On-Line Partitioning of Partially Ordered Sets

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William T. (Tom) Trotter received his Ph.D. in 1969 from the University of Alabama, where he worked under the supervision of William J. Gray. His Ph.D. dissertation and graduate work was concentrated in topology, analysis and algebra, but after the highly successful 1971 Bowdoin Conference, he has worked exclusively in the areas of combinatorics and graph theory. Trotter is the author of more than 70 research publications and has spoken at numerous international and national conferences. He has served as Assistant Dean and Department Chair at the University of South Carolina and is currently Chair of the Department of Mathematics at Arizona State University.

Combinatorial mathematics has as one of its distinguishing characteristics a degree of elegance that is somewhat surprising in view of the concrete nature of the subject. Even as combinatorial mathematics matures and the research techniques required to make important contributions become more sophisticated, elegant combinatorial theorems accessible to undergraduate students continue to be discovered. The best of these spark more intensive investigations which, in turn, yield theorems of great depth and significance. And so it goes.

In this article, we present an example from the combinatorial theory of partially ordered sets. Our example is representative of a class of combinatorial problems known as extremal problems, an area of combinatorial mathematics which includes some notoriously difficult problems. Extremal problems on partially ordered sets are particularly challenging, but our example yields an elegant solution that can be readily understood without special background knowledge of combinatorics. As an added bonus, our example includes a discussion of on-line (recursive) algorithms, an important topic in discrete optimization.

Partially Ordered Sets

Let us briefly review some of the notation and terminology that will be used in this article. A *partially ordered set* or *poset* consists of a pair $(X, \leq)$, where X is a set (usually finite in this article) and $\leq$ is a binary relation on X that satisfies the following three requirements:

(i) $x \leq x$ for every $x \in X$ [reflexivity]

(ii) if $x \leq y$ and $y \leq x$, then $x = y$ for all $x, y \in X$ [antisymmetry]

(iii) if $x \leq y$ and $y \leq z$, then $x \leq z$ for all $x, y, z \in X$ [transitivity].

A binary relation satisfying (i)–(iii) is called a *partial order* on X. (For example, a partial order on any set of positive integers is defined by $x \leq y$ if x divides y. As a second example, let X be any family of sets and define $x \leq y$ if x is a subset of y.) In the remainder of this article, we will adopt the convention of using a single symbol, such as X, to denote a poset—it being understood that X is a set on which a partial order $\leq$ has been specified.

The notation $y \geq x$ means exactly the same as $x \leq y$. When $x \leq y$ and $x \neq y$, we write $x < y$ or $y > x$. We say that x is *comparable* to y when either $x \leq y$ or $x \geq y$; otherwise we say x and y are *incomparable* and write $x \| y$. Of course, incomparable points must be distinct.

A subset $C \subset X$ is called a *chain* if x and y are comparable for every $x, y \in C$. The maximum number of points in a chain is called the *length* of the poset X. Dually, a subset $A \subset X$ is called an *antichain* if x and y are incomparable for every distinct pair $x, y \in A$. The maximum number of points in an antichain is called the *width* of X. Note that $\{x\}$ is both a chain and an antichain for each $x \in X$.

An element $x \in X$ is called a *maximal* point if there is no point $y \in X$ so that $x < y$. The set of maximal points is denoted MAX(X). *Minimal* points are defined analogously, and MIN(X) denotes the set of minimal points. MAX(X) and MIN(X) are each antichains of X.

We say *x covers y* (written $x\!: > y$) if $x > y$ and there is no point $z \in X$ with $x > z > y$. To describe a poset, it is enough to specify the set X and to list the ordered pairs $(x, y) \in X \times X$ for which $x\!: > y$. When X is relatively small, we can describe the poset with a drawing in the Euclidean plane of the graph whose vertex set is X and whose edge set consists of the pairs $\{x, y\}$ where $x\!: > y$. In this graph, it is customary to use straight line segments for edges, and it is required that the vertical placement of x be higher in the plane than that of y when $x\!: > y$. In Figure 1, we see the diagram of a poset X. Here, MIN(X) $= \{3, 12, 13, 14\}$ and MAX(X) $= \{1, 2, 6, 7, 8\}$, so that the width of X is at least five. However, the width is actually six, and $A = \{3, 4, 5, 6, 7, 8\}$ is a 6-element antichain. Also, the length is four, and $C = \{2, 4, 9, 13\}$ is a 4-element chain.

