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Interval Graphs, Interval Orders, and Their Generalizations

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Abstract. We survey some recent results on interval graphs and interval orders concentrating on algorithmic questions and generalizations involving multiple intervals, intervals in higher dimensions, intervals with tolerances, and other geometric figures. We include some open problems and discuss directions for future research.

1. Introduction. Interval graphs and their generalizations have been investigated intensively by researchers in the mathematical, social and biological sciences for more than 25 years. Understandably, a good fraction of this interest stems from the wide range of applications of interval graphs. On the other hand, the study of interval graphs has yielded a substantial body of mathematical theory of independent interest. To gain some appreciation for both aspects of the subject and for details on its history, we encourage the reader to consult the books [16], [27] by Peter Fishburn and Martin Golumbic and the special volume [28] of Discrete Mathematics edited by Golumbic. These references also provide an extensive bibliography of papers in this area.

The purpose of this article is to survey recent research on interval graphs and interval orders concentrating on algorithmic questions and generalizations involving multiple intervals, higher dimensional analogues, and other geometric figures. Our goal is to demonstrate that interval graphs and their generalizations continue to provide interesting results even as new problems arise.

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In view of our earlier remarks about the scope of the research conducted on interval graphs, we confess that this article cannot include a full discussion of some interesting and important work. To those authors whose work is not mentioned, we apologize in advance.

In the interests of brevity, we provide in the article only the basic definitions and terminology necessary to understand the results and problems discussed. Additional background material is given in the books [16], [27], [61], the papers [12], [14], [24], [25], [26], [34], [36] [37], [49] [78] [79] [81] and in some recent theses [6], [43], [47], [55], [58], [64], [73] and [75].

Let $F = \{I_x : x \in V\}$ be an indexed family of nondegenerate closed[1] intervals of the real line **R**. It is natural to associate with the family F a graph G(F) and a partially ordered set P(F) each having the index set V as point set. In the graph G(F), the edges are distinct pairs xy from V for which the intervals I_x and I_y have nonempty intersection. In the partially ordered set P(F), the ordering is defined by $x < y$ when I_x and I_y are disjoint, and the right end point of I_x is less than the left end point of I_y. The graph G(F) is called an <u>interval graph</u> and the partially ordered set P(F) is called an <u>interval order</u>.

2. Algorithmic Questions. Recall that the chromatic number of a graph is at least as large as the maximum clique size. Interval graphs enjoy the special property that these two parameters are always equal. This phenomenon can be easily established by noting that interval graphs are rigid circuit graphs [11], i.e.,they do not contain induced cycles on four or more vertices. In fact, interval graphs possess additional properties which are very useful in designing algorithms for coloring. To be more precise, suppose $G = (V,E)$ is an interval graph. Choose a representation $F = \{I_x : x \in V\}$ with all end points distinct. For each $x \in V$, let $I_x = [a(x), b(x)]$. Label the vertices $v_1, v_2, \ldots, v_n$ so that $a(v_1) < a(v_2) < \ldots < a(v_n)$. If the first-fit (or greedy) algorithm is then used to color the vertices of G using this ordering of the vertices, then an optimum coloring is acheived. By this we mean that if the maximum clique size of G is m, then exactly m different colors will be used on the vertices of G.

A graph G is <u>perfect</u> if every induced subgraph of G has chromatic number equal to its maximum clique size. Of course, interval graphs are perfect. The fact that the

[1] We may assume wihout loss of generality that the intervals are closed as long as the family F is finite. If desired, we may also assume that all end points are distinct.

ordering by left end points allows the first-fit algorithm to produce an optimum coloring is an even stronger statement. It shows that interval graphs are <u>perfectly orderable</u> as defined by Chvatal [5], i.e. there is an ordering of the vertices on which first-fit is optimal on the graph as well as each of its induced subgraphs. Three algorithmic questions are immediate.

Q_1: Suppose the vertices of an interval graph are labelled in some arbitrary fashion (not necessarily by left end points) as $v_1, v_2, \ldots, v_n$. The first-fit algorithm is then used to color the vertices in this order. If the maximum clique size of G is m, how many colors are used (The answer does not depend on n)?

