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Angle Orders and Zeros

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Abstract. An *angle order* is a partially ordered set whose points can be mapped into unbounded angular regions in the plane such that x is less than y in the partial order if and only if x's angular region is properly included in y's. The *zero augmentation* of a partially ordered set adds one point to the set that is less than all original points. We prove that there are finite angle orders whose augmentations are not angle orders. The proof makes extensive use of Ramsey theory.

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1. Introduction

This paper answers the question raised in Fishburn and Trotter [5] and noted as Problem 23 in Trotter [11] of whether every finite angle order augmented by adding a least element below all others (a *zero*) is also an angle order. We answer in the negative. Our proof constructs a particular angle order Γ_n and shows that the zero augmentation of Γ_n, written as $\Gamma_n + 0$, is not an angle order when n is large. Because the proof uses Ramsey theory [2, 6] extensively to deduce regular patterns in a potentially chaotic whole, it is uninformative about the cardinality of the smallest angle order whose zero augmentation is not an angle order.

Our question is motivated by recent studies [3, 4, 5, 8, 10, 12] of finite partially ordered sets $(X, <_0)$ that are representable by closed, connected regions in the Euclidean plane ordered by proper inclusion. We assume that $0 < |X| < \infty$ and that $<_0$ is an asymmetric and transitive binary relation on X. The *inverse* of $(X, <_0)$ is $(X, <_0^*)$ where $x <_0^* y$ if $y <_0 x$, and the *zero augmentation* of $(X, <_0)$ is

$$(X, <_0) + 0 = (X \cup \{z\}, <_0 \cup \{(z, x) : x \in X\}), \quad z \notin X.$$

Let $\mathscr{R}$ denote an infinite collection of closed, connected regions in the plane. Call

$(X, <_0)$ an *$\mathscr{R}$-order* if there exists $f: X \to \mathscr{R}$ such that, for all $x, y \in X$,

$$x <_0 y \Leftrightarrow f(x) \subset f(y),$$

and let $\langle \mathscr{R} \rangle$ denote the class of all $\mathscr{R}$-orders. We say that $\langle \mathscr{R} \rangle$ is *invertible* if

$$(X, <_0) \in \langle \mathscr{R} \rangle \Rightarrow (X, <_0^*) \in \langle \mathscr{R} \rangle,$$

and that $\langle \mathscr{R} \rangle$ is *zero-augmentable* if

$$(X, <_0) \in \langle \mathscr{R} \rangle \Rightarrow (X, <_0) + 0 \in \langle \mathscr{R} \rangle.$$

Invertibility and zero-augmentability have been studied for various $\mathscr{R}$-orders, including:

circle orders [3, 10, 12]: $\mathscr{R}$ is the collection of circular disks;
regular N-gon orders [4, 8, 12]: with $N \geqslant 3$ fixed, $\mathscr{R}$ is the collection of regular N-gons whose two lowest corners lie on a line parallel to the abscissa;
N-gon orders [4, 10, 12]: with $N \geqslant 3$ fixed, $\mathscr{R}$ is the collection of N-sided convex polygons;
angle orders [5, 8, 12]: $\mathscr{R}$ is the collection of unbounded angular regions. An *angular region* consists of a *vertex* $v \in \mathbb{R}^2$ and all points on the rays from v clockwise from an *initial ray* r_1 to a *terminal ray* r_2, with $0 < \theta < 2\pi$ for the clockwise angle θ from r_1 to r_2.

The first three cases have bounded and convex regions, but the unbounded angular regions are convex only if $\theta \leqslant \pi$. The angular region is a closed half-plane when $\theta = \pi$. When $\theta > \pi$, the closure of the complement of the angular region is a convex angular region whose angle is $2\pi - \theta$.

The classes of circle orders and regular N-gon orders are invertible and zero-augmentable [8, 10, 12]. Here, invertibility implies zero-augmentability: invert the order, add a large region that includes the others, then re-invert to transform the large region into a zero for the original order. On the other hand, for each $N \geqslant 3$ the class of N-gon orders is neither invertible nor zero-augmentable [4]. This is true even if all N-gons are regular, but not similarly oriented.

The situation for angle orders is quite different, owing in part to unboundedness. Since angular region α is properly included in angular region β exactly when the closure of α's complement properly includes the closure of β's complement, the class of angle orders is invertible. But the double inversion technique of the preceding paragraph cannot be used to deduce zero-augmentability from invertibility, for it may be impossible to add an angular region that includes the angular regions used to represent an angle order. Indeed, as announced earlier:

THEOREM 1. *The class of angle orders is not zero-augmentable.*

This is proved in ensuing sections. We construct an angle order Γ_n, suppose that $\Gamma_n + 0$ is an angle order, and obtain a contradiction when n is large. The proof is facilitated by three basic results from Ramsey theory [2, 6] and a related result, Lemma 4, from the theory of tournaments [1, 7].

LEMMA 1. *Every linear arrangement of m^2+1 distinct numbers has either an increasing subsequence or a decreasing subsequence of at least $m+1$ numbers.*

LEMMA 2. *For each positive integer m there is an integer $M(m)$ such that among any $M \geqslant M(m)$ points in the plane there are m which form a convex m-gon.*

Remark. Some of the m among M points that delineate the m-gon could lie on sides between others. The extreme case has all m points collinear, so the interior of the convex m-gon could be empty.