If $Y \subset X$ and $\leq$ is a partial order on X, then the restriction of $\leq$ to Y is a partial order on Y. Equipped with this partial order, we will refer to Y as a *subposet* of X. For example, the poset Y in Figure 2 is a subposet of the poset X in Figure 1.

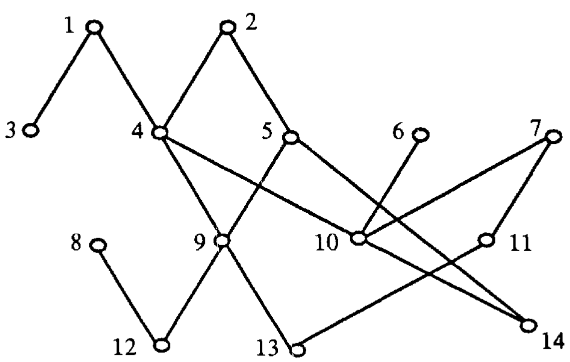

Figure 1. A poset X of width 6 and length 4.

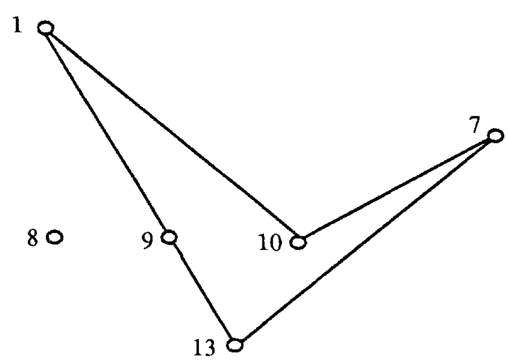

Figure 2. A subposet Y of the poset X.

Dilworth's Theorem and Its Dual

The poset X in Figure 1 can be partitioned by the 7 chains

$$\{1,3\} \cup \{2,4\} \cup \{5,14\} \cup \{6,10\} \cup \{7,11\} \cup \{8,12\} \cup \{9,13\}. \qquad (1)$$

It can also be partitioned by the 6 chains

$$\{1,3\} \cup \{4,9,13\} \cup \{2,5,14\} \cup \{6,10\} \cup \{7,11\} \cup \{8,12\}. \qquad (2)$$

Therefore, it seems reasonable to ask: What is the least number of chains into which a poset can be partitioned? As (1) and (2) illustrate, a poset X of width n cannot be partitioned into fewer than n chains since distinct points from an n-element antichain cannot be assigned to the same chain. The following theorem, due to R. P. Dilworth [1], provides the surprising answer that the least number of chains always equals the width of X. (The proof below is patterned after Perles [4]. See Tverberg's paper [5] for another short proof.)

Theorem 1. *A poset X of width n can be partitioned as $X = C_1 \cup C_2 \cup \cdots \cup C_n$, where C_i is a chain for each $i = 1, 2, \ldots, n$.*

Proof. We proceed by induction on $|X|$, the number of points in X. Obviously, the theorem is true when $|X| = 1$ and $|X| = 2$. Now assume that for some integer $k \geq 3$, the theorem is true for every poset having fewer than k points, and let X be any poset with $|X| = k$. We may assume that X has width $n \geq 2$ (since the theorem clearly holds when X is a chain).