Q_2: How may colors are required to color an interval graph with maximum clique size m in an on-line fashion? By "on-line," we mean that the vertices are received one at a time. When the new vertex arrives, all adjacencies to previous vertices are given. Based on this information, a color must then be assigned to this vertex and this assignment is permanent. Only then is the next vertex given.

Q_3: For which graphs is there a bounded number of permutations of the vertices so that for every induced subgraph, the first fit algorithm provides an optimum coloring when applied to one of the permutations? Of course, this question also makes sense if we replace "bounded" by "small" and replace "optimum" by "nearly optimum.

The second question was answered completely by Kierstead and Trotter [45]. They showed that there exists an on-line algorithm which will color an interval graph with maximum clique size m using 3m-2 colors. They also showed that this result was sharp. The optimum algorithm they produced is polynomial in the number of vertices; however, this algorithm is not first-fit.

It is surprising the Q_1 remains open. A. Gyarfas and J. Lehel [35] have shown that the first-fit algorithm will not use more than 3mlogm colors, but no one has been able to show that the correct answer is not linear in m. Perhaps this is another relatively innocent looking problem with solution mf(m) where f(m) is a slow growing function in the spirit of the iterated log or inverse Ackerman functions. Question 3 is wide open, although the merit of the answer will no doubt depend on what implications are obtained.

The concept of a coloring can be generalized in many ways. In [56], Opsut and Roberts discuss the reduction of some of these problems to linear programming problems. Of course, this reduction to a polynomial algorithm is only valid when the underlying graph satisfies some special properties, for instance, being an interval graph.

3. Multiple Intervals. Define the <u>interval number</u> of a graph G, denoted i(G), as the least t for wich G is the intersection graph of a family of sets with each set the union of t intervals of the real line. An interval graph has interval number 1, while the interval number of a cycle of 4 or more vertices is 2.

In [80] Trotter and Harary showed that $i(K_{m,n}) = \lceil (mn+1)/(m+n) \rceil$ and conjectured that if G is any graph on n vertices, then $i(G) \leq \lceil (n+1)/4 \rceil$. This conjecture was settled in the affirmative by J. Griggs [32] (see also [1]). The inequality is best possible as is evidenced by the complete bipartite graph with balanced sides. For multipartite graphs, Hopkins and Trotter [40] showed that if $n_1 \geq n_2 \geq \ldots \geq n_p$ then $i(K_{n_1,n_2}) \leq i(K_{n_1,n_2,\ldots,n_p}) \leq i(K_{n_1,n_2}) + 1$. Subsequently, Hopkins, Trotter and West [41] showed that $i(K_{n_1,n_2}) = i(K_{n_1,n_2,\ldots,n_p})$ unless $(n_1,n_2) = (7,5)$ or $n_1 = n_2^2 - n_2 - 1$. In these two cases, the upper bound may be obtained.

In [33], Griggs and West showed that if the maximum degree in G is d, then $i(G) \leq \lceil (d+1)/2 \rceil$. They also showed that this inequality is tight if G is regular and triangle-free. They also showed that there exists an absolute constant c so the $i(G) \leq c\sqrt{e}$ where e is the number of edges in G. The best possible value of c is not known.

In [69], Scheinerman and West showed that the interval number of a planar graph is at most 3. This result is easily seen to be best possible. Scheinerman [67] has shown that there exists an absolute constant c for which any graph of genus g has interval number at most $c\sqrt{g}$. The complete bipartite graphs show that this result is best possible up to the value of c. Perhaps it is possible to obtain an exact answer in the spirit of the Heawood map coloring formula.

To the best of my knowledge, no one has investigated algorithmic questions for graphs with bounded interval number. Even the special case $i(G) \leq 2$ would be challenging since these graphs need not be perfect. Since forests have interval number at most 2, it will however be necessary to provide additional restictions on the class if we seek an algorithm for coloring graphs with a number of colors bounded by a function of the maximum clique size. Also no one has investigated generalizations of interval orders involving multiple intervals. Here it is not clear what the appropriate definition should be.