LEMMA 3. *For each positive integer m there is an integer $N(m)$ such that, for every complete graph K_N with $N \geqslant N(m)$ points whose edges are three colored (each edge red, green, or blue), all edges of some complete subgraph K_m have the same color.*

LEMMA 4. *For each positive integer m there is an integer $T(m)$ such that, for every asymmetric and complete ($x \neq y \Rightarrow x <_1 y$ or $y <_1 x$) binary relation $<_1$ on $T \geqslant T(m)$ points, there are m points on which $<_1$ is transitive.*

2. A Special Angle Order

This section defines the angle order Γ_n used to prove Theorem 1. Our proof by contradiction for $\Gamma_n + 0$ begins in the next section. A few definitions will aid our description of Γ_n.

An angular region A with angle θ clockwise from r_1 to r_2 is *little* if $\theta < \pi$, *half-planar* if $\theta = \pi$, and *big* if $\theta > \pi$: see Figure 1. Let $\mathbf{n} = \{1, 2, \ldots, n\}$. We say that a subset I of $\mathbf{n}$ is *contiguous* if I has the form $\{i, i+1, \ldots, j\}$ for $1 \leqslant i \leqslant j \leqslant n$ or $\{i, i+1, \ldots, n, 1, \ldots, j\}$ for $1 \leqslant j < i \leqslant n$. For each $i \in \mathbf{n}$, $\hat{i}$ is the unique integer in $\mathbf{n}$ for which $|i - \hat{i}| = n/2$. *We assume henceforth that n is even.*

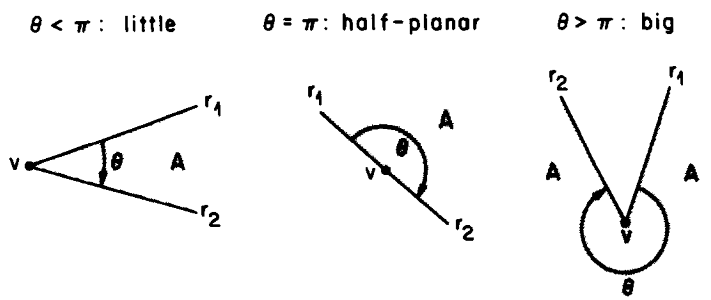

Fig. 1. Angular regions.

We define Γ_n by an angular region representation in the plane. With $\Gamma_n = (X, <_0)$, we take $X = X_1 \cup X_2 \cup X_3$ where

$$X_1 = \{I \subseteq \mathbf{n} : 1 \leqslant |I| \leqslant n-1, \ I \text{ is contiguous}\},$$
$$X_2 = \{\{i,j\} : \{i,j\} \text{ is a noncontiguous 2-set in } \mathbf{n}\},$$
$$X_3 = \{i : i \in \mathbf{n}\}.$$

The X_j are mutually disjoint, $|X_1| = n(n-1)$, $|X_2| = n(n-3)/2$ and $|X_3| = n$. The singletons $\{i\} \in X_1$ are distinguished from the single elements $i \in X_3$.

The angular regions for $x_1 \in X_1$, $x_2 \in X_2$ and $x_3 \in X_3$ will be denoted by $\alpha(x_1)$, $\beta(x_2)$ and $\gamma(x_3)$ respectively, with $x <_0 y$ if and only if the angular region for x is properly included in the angular region for y. All angular regions for Γ_n are positioned with reference to a fixed circular disk C with radius 1 centered at the origin $\mathbf{0} = (0, 0)$. We partition C into $2n$ equal wedges by rays from $\mathbf{0}$ at successive angles of π/n and label the $2n$ ray-perimeter intersection points as $1, 1', 2, 2', \ldots, n, n'$ in clockwise succession. Also let θ_0 denote a fixed angle (dependent on n) that is much smaller than π/n.

The vertex of $\alpha(\{i\})$ for each singleton $\{i\}$ in X_1 is point i on C's perimeter, $\theta(\{i\}) = \theta_0$, and the little $\alpha(\{i\})$ extends away from C and is bisected by the ray from $\mathbf{0}$ through i. The vertex of $\alpha(\{\text{not } i\})$ for each $(n-1)$-set $\{\text{not } i\} = \mathbf{n}\backslash\{i\}$ in X_1 is $\mathbf{0}$, $\theta(\{\text{not } i\}) = 2\pi - \theta_0$, and the big $\alpha(\{\text{not } i\})$ has rays at angle $\theta_0/2$ on either side of the ray from $\mathbf{0}$ through i. [The *complement of* $\alpha(\{\text{not } i\})$ properly includes $\alpha(\{i\})$.] Every other $I \in X_1$, with $2 \leqslant |I| \leqslant n-2$, has vertex $\mathbf{0}$ for $\alpha(I)$. When $I = \{i, \ldots, j\}$, $r_1(I)$ is the ray from $\mathbf{0}$ through $(i-1)'$ [n' if $i = 1$] and $r_2(I)$ is the ray from $\mathbf{0}$ through j'. Such an $\alpha(I)$ is half-planar if $j = \hat{i}$. Figure 2 illustrates the angular regions for X_1.

The angular region $\beta(\{i,j\})$ for a noncontiguous pair $\{i,j\}$ in X_2 is formed from the lines tangent to C at points i and j. Its vertex is the intersection point of those lines (if $j = \hat{i}$, tilt one of the tangents very slightly at its point of tangency so that the two lines eventually cross). Its rays are the half-lines tangent to C, and $\theta(\{i,j\})$ is the big angle thus formed so that almost all of C is exterior to $\beta(\{i,j\})$.