For each chain C in X, it is easy to see that the width of the subposet $X - C$ is either n or $n-1$. Suppose that X contains a chain C such that the width of $X - C$ is $n-1$. Then $C \neq \varnothing$, and $|X - C| < k$. By the inductive hypothesis, there is a partition of $X - C$ into $n-1$ chains. However, this implies that X can be partitioned into n chains. So it remains only to consider the case that the width of $X - C$ is n for every chain C contained in X. Consequently, for every $x \in \mathrm{MAX}(X)$, there exists $y \in \mathrm{MIN}(X)$ so that $x > y$; otherwise $\{x\}$ is a chain in X and the width of $X - \{x\}$ is $n-1$.

Choose an arbitrary pair $x \in \mathrm{MAX}(X)$ and $y \in \mathrm{MIN}(X)$ with $x > y$. Since the width of $X - \{x, y\}$ is n, we may choose an n-element antichain $A = \{a_1, a_2, \ldots, a_n\}$ contained in $X - \{x, y\}$. Now let

$$\mathscr{U}(A) = \{u \in X\colon u > a \text{ for some } a \in A\}$$

and

$$\mathscr{D}(A) = \{d \in X\colon d < a \text{ for some } a \in A\}.$$

Observe that both $\mathscr{U}(A)$ and $\mathscr{D}(A)$ are nonempty, since $x \in \mathscr{U}(A)$ and $y \in \mathscr{D}(A)$. Moreover, each of the subposets $A \cup \mathscr{U}(A)$ and $A \cup \mathscr{D}(A)$ has width n and fewer than k points. Therefore (by our inductive hypothesis), there exist chain partitions

$$A \cup \mathscr{U}(A) = C_1 \cup C_2 \cup \cdots \cup C_n$$

and

$$A \cup \mathscr{D}(A) = D_1 \cup D_2 \cup \cdots \cup D_n.$$

Without loss of generality, we may assume that $a_i \in C_i \cap D_i$ for $i = 1, 2, \ldots, n$. This in turn implies that $X = (C_1 \cup D_1) \cup (C_2 \cup D_2) \cup \cdots \cup (C_n \cup D_n)$ is a partition of X into n chains. This completes the proof of our theorem.

The proof we have given for Theorem 1 provides an algorithm for partitioning a width n poset into n chains, but this algorithm is not very efficient. Good algorithms exist because the problem of partitioning a poset into chains can be formulated as a network flow problem, an important instance of a linear programming problem.

Theorem 1 has a dual version in which we ask for a minimum partition of a poset into antichains. This version has an elementary solution.

Theorem 2. *A poset X of length n can be partitioned as $X = A_1 \cup A_2 \cup \cdots \cup A_n$, where A_i is an antichain for each $i = 1, 2, \ldots, n$.*

Proof. For each $x \in X$, let depth(x) be the length of the longest chain C in X having x as its least element. Then for each i ($i = 1, 2, \ldots, n$), let $A_i = \{x \in X\colon \text{depth}(x) = i\}$. It is clear that each A_i is an antichain, and that $\{A_1, A_2, \ldots, A_n\}$ partitions X.

On-Line Antichain Partitioning Problems

In this section, we consider the problem of providing a minimum "on-line" partition of a poset into antichains: Given a positive integer n, find the least integer $t = t(n)$ for which any poset of length at most n can be partitioned on-line into t antichains. In the next section, we will discuss the dual problem of providing a minimum on-line partition into chains. We begin with the antichain problem because the arguments given in the preceding section suggest that it may be the easier of the two. In fact, it is.

The on-line antichain partitioning problem can be formulated as a two-person game involving three positive integers, k, n, t with $t \geq n$. We call this game the *(k, n, t)-game*. In the (k, n, t)-game, Player A constructs a k-element poset X of length at most n, and Player B attempts to partition X into t antichains. However, both the construction and the partition are to be done one point at a time with the players alternating turns. The moves of both players are permanent and cannot be modified at a later time.

We consider the game as being played in a series of rounds. Prior to the ith round, Player A has already constructed a poset on $i - 1$ points, and Player B has partitioned this poset into t antichains. During round i, Player A introduces a new point x to the poset and describes the comparabilities between x and the points from previous rounds. Player B responds by assigning x to one of the t sets that will partition X. Since the t sets are to be antichains of X, Player B must be sure that the new point x is incomparable to all of the points in the set to which it is assigned.