4. Higher Dimensional Analogues. F. Roberts [59] defined the boxicity of a graph G, denoted Box (G), as the least t for which G is the intersection of the boxes in $\mathbf{R}^t$ (A box is the cartesian product of t nondegenerate closed intervals in R). Roberts showed that the boxicity of a graph on n vertices does not exceed $\lfloor n/2 \rfloor$ and Trotter [77] and Witsenhausen [90] completely characterized those graphs for which the inequality is tight. One such example is the complement of a matching. Recently, C. Thomassen [76] has shown that the boxicity of a planar graph does not exceed 3. In fact, he showed that intersecting boxes may be required to intersect only along a face. Cozzens and Roberts [8] obtained bounds on boxicity by studying edge coverings of $\bar{G}$ by complements of interval graphs. Cozzens and Roberts [9] obtained such bounds by relating the boxicity to the notion of a set of linear orders which forms a k-suitable of arrangements, an idea introduced by Joel Spencer [74].

Feinberg [15] defined the <u>circular dimension</u> of a graph G, here denoted cd(G), as the least t for which there exists a mapping which assigns to each vertex $x \in V(G)$ a sequence $A_x(1), A_x(2), \ldots, A_x(t)$ of arcs of the unit circle so that $xy \in E(G)$ exactly when $A_x(i) \cap A_y(i) \neq \phi$ for $i=1,2,\ldots,t$. Feinberg gave a formula for the circular dimension of a complete multipartite graph. Evidently, $cd(G) \leq Box(G)$ although if G is the complement of a matching of m edges [15], then Box(G) = m and cd(G) = 1. It is not known whether the maximum value of the circular dimension of a graph on n vertices is o(n). Other recent results about circular dimension are developed as part of a general theory of dimensional properties of graphs by Cozzens and Roberts [10]

It is a relatively straightforward calculation (see [13], [71] and [77] for example) to show that there exists an absolute constant c so that almost all labelled graphs on n vertices have interval number, boxicity, and cicular dimension all exceeding cn/logn. More sophisticated counting arguments are used by Scheinerman [65], [66] in his development of a theory of random interval graphs.

The concept of boxicity extends to partially ordered sets. Bogart and Trotter [3] defined the <u>interval dimension</u> of a partially ordered set P, denoted Idim(P), as the least t for which there exists a mapping assigning to each point $x \in P$ a sequence $[a_x(j), b_x(j)]$, $j=1,2,\ldots, t$ of non-degenerate closed intervals of R so that $x < y$ in P if and only if $b_x(j) < a_y(j)$ for $j=1,2,\ldots, t$. Bogart and Trotter [2], [3] also produced a number of inequalities for the interval dimension of a partially ordered set in terms of the cardinality and the width of certain subposets.

Scheinerman introduced a generalization of interval graphs involving intervals in posets of higher dimension. Recall that the <u>dimension</u> of a poset P is the least t for which P is the intersection of t linear extensions. When P is a poset and $a < b$ in P, define the <u>interval</u> [a,b] by $[a,b] = \{x \varepsilon P: a \leq x \leq b \text{ in } P\}$. Then define the <u>poset boxicity</u> of G, here denoted pBox(G), as the least t so that G is the intersection graph of intervals in a poset of dimension t. A graph with poset boxicity 1 is an interval graph.

The concept of poset boxicity is not very well understood at this time. Trotter and West [82] have shown that there exists an absolute constant c so that the poset boxicity of a graph on n vertices is at most cloglogn. This result is probably far from best possible. In fact, it is surprisingly difficulty to show that there exist graphs with arbitarily large poset boxicity [46].

Let G = (V,E) be an arbitrary graph. Define the split of G, denoted split(G), as the poset of height 1 having V as minimal elements and E as maximal elements and ordering $x<e$ when vertex x is an end point of edge e. W. Schnyder [70] proved the following striking result. A graph G is planar if and only if the dimension of split(G) is at most 3. It would be quite interesting to develop characterizations of planar graphs strengthening the previously mentioned results of Scheinerman, and West [69] and Thomassen [76].