Finally, the vertex of $\gamma(i)$ for each $i \in X_3$ is positioned on the ray from $\mathbf{0}$ through i just outside of C so that it lies in the interior of $\alpha(\{i\})$. The rays of $\gamma(i)$ make angle $\theta_0/2$ on either side of the line between i and $\mathbf{0}$ (back toward $\mathbf{0}$), and $\theta(i)$ is the *big* angle thus defined with $\mathbf{0}$ and all of $\alpha(\hat{i})$ outside of $\gamma(i)$. The angular regions for X_2 and X_3 are illustrated in Figure 3.

Recalling that θ_0 is presumed to be very small compared to π/n, we note parts of $<_0$ and its symmetric complement $\sim_0$ ($x \sim_0 y$ if neither $x <_0 y$ nor $y <_0 x$) that are implied by the inclusions and noninclusions in the preceding construction of Γ_n. First, $<_0$ restricted to X_1 agrees with $\subset$:

$$I <_0 J \Leftrightarrow \alpha(I) \subset \alpha(J) \Leftrightarrow I \subset J.$$

Next, for each $\{i,j\} \in X_2$ and all singletons in X_1,

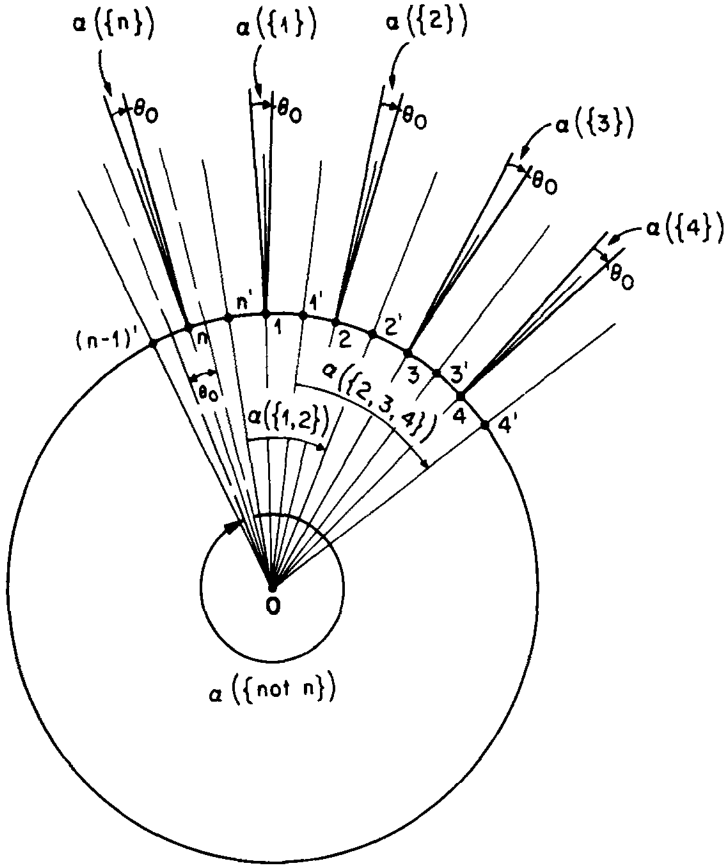

Fig. 2. Angular regions for X_1.

$$\{i\} <_0 \{i,j\} \text{ and } \{j\} <_0 \{i,j\},$$

$$\{k\} \sim_0 \{i,j\} \text{ for every } k \in \mathbf{n}\backslash\{i,j\}.$$

Finally, for each $i \in X_3$,

$$i \sim_0 \{i\} \text{ and } i \sim_0 \{\hat{\imath}\},$$

$$\{j\} <_0 i \text{ for every } j \in \mathbf{n}\backslash\{i, \hat{\imath}\},$$

$$i <_0 \{\text{not } \hat{\imath}\}$$

$$i \sim_0 \{\text{not } j\} \text{ for every } j \in \{\text{not } \hat{\imath}\}.$$

For X_3, $\alpha(\{j\}) \subset \gamma(i)$ for every $j \notin \{i, \hat{\imath}\}$ while neither $\alpha(\{i\})$ nor $\alpha(\{\hat{\imath}\})$ is included in $\gamma(i)$. Moreover, $\gamma(i) \subset \alpha(\{\text{not } \hat{\imath}\})$ since the ray pairs of $\gamma(i)$ and $\alpha(\{\text{not } \hat{\imath}\})$ are parallel and the vertex of $\gamma(i)$ lies in $\alpha(\{\text{not } \hat{\imath}\})$. However, for every $j \neq \hat{\imath}$, the rays of $\gamma(i)$ and $\alpha(\{\text{not } j\})$ cross.

