. Clearly every linear extension in Σ is super greedy and the cardinality is correct. Thus it suffices to show that for every nonforced pair (x, y) there exists a linear extension Σ which puts x over y. We break the argument into two cases: $x \in D$ and $x \in A$.

Case 1: $x \in D$. Then $y \notin B_d$. Thus $y < x$ in $\mathscr{R}_x = \mathscr{L}(B_x, \emptyset, x)$ by property (3.1).

Case 2: $x \in A$. Say $x \in B_d$. Then $y < x$ in $\mathscr{R}_d = \mathscr{L}(B_d, \emptyset, d)$ by property (3.5) or (3.3) depending on whether or not $y \in B_d$. □

Proof of Theorem 2.b.i. We must show that $\mathscr{P}$ has a super greedy realizer Σ of cardinality at most $2n$, where $n = \text{width}(P - \text{MIN}(P))$. Partition $P - A$ into chains $E_1, \ldots, E_n$. Note that each E_i is also an A-chain. For $i = 1, \ldots, n$ define:

$$\mathscr{R}_i = \mathscr{L}(A, E_i, \emptyset) \quad \text{and} \quad \mathscr{R}_{n+i} = \mathscr{N}(A, E_i).$$

We claim that $\Sigma = \{\mathscr{R}_i : i = 1, \ldots, 2n\}$ is a super greedy realizer of $\mathscr{P}$ of cardinality $2n$. Clearly each linear extension in Σ is super greedy and the cardinality is correct. Thus it suffices to show that for every nonforced pair (x, y) there exists a linear extension in Σ which puts x over y. We break the argument into three cases: $x \in U$, $x \in A$ and $y \in U$, and $x, y \in A$.

Case 1: $x \in U$. Say $x \in E_i$. Then $y < x$ in $\mathscr{R}_{n+i} = \mathscr{N}(A, E_i)$ by property (5.1).

Case 2: $x \in A$ and $y \in U$. Say $u \in E_i$. Then $y < x$ in $\mathscr{R}_i = \mathscr{L}(A, E_i, \emptyset)$ by property (3.2).

Case 3: $x, y \in A$. Then $y < x$ in $\mathscr{R}_{n+1} = \mathscr{N}(A, E_1)$ by property (5.2). □

Proof of Theorem 2.c.i. We must show that $\mathscr{P}$ has a super greedy realizer of cardinality at most $n + 1$, where $n = \text{width}(P - \text{MAX}(P))$. Partition $P - A$ into chains $E_1, \ldots, E_n$. For $i = 1, \ldots, n$ define:

$$\mathscr{R}_0 = \mathscr{S}(A);$$
$$A_i = \{a \in A : \mathscr{R}_0\text{-max}\{d \in D : d <: a\} \in E_i\}; \quad \text{and}$$
$$\mathscr{R}_i = \mathscr{T}(A_i, E_i).$$

Note that A_i is indeed rooted in E_i. By assumption (A2) $\{A_i : i = 1, \ldots, n\}$ is a partition of A. We claim that $\Sigma = \{R_i : i = 0, 1, \ldots, n\}$ is a super greedy realizer of $\mathscr{P}$ of cardinality $n + 1$. Clearly each linear extension in Σ is super greedy and the cardinality is correct. Thus it suffices to show that for every nonforced pair (x, y) there exists a linear extension in Σ which puts x over y. We consider four cases: $x \in E_i$ and $y \in A_i$, $x \in E_i$ and $y \notin A_i$, $x, y \in A_i$, and $x \in A_i$ and $y \in A_j$, where $i \neq j$. Note that it is not possible to have $x \in A$ and $y \in D$.

Case 1: $x \in E_i$ and $y \in A_i$. Then $y < x$ in $\mathscr{R}_0 = \mathscr{S}(A)$ by property (6.1).

Case 2: $x \in E_i$ and $y \notin A_i$. Then $y < x$ in $\mathscr{R}_i = \mathscr{T}(A_i, E_i)$ by property (7.1).

Case 3: $x, y \in A_i$. Then $y < x$ in $\mathscr{R}_i = \mathscr{T}(A_i, E_i)$ by property (7.3).

Case 4: $x \in A_i$ and $y \in A_j$, where $i \neq j$. Then $y < x$ in $\mathscr{R}_i = \mathscr{T}(A_i, E_i)$ by property (7.2). □

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