Associated with a digraph D is a related graph G called a competition graph. In G, two vertices x and y are adjacent when there is a third vertex z for which D contains directed edges from x to z and from y to z. An interesting open problem is to characterize diagraphs whose competition graphs are interval graphs. There is a great deal of empirical evidence that real world digraphs arising from ecological food webs have this property. For every G, there is a minimum number k(G) of isolated vertices whose addition to G yields a competition graph. However, Opsut [54] showed that the determination of k(G) is NP complete. The theses [55], [58] by Opsut and Raychaudhuri also contain lists of open problems involving competition graphs, set coloring, powers of graphs and other topics.

5. Other Geometric Figures. For graphs, the most natural generalization of interval graphs to other geometric figures (other than cartesian products) is to consider spheres (discs, balls) in Euclidean space. Define the sphericity of G, here denoted sph(G), at the least t for which G is the intersection graph of a family of spheres in Euclidean t-dimensional space $\mathbf{R}^t$. We call the parameter unit sphericity if we require each sphere to have radius 1.

In a series of papers [50], [51], and [52], H. Maehara and P. Fishburn [18] have investigated the unit sphericity of a graph and the relationship between this parameter and other graph parameters. As discussed in [63], there are a number of interesting applications of graphs with sphericity 1. For graphs, other natural variations include line segments or strings in the plane [46], chords of a circle (see Chapter 11 of [27]), arcs on a circle [83], [84], [85], [86] cubes in n-space [59], boundaries of rectangles [75], and convex bodies in the plane [38].

For posets, there are several natural variations. Define the spherical dimension of a poset P, denoted by sph dim(P), as the least t for which there exists a mapping which assigns to each $x \in P$ a sphere S(x) in $\mathbf{R}^t$ so that $x<y$ in P if and only if S(x) is a subset of S(y). Note that this concept is actually a generalization of the notion of dimension. In particular sph dim (P) ≤ 1 exactly when dim(P) ≤ 2.

The case where sph dim(P) ≤ 2 is particularly interesting. These posets are called circle orders, although we

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should really call them disc orders. Fishburn [17] has shown that every interval order is a circle order. Recently, several researchers have been trying to determine whether every 3-dimensional poset is a circle order. In the positive direction, Scheinerman and Weirman [68] have shown that if P is any finite 3-dimensional poset and $p \geq 3$, then there exist a mapping which assigns to each x p a convex region S_x in the plane so that each S_x is a regular p-gon (and the bottom side of S_x is horizontal) with $x \leq y$ in P if and only if S_x is a subset of S_y. On the other hand, these same authors have shown that the countably infinite poset Z^3 is not a circle order. These results almost seem to contradict each other.

In another direction, Fishburn and Trotter [23] consider angle orders. These are posets which arise from considering angular regions in the plane ordered by inclusion. They showed that every poset P satisfying $\dim(P) \leq 4$ is an angle order. They also conjectured that there exists a poset P with $\dim(P) = 5$ which is not an angle order. This conjecture has just been settled in the affirmative by R. Jamison [42].

Every interval order is also an angle order [15]. There is a natural notion of angle dimension for posets, and it is probably the case that for every number k, there exists a poset P and a point $x \in P$ for which the angle dimension of P exceeds the angle dimension of $P - x$ by more than k.

Given a collection C of real valued (piecewise linear) functions defined on the unit interval [0,1], define a partial order on C by $f \leq g$ when $f(x) \leq g(x)$ for all x. As observed in [29], every partial order arises in this manner. In particular, the 2-dimensional posets result from collections of linear functions. For each distinct pair $f, g \in C$, let $X(f,g) = |\{x \in [0,1] : f(x) = g(x)\}|$. Then let $X(C)$ be the maximum of $X(f,g)$ taken over all distinct pairs. Finally, define the crossing number of a poset P, denoted $X(P)$, as the minimum value of $X(C)$ where C is a representation of P. J. Sidney, S. Sidney, and J. Urrutia [72] proved that $X(P) \leq \dim(P) - 1$ and that this inequality is best possible. The techniques in [72] and Urrutia's paper [87] can be used to provide a relatively simple proof that there exist 5-dimensional posets which are not angle orders.