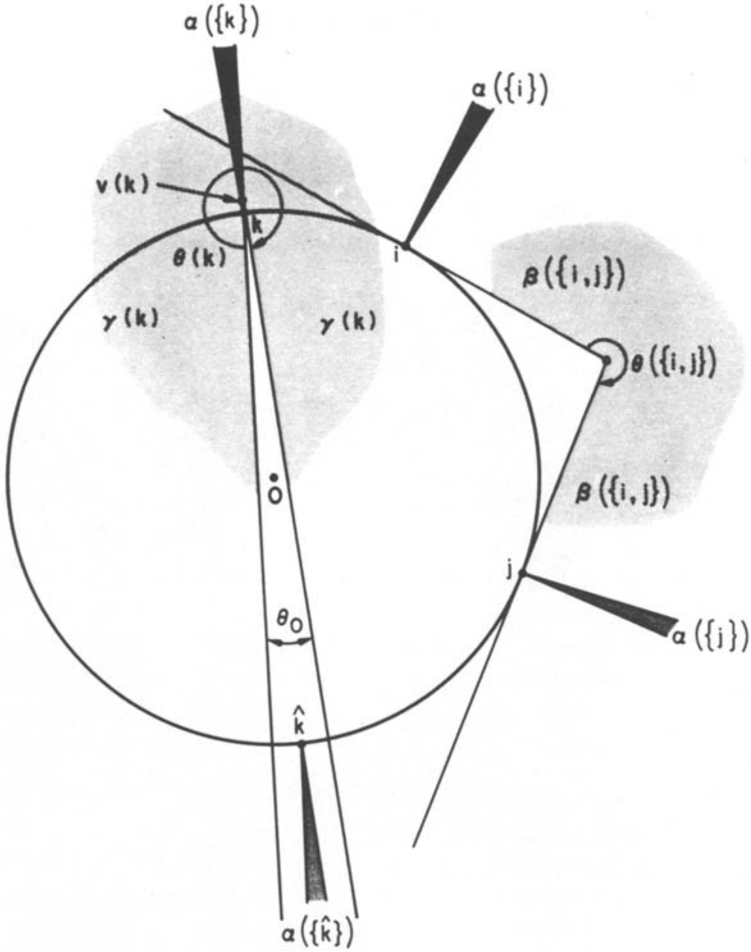

Fig. 3. Angular regions for X_2 and X_3.

3. Supposition and Singleton Regions

The preceding partial specification of $<_0$ and $\sim_0$ identifies the most important parts of $\Gamma_n = (X, <_0)$ for our ensuing proof of

THEOREM 2. *There is an integer* n_0 *such that* $\Gamma_n + 0$ *is not an angle order when* n *is even and* $n \geqslant n_0$.

Since Γ_n is an angle order by construction, we prove Theorem 1 by proving Theorem 2. Proceeding with a proof by contradiction, *we suppose henceforth that* $\Gamma_n + 0$ *is an angle order for every even* n. It will be shown that this is false. In particular, $\Gamma_n + 0$ is not an angle order when n is large.

Given our focal supposition, we presume an angular representation for $\Gamma_n + 0$, *denote the added zero and its angular region by* $\emptyset$, *and position the whole so that* $\emptyset$ *has vertex* $\mathbf{0} = (0, 0)$ *and includes the negative ordinate in its interior. We also sometimes refer to an angular region of the presumed representation of* $\Gamma_n + 0$ *by the name of its element in* X, *such as* $\{i\}$ *for its own angular region.*

This section derives constraints on the singletons $\{i\}$ in X_1 that subsequent sections exploit to obtain the desired contradiction. We note where the section is headed and then show how we get there.

Several definitions are needed. Let v_i and R_i be the vertex of $\{i\}$ and the ray from $\mathbf{0}$ through v_i respectively. Let $i <_2 j$ mean that, as we proceed clockwise around $\mathbf{0}$ from the nonpositive ordinate, we meet R_i before R_j. Let S denote a nonempty subset of $\mathbf{n}$. We say that S is *concave to* $\mathbf{0}$ if $R_i \neq R_j$ for distinct $i, j \in S$ and, for any three $i, j, k \in S$ for which $i <_2 j <_2 k$, the line segment between $\mathbf{0}$ and v_j does not cross over the line segment between v_i and v_k. (The two line segments can touch if v_j is on the segment between v_i and v_k.) Similarly, S is *convex to* $\mathbf{0}$ if all R_i for $i \in S$ are different and $i <_2 j <_2 k$ for $i, j, k \in S$ implies that the line segment between $\mathbf{0}$ and v_j crosses the line segment between v_i and v_k, or touches that line segment. Figure 4 illustrates these definitions.

A subset S of $\mathbf{n}$ is said to be a *natural semiset* if

(a) it is either concave to $\mathbf{0}$ or convex to $\mathbf{0}$,
(b) all $i \in S$ lie on a less-than-semicircular arc of the perimeter of C (see Section 2), and
(c) the clockwise order of the $i \in S$ along this arc equals $<_2$ on S or the inverse of $<_2$ on S.

For example, if $n = 100$ and the v_i on the upper left of Figure 4 have indices 89, 93, 2, 3, 7, 11, 12, 20 in order left to right, then $\{2, 3, 7, 11, 12, 20, 89, 93\}$ is a natural semiset.

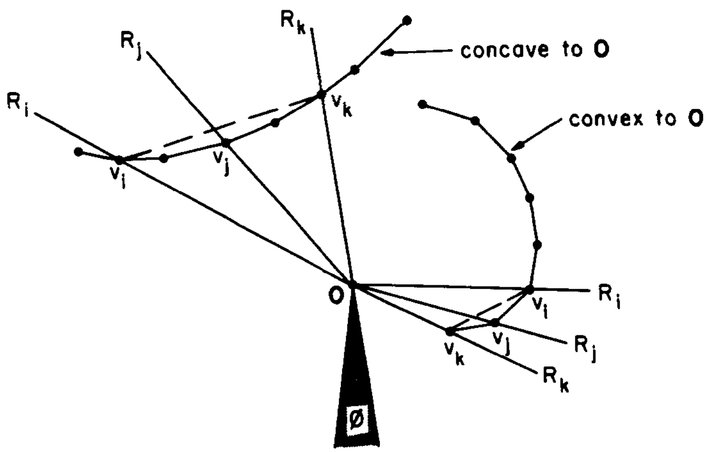

Fig. 4. Sets concave to $\mathbf{0}$ and convex to $\mathbf{0}$.