The game ends at the ith round ($1 \leq i \leq k$), and Player A is declared the winner, if Player B has no legitimate move—that is, the new point x is comparable to at least one point in each of the t sets (so Player B cannot complete his partition of X into t antichains). Player B is declared the winner if he makes a legitimate move at each of the k rounds in the construction of the poset. Thus, at round k, Player B will produce a partition of X into t antichains.

Before giving an illustration of this game, we introduce the following terminology. The phrase "posets of length n can be partitioned on-line into t antichains" will indicate that for each $k \geq 1$, there is a strategy for Player B to win the (k, n, t)-game. Similarly, "posets of length n cannot be partitioned on-line into t antichains" will indicate that there is some $k \geq 1$ for which Player A has a winning strategy in the (k, n, t)-game. Note that if Player A has a winning strategy in the (k, n, t)-game, then Player A has a winning strategy in the (m, n, t)-game for every $m \geq k$.

Example. To establish that posets of length 2 cannot be partitioned on-line into 2 antichains, we show that Player A has a winning strategy for the (4,2,2)-game. At round 1, Player A presents Player B with the 1-element poset x. Player B responds by assigning x to one of two antichains. At round 2, Player A introduces the new point y with $x \| y$. In responding, Player B has two options: he may assign the new point y to the antichain containing x, or he may assign y to the other antichain. If Player B assigns x and y to different antichains, then at step 3, Player A introduces z with $z > x$ and $z > y$. Player B has no legal move and the game ends with Player A the winner. If, on the other hand, Player B assigns x and y to the same antichain, then at round 3, Player A makes $z > x$ and $z \| y$. Player B's only move is to assign z to the other antichain. At round 4, Player A introduces w with $w < z$, $w < y$, and $w \| x$. Again, Player B has no legal move.

Although posets of length 2 cannot be partitioned on-line into 2 antichains, they can be partitioned on-line into 3 antichains. The general result is given by the following theorem which is due to J. Schmerl, although the proof we give for the lower bound is due to E. Szemerédi (see [3]).

Theorem 3. *Posets of length n can be partitioned on-line into $n(n+1)/2$ antichains.*

Proof. Let $n \geq 1$ and set $t = n(n+1)/2$. We shall provide a strategy for Player B which will unable him to win the (k, n, t)-game against any strategy by Player A. Observe that there are exactly t ordered pairs of the form (r, s) where $1 \leq r$, $s \leq n$ and $r + s \leq n + 1$. The t antichains that will make up the partition of X will be denoted $\{A(r, s)\colon 1 \leq r,\ s \leq n \text{ and } r + s \leq n + 1\}$.

In view of the rules governing the game, we need give only the method by which at each round Player B determines the pair (r, s) and therefore the set $A(r, s)$ to which x is assigned. To decide this, Player B examines the poset constructed thus far by Player A and defines $r = r(x)$ as the number of points in the largest chain $\mathscr{U}(x)$ having x as its least element. Then let $s = s(x)$ be the number of points in the largest chain $\mathscr{D}(x)$ having x as its greatest element (see Figure 3). It is important to note that $r(x)$ and $s(x)$ are determined once and for all when x enters the poset. Thus, although it may happen at a future round that x is the least element of a chain having more than $r(x)$ points, the value $r(x)$ remains fixed.