6. Tolerances and Thresholds.

In [21], Golumbic, Monma, and Trotter introduced the concept of a tolerance graph. These graphs arise from an indexed family $F = \{I_x : x \in V\}$ of intervals and a set $\{t_x : x \in V\}$ of nonnegative real numbers called tolerances. From these families, we determine a graph whose vertex set is V by making xy an edge when the length of the overlap of I_x and I_y is at least as large as the minimum of the tolerances t_x and t_y. Using $|I|$ to denote the length of the interval I, we can write symbolically that $xy \in E(G)$ exactly when $|I_x \cap I_y| > \min\{t_x, t_y\}$.

In [30], it is shown that every interval graph is a tolerance graph. So are permutation graphs. It is also shown that tolerance graphs are perfect. This is accomplished by showing that the complement of a tolerance graph is perfectly orderable. It is not known how difficult it is to recognize tolerance graphs and the problem of providing a forbidden subgraph characterization appears very difficult.

In another direction Monma, Reed, Saks and Trotter [53] have investigated generalized threshold graphs. The graphs arise from two families $\{w_x : x \in V\}$, $\{t_x : x \in V\}$ of real numbers with the rule $xy \in E(G) \iff w_x + w_y \geq \min\{t_x, t_y\}$. They show that the complement of a generalized threshold graph is a tolerance graph. However, they also provide a polynomial time algorithm for recognizing generalized threshold graphs.

The general notions of tolerance and threshold as well as other more subtle rules for adjacency appear to be ripe areas for future investigation. The appropriate definitions will (and should) be motivated by considering physical models. However, we do not know of appropriate definitions for partial orders with tolerances.

7. Restriction on Lengths. The interval count [46] of an interval graph G, denoted here $i_c(G)$, is the least t for which G has a representation using intervals of exactly t different lengths. Interval graphs with $i_c(G) = 1$ are called unit interval graphs. These graphs have received a great amount of attention. For example Roberts [60] showed that an interval graph has $i_c(G) = 1$ if and only if it does not contain $K_{1,3}$. On the other hand, it is easy to see that for each $k \geq 2$, there are infinitely many forbidden subgraphs for the class of interval graphs with $i_c(G) \leq k$. In [47], an example is given where the removal of a vertex can decrease the interval count of a graph by 2. Certainly, the removal of a vertex can decrease the interval count by an arbitrarily large amount, but no one has yet found the proof.

Several years ago, I conjectured that if $1 < r < s < t$ and if G was any interval graph with $i_c(G) = 2$ having two representations - one using only intervals of length 1 and r and another using only intervals of length 1 and t then G also had a representation using only intervals of length 1 and s. P. Fishburn [19], [20] disproved this conjecture. In [14], Fishburn and Graham investigate the problem of characterizing interval graphs having representations in which all intervals have lengths from the interval [1,r]. They show that if r is a rational number, then the number of forbidden subgraphs is bounded and is a unique graph when r is an integer. In his thesis [31], Greenough investigated the minimum positive integer r(P) so that an interval order P could be represented using only intervals with integer end points from the first r(P) positive integers. Greenough also gave simple formulas for the number of unlabelled interval orders with a given value of r(P). However, to my knowledge Greenough's interesting ideas have not received further attention. In a slightly different direction it would be nice to extend Hanlon's fundamental work [39] on counting interval graphs to related structures. Even a parallel result for circular arc graphs would be an important step.

Note Added in Proof: Since this paper was prepared, I have learned of some additional references for the on-line interval graph coloring problems discussed in Section 2. At the 1973 British Conference, D. R. Woodall posed question Q_1 using slightly different terminology. In his JCT (A) 21 (1976) 222-229 paper, Witsenhausen showed that the first fit algorithm may use $(4-\varepsilon)m$ colors on an interval graph with maximum clique size m. The same problem has been investigated by M. Chrobak and M. Shusarek who indicate in a preprint "On Some Packing Problems Related to Dynamic Storage Allocation" that they can improve this lower bound to 44m/10. Apparently, the dynamic storage problems associated with this problem were first investigated by J. M. Robson in J. ACM 18 (1971) 416-423.

Second Note: H. Kierstead has solved question Q_1 by showing that the first fit algorithm will color an interval graph of maximum clique size m in cm colors where c is an absolute constant. Kierstead's argument yields the value c = 40 and with additional work he can improve this to c = 26. The best possible value of c remains unknown.

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1. Introduction

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