Finally, a subset S of $\mathbf{n}$ is defined to be *inclusion uniform* if one of the following three things holds:

(1) for all distinct $i, j \in S$, $v_i \notin \{j\}$;
(2) for all distinct $i, j \in S$, $v_i \in \{j\}$;
(3) either for all distinct $i, j \in S$, $i <_2 j \Rightarrow v_i \in \{j\}$ and $v_j \notin \{i\}$, or for all distinct $i, j \in S$, $i <_2 j \Rightarrow v_i \notin \{j\}$ and $v_j \in \{i\}$.

Figure 5 illustrates these three types of inclusion uniformity. Recall that $\{i\}$ and $\{j\}$ are being used to denote the angular regions of the named singletons.

We can now state the main conclusion of this section.

THEOREM 3. *For each positive integer m there is an integer $n_0(m)$ such that, for all even $n \geqslant n_0(m)$, an angular representation that is order-isomorphic to the presumed*

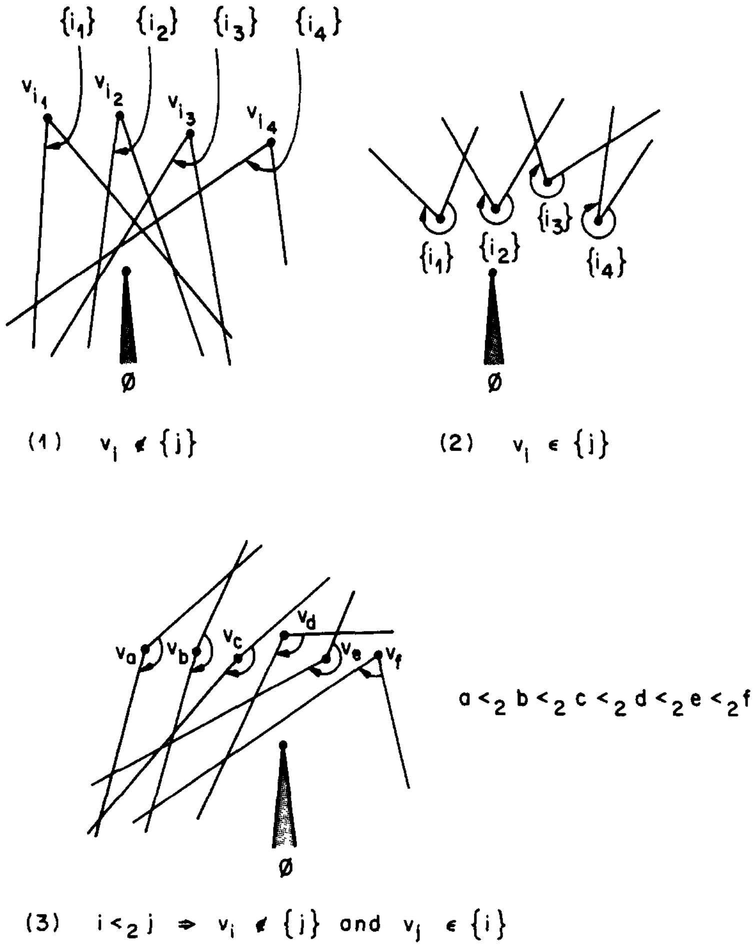

Fig. 5. Three types of inclusion uniformity.

angular representation of $\Gamma_n + 0$ *has an inclusion uniform natural semiset* $S \subseteq \mathbf{n}$ *with* $|S| \geqslant m$.

This is proved in the rest of this section. Its order-isomorphic aspect comes into play only if one ray from $\mathbf{0}$ contains a number of the v_i, in which case we reposition $\emptyset$ slightly to ensure a suitably large concave or convex S. The next section uses Theorem 3 as the point of departure for the completion of our proof of Theorem 2.

To prove Theorem 3 we begin with three lemmas on angular regions for singletons. In addition to v_i and R_i as defined earlier, let θ_{1i} and θ_{2i} be the clockwise angles around v_i from the downward vertical at v_i to the initial ray of $\{i\}$ and the terminal ray of $\{i\}$, respectively. Also let $\theta(\{i\})$ be the angle from the initial to terminal rays of $\{i\}$: see Figure 6.

LEMMA 5. $\theta_{2i} < \theta_{1i}$ *for every* $i \in \mathbf{n}$.

LEMMA 6. $\max_{i \in \mathbf{n}} \theta_{2i} < \min_{i \in \mathbf{n}} \theta_{1i}$.

LEMMA 7. $|\{i \in \mathbf{n} : v_i = v\}| \leqslant 2$ *for every* $v \in \mathbb{R}^2$.