We now show that each $A(r, s)$ is an antichain. Suppose point x enters in round i, and a distinct comparable point y enters in round $j > i$. If $x \in A(r, s)$, there is an

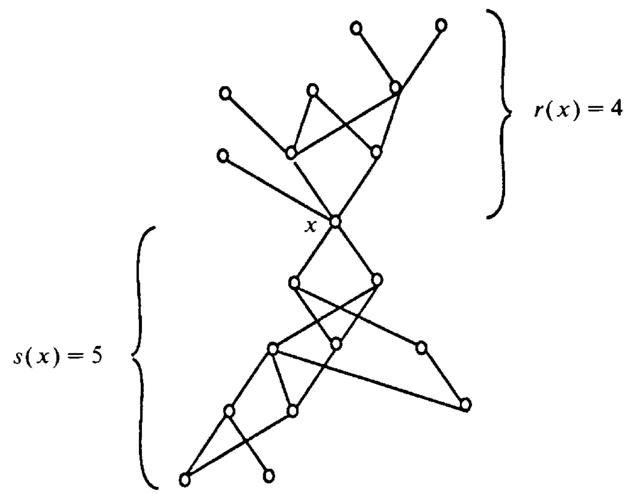

Figure 3

s-element chain $\mathscr{D}(x)$ having x as its greatest element and an r-element chain $\mathscr{U}(x)$ having x as its least element. Now consider what happens when the comparable point y enters: If $x < y$, then $\mathscr{D}(x) \cup \{y\}$ is an $s+1$-element chain having y as its greatest element; if $y < x$, then $\mathscr{U}(x) \cup \{y\}$ is an $r+1$-element chain having y as least element. In either case, Player B cannot assign y to $A(r, s)$. Thus, no two comparable points can be in the same $A(r, s)$. This completes our proof that a poset of length n can be partitioned on-line into $n(n+1)/2$ antichains.

It remains to be shown that posets of length n cannot be partitioned on-line into fewer than $n(n+1)/2$ antichains. For $t < n(n+1)/2$, we will show that there exists an integer k such that Player A has a winning strategy for the (k, n, t)-game. In fact, surely for $|X| = k < n(n+1)/2$, fewer than $n(n+1)/2$ antichains are needed. We will actually prove the stronger result: For every pair n, t of positive integers, there exists an integer $k = k(n, t)$ that provides Player A a strategy $S(n, t)$ for constructing a k-element poset X_k of length n, and this strategy will either (1) win the (k, n, t)-game, or (2) force Player B to use at least $n(n+1)/2$ of the t antichains in covering X_k while using at least n antichains in covering the maximal elements of X_k. In particular, note that if $t < n(n+1)/2$, then option (1) must apply. The argument proceeds by induction on n. Note that the statement is trivial when $n = 1$, since Player A need only present B with a 1 point poset.

Next, we assume the statement is true for all n satisfying $1 \le n \le m$. Now suppose t is any positive integer. Let $k = k(m, t)$ and $k' = 3k + 1$. We now describe a strategy $S(m+1, t)$ for Player A to construct a k'-element poset $X_{k'}$ of length $m+1$, where $S(m+1, t)$ will either (1) win the $(k', m+1, t)$-game, or (2) force Player B to use at least $(m+1)(m+2)/2$ antichains in covering $X_{k'}$ while using at least $m+1$ antichains in covering the maximal elements of $X_{k'}$.

First, Player A follows strategy $S(m, t)$ to construct a k-element poset having length m. Clearly, we may assume that Player B is able to respond during each of the first k rounds of the game. For convenience, we let Y_1 denote the k-element poset constructed thus far. We know that Player A has forced Player B to use at least $m(m+1)/2$ antichains in covering the points of Y_1 and to use at least m different antichains in covering the maximal elements of Y_1.

Without regard for the results of the first k rounds, Player A begins from scratch to follow strategy $S(m, t)$ a second time to construct a k-element poset Y_2. There are no comparabilities between points in Y_1 and points in Y_2. Since the length of the poset determined by $Y_1 \cup Y_2$ is m, we may assume that Player B is able to respond during each of the first $2k$ rounds. Therefore, Player B is forced to use at least $m(m+1)/2$ antichains in covering the points in Y_2 and to use at least m different antichains in covering the maximal elements of Y_2. It is important to remember that even though Player A follows the same strategy in constructing Y_1 and Y_2, they need not be isomorphic because Player B may not partition them in the same fashion.