Proof of Lemma 5. $\theta_{1i} \neq \theta_{2i}$ since $0 < \theta(\{i\}) < 2\pi$. If $\theta_{1i} < \theta_{2i}$ then some part of the negative ordinate lies outside $\{i\}$. This contradicts $\emptyset \subset \{i\}$. Hence $\theta_{2i} < \theta_{1i}$. □

Proof of Lemma 6. Suppose to the contrary that $\theta_{1i} \leqslant \theta_{2j}$ for some i and j. This and Lemma 5 imply $\theta_{2i} < \theta_{1i} \leqslant \theta_{2j} < \theta_{1j}$. Let C' be a circular disk centered at $\mathbf{0}$ that

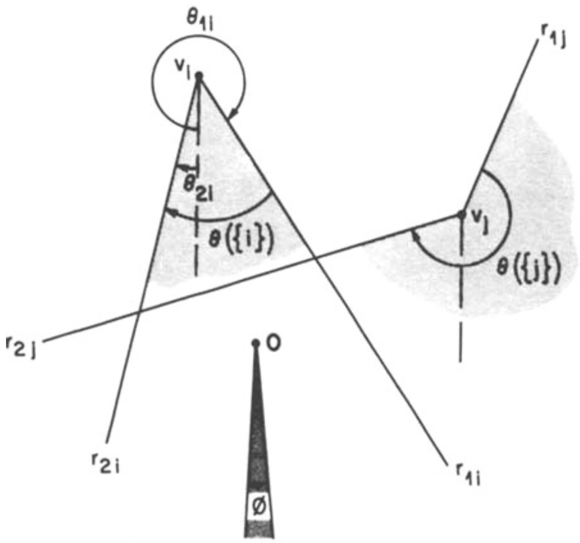

Fig. 6. Angular regions for singletons.

contains v_i, v_j and all crossing points of the rays of $\{i\}$ and $\{j\}$. Then every point on the perimeter of C' that is not in $\{i\}$ must be in $\{j\}$. Moreover, all points *not* in $\{i\} \cup \{j\}$ lie in C'. Then $\{i, j\}$, whether this pair is in X_1 or X_2, must cover the entire plane to have $\{i\} \subset \{i, j\}$ and $\{j\} \subset \{i, j\}$, i.e., for $\{i\} <_0 \{i, j\}$ and $\{j\} <_0 \{i, j\}$. Since this contradicts $0 < \theta < 2\pi$ for angular regions, we conclude that $\theta_{2j} < \theta_{1i}$ for all i and j. □

Proof of Lemma 7. Suppose to the contrary that $v_i = v_j = v_k = v$ for three singletons in X_1. Assume for definiteness that $\theta_{2i} = \max\{\theta_{2i}, \theta_{2j}, \theta_{2k}\}$, and that θ_{1i} or θ_{1j} equals $\max\{\theta_{1i}, \theta_{1j}, \theta_{1k}\}$. Since the three vertices are identical, it follows from Lemma 6 that $\{k\} \subseteq \{i\} \cup \{j\}$. Moreover, whether $\{i, j\}$ is in X_1 or X_2, we have $\{i\} \cup \{j\} \subseteq \{i, j\}$ since $\{i\} <_0 \{i, j\}$ and $\{j\} <_0 \{i, j\}$. Hence $\{k\} \subseteq \{i, j\}$. This contradicts $\{k\} \sim_0 \{i, j\}$ unless the angular regions of $\{k\}$ and $\{i, j\}$ are identical. But then $\{i\} \subset \{k\}$, contrary to $\{i\} \sim_0 \{k\}$. □

We now use the Ramsey theory results at the end of Section 1 to complete the proof of Theorem 3.

Proof of Theorem 3. Lemma 7 implies that either there are at least $n/4$ distinct v_i to the left of the ordinate, or at least $n/4$ distinct v_i on or to the right of the ordinate.

Assume for definiteness that the latter region contains $n_1 \geqslant n/4$ distinct v_i. Apply Lemma 2 to these v_i. Given n_2 as large as we please, it follows with suitably large n and hence n_1 that n_2 of the n_1 v_i form a convex n_2-gon in the right half-plane. Let r_u and r_l denote the upper and lower rays respectively from $\mathbf{0}$ tangent to the n_2-gon. If more than a few v_i are on $r_u \cup r_l$, move the vertex of $\emptyset$ downward slightly (do not change its rays' angles) so that the new $\emptyset$ is properly included in the original $\emptyset$ and each of the new r_u and r_l contains a single v_i from the n_2. Then translate *all* angular regions uniformly upward to reposition the vertex of $\emptyset$ at $\mathbf{0}$. After this order-isomorphic shift (if needed), each of r_u and r_l touches the n_2-gon at only one v_i of the n_2. It is then easily seen that the i for the v_i on the boundary of the n_2-gon from r_u counterclockwise to r_l form an $S_1 \subseteq \mathbf{n}$ that is concave to $\mathbf{0}$, while the i for the v_i on the boundary from r_u clockwise to r_l form an $S_1 \subseteq \mathbf{n}$ that is convex to $\mathbf{0}$. And one of these S_1's has at least $n_2/2$ members.

Taking the larger of the concave S_1 and the convex S_1, it follows from Lemma 1 that we obtain a natural semiset $S_2 \subseteq S_1$ with at least $n_3 = \lfloor \sqrt{n_2/2}/2 \rfloor$ members. By taking n suitably large, we get n_3 as large as we please.

Let K be the complete graph with n_3 points labelled by the $i \in S_2$. For all distinct $i, j \in S_2$, color edge $\{i, j\}$ red if $v_i \notin \{j\}$ and $v_j \notin \{i\}$, color edge $\{i, j\}$ green if $v_i \in \{j\}$ and $v_j \in \{i\}$, and color edge $\{i, j\}$ blue otherwise. Fix m_0 as large as desired. Then, with n_3 suitably large, Lemma 3 yields a monochromatic complete subgraph K_{m_0}. If its color is red or green, we have an inclusion uniform natural semiset with m_0 elements.

Suppose the monochromatic K_{m_0} is blue. For distinct i and j in the corresponding m_0-point subset of S_2, take $i <_1 j$ if $v_i \in \{j\}$ and $v_j \notin \{i\}$. Fix m_1 as large as desired. Then, with m_0 suitably large, Lemma 4 implies that $<_1$ is a linear order on some m_1-point subset $S_3 \subseteq S_2$. Moreover, by Lemma 1, there is an $S_4 \subseteq S_3$ with $|S_4| \geqslant \lfloor\sqrt{m_1/2}\rfloor$ such that $<_1$ restricted to S_4 coincides with either $<_2$ or its dual restricted to S_4. By type (3) inclusion uniformity, S_4 is an inclusion uniform natural semiset of $\mathbf{n}$.

Finally, with m as in Theorem 3, choose m_0 suitably large so that $\lfloor\sqrt{m_1/2}\rfloor \geqslant m$ for the preceding paragraph. Then, by the two preceding paragraphs, some $S \subseteq \mathbf{n}_0$ with $|S| \geqslant m$ is an inclusion uniform natural semiset when n_0 is suitably large. □

4. Out on a Limb

This section and the next bring X_3 into the picture to contradict the supposition behind Theorem 3 by proving that m therein is bounded above. In this section we establish a bounding result for a certain pattern of singletons and $(n-1)$-element subsets of $\mathbf{n}$ in X_1. The next section then uses this to analyze the several types of inclusion uniform natural semisets.

Continuing under the supposition that $\Gamma_n + 0$ is an angle order for each even n, *henceforth let S denote an inclusion uniform natural semiset of* $\mathbf{n}$ *with m members.* With no loss of generality take $S = \{a_1, a_2, \ldots, a_m\}$ with $1 \leqslant a_1 < a_2 < \cdots < a_m \leqslant n/2$ and $a_1 <_2 < a_2 <_2 \cdots <_2 a_m$ by the definition of $<_2$ early in the preceding section.

As before, $\hat{a}_i \in \mathbf{n}$ with $|a_i - \hat{a}_i| = n/2$. Since the $a_i \in S$ lie in less than a semicircular arc of C, $\hat{a}_i \notin S$ and $\{\hat{a}_i\} \subset \{\text{not}[a_1, a_m]\}$ in the presumed representation. Here and later we use $[a, b]$ to denote the contiguous subset of $\mathbf{n}$ clockwise on C's perimeter from a to b inclusive. We continue to let members of $X_1 \cup X_2$ double for their angular regions, but *now use $\delta(i)$ to denote the angular region for $i \in X_3$ in the presumed representation of* $\Gamma_n + 0$. For example, with

$$A \approx_0 B \text{ if neither } A \subset B \text{ nor } B \subset A,$$

the end of Section 2 gives

$$\delta(a_i) \approx_0 \{a_i\}, \quad \delta(a_i) \approx_0 \{\hat{a}_i\}$$
$$\{j\} \subset \delta(a_i) \text{ for every } j \in \mathbf{n}\backslash\{a_i, \hat{a}_i\}$$
$$\delta(a_i) \subset \{\text{not } \hat{a}_i\}$$
$$\delta(a_i) \approx_0 \{\text{not} j\} \text{ for every } j \in \mathbf{n}\backslash\{\hat{a}_i\}.$$

There are two basic ways to position $\{\text{not } a_i\}$ relative to $\{a_i\}$ when $1 < i < m$:

way 1: one or both rays of $\{\text{not } a_i\}$ cross the two-piece linear curve from $v_{a_{i-1}}$ to v_{a_i} to $v_{a_{i+1}}$,

way 2: the rays of $\{\text{not } a_i\}$ do not cross the indicated two-piece linear curve.