If Player B has used at least $m+1$ antichains to cover the antichain $M_0 = \mathrm{MAX}(Y_1) \cup \mathrm{MAX}(Y_2)$, then Player A employs strategy $S(m, t)$ a third time to construct a k-element poset Y_3. However, this time Player A makes $y_3 < a$ and $y_3 \| y$ for every $y_3 \in Y_3$, $a \in M_0$, and $y \in (Y_1 \cup Y_2) - M_0$. The poset is completed by adding an element x_0 incomparable to all other points (see Figure 4). Note that the length of the resulting poset is $m+1$, so we may again assume that Player B has responded at each round. Since $M_0 \cup \{x_0\}$ is the set of maximal elements, Player B has used at least $m+1$ different antichains in covering the maximal elements. Finally, we observe that any antichain used by Player B to cover a point in Y_3 cannot have been used to cover a point of M_0. This implies that Player B will have used at least $m(m+1)/2 + (m+1) = (m+1)(m+2)/2$ antichains in covering the points of the poset. This is exactly what we claimed.

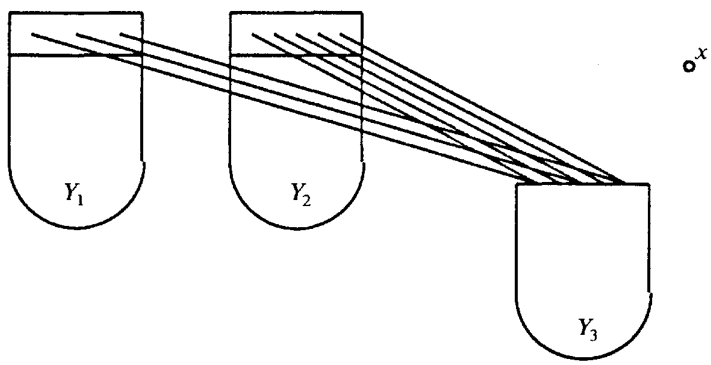

Figure 4

Next, we turn to the case where Player B has used only m antichains in covering M_0. In this case, at round $2k+1$, Player A adds the point y_0 with $y_0 \| y_1$ for every $y_1 \in Y_1$ and $y_0 > y_2$ for every $y_2 \in Y_2$. It follows that Player B will use $m+1$ antichains in covering the set $M_1 = \mathrm{MAX}(Y_1) \cup \{y_0\}$. Then Player A follows strategy $S(m, t)$ to build a k-element poset Y_3 with $y_3 < a$ and $y_3 \| y$ for every $y_3 \in Y_3$, $a \in M_1$, $y \in (Y_1 - M_1) \cup Y_2$. As before, we conclude that Player B will use at least $(m+1)(m+2)/2$ antichains in covering the resulting poset while using at least $m+1$ antichains in covering the maximal elements. This observation completes the proof.

The On-Line Chain Partitioning Problem

In comparing Theorems 1 and 2, it is clear that the antichain partitioning problem is easier than the chain partitioning problem. This is also true for the on-line versions. H. Kierstead [2] has shown that posets of width n can be partitioned on-line into $(5^n - 1)/4$ chains. But it is not known whether this bound is the best possible. The correct value when $n = 2$ is not known. It is either 5 or 6. (As a challenging exercise, the reader may enjoy trying to devise a strategy for partitioning on-line a poset of width 2 into 6 chains.) The argument in the preceding section can be modified to show that posets of width n cannot be partitioned on-line into fewer than $n(n+1)/2$ chains. However, it is an open (and probably difficult) problem to determine whether the on-line chain partitioning problem is polynomial or exponential in the width. The interested reader is encouraged to consult Kierstead's survey article on recursive combinatorics [3] for additional information concerning optimization problems done in an on-line fashion.

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God Made the Integers but Who's Keeping Track of Them?

The Pooka MacPhellimey, a member of the devil class, sat in his hut in the middle of a firwood meditating on the nature of the numerals and segregating in his mind the odd ones from the even.

Flann O'Brien, *At Swim-Two-Birds*, *Plume Books*