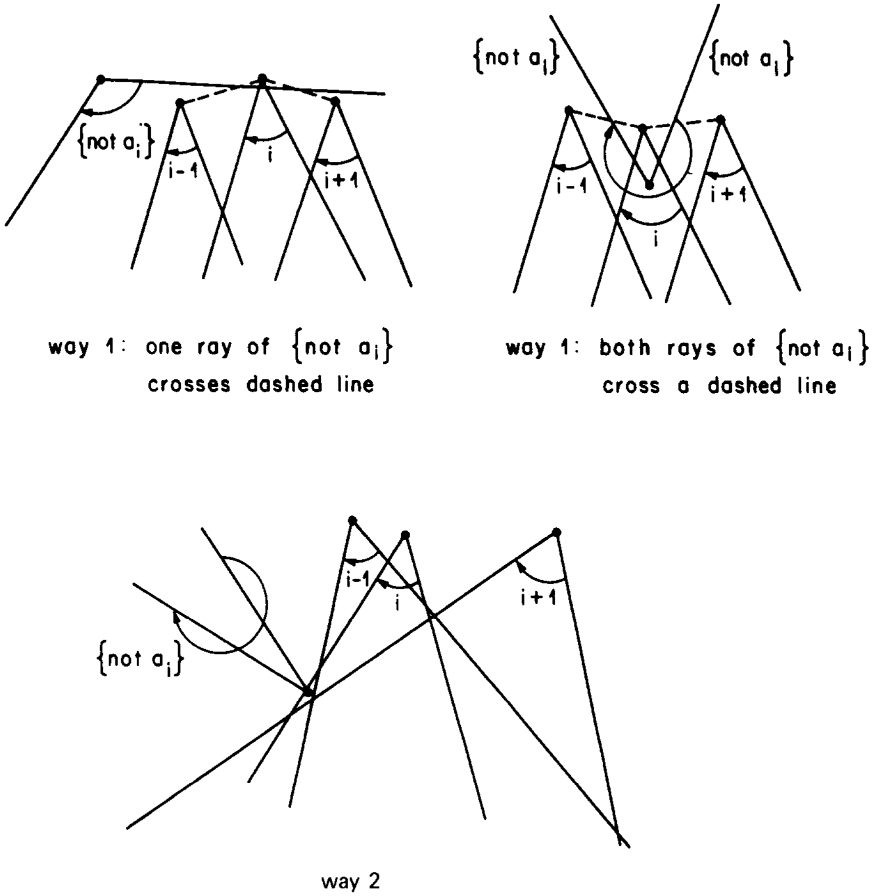

Fig. 7. Singletons and $(n-1)$-sets.

These are illustrated in Figure 7. While various mixtures of these ways might occur, it is easily seen that only way 2 is possible when type (3) with its nested singleton vertices holds for S.

LEMMA 8. *The number of* $a_i \in S$ *for which way* 2 *occurs is bounded above.*

We prove Lemma 8 in the rest of this section by a series of steps that consider positions for the angular regions of elements in X. Assuming that way 2 occurs for a number of $a_i \in S$ (about 30 will suffice), we eventually arrive at a contradiction.

Let way 2 hold for a large number of a_i. Some of the cuts of $\{a_i\}$ by $\{\text{not } a_i\}$ could occur on terminal rays of the $\{a_i\}$ (bottom part of Figure 7) while others occur on initial rays. Take the majority occurrence of these two cases and reduce S by retaining only the a_i thus involved. Let S refer to the reduced set which, like the original, is an inclusion uniform natural semiset.

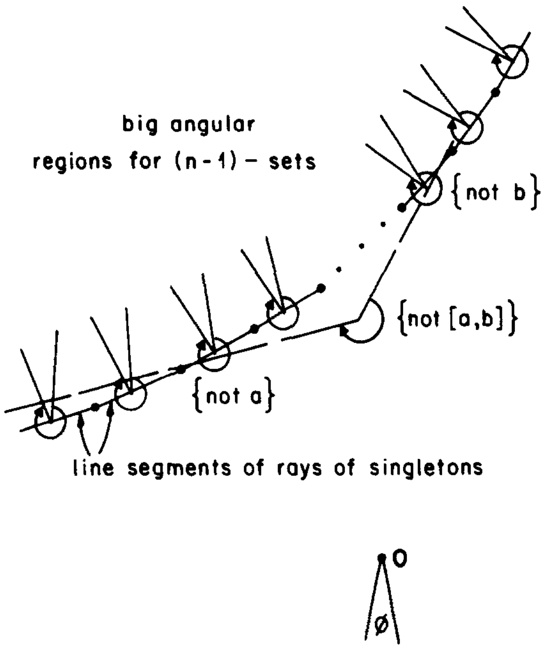

Fig. 8. Way 2 for a succession of singletons.

Figure 8 illustrates how the $(n-1)$-sets for S thus reduced penetrate rays of their corresponding singletons. Because $\{a_j\} \subset \{\text{not } a_i\}$ when $j \neq i$, and each singleton ray segment shown extends indefinitely in one direction or the other, the singleton ray segments are concave to $\mathbf{0}$. The angular regions for the singletons lie below those ray segments and of course include $\emptyset$.

Figure 8 also illustrates $\{\text{not}[a, b]\}$ for $a < b$ in S. Its placement is dictated by $\{a_i\} \subset \{\text{not}[a, b]\} \subset \{\text{not } a_j\}$ for $a_i \in S\backslash[a, b]$ and $a_j \in [a, b]$. Careful viewers will note a potential inaccuracy in this part of the figure (if ray segments for singletons outside $[a, b]$ extend toward the middle of the figure, the rays of $\{\text{not}[a, b]\}$ must lie above them), but such artistic license should not be troublesome.

The configuration of Figure 8 forces two other things. First, since $\{j\} \subset \delta(\hat{a}_i)$ for all $j \notin \{a_i, \hat{a}_i\}$ along with $\{a_i\} \approx_0 \delta(\hat{a}_i)$ and $\delta(\hat{a}_i) \subset \{\text{not } a_i\}$, the rays of $\delta(\hat{a}_i)$ must enclose those of $\{\text{not } a_i\}$ and cut the $\{a_i\}$ ray segment with vertex beneath that segment. Each $\delta(\hat{a}_i)$ is big, except possibly for the ones on the ends, and because of regions like $\{\text{not}[a, b]\}$, which bear $\approx_0$ to $\delta(\hat{a}_i)$, at least one ray of $\delta(\hat{a}_i)$ must cross a ray of each $\{\text{not}[a, b]\}$ for $a < a_i < b$.

Second, since each $\{\text{not } [a, b]\}$ must properly include every singleton $\{\hat{a}_i\}$ ($\hat{a}_i \notin [a, b]$ because of the semicircular feature of S), the $\{\hat{a}_i\}$ must lie wholly to the right of and/or beneath the rays of the $\{\text{not}[a, b]\}$. In addition, since $\{\hat{a}_i\} \approx_0 \delta(\hat{a}_i)$ and $\{\hat{a}_i\} \subset \delta(\hat{a}_j)$ when $j \neq i$, $\{\hat{a}_i\}$ must cross a ray of $\delta(\hat{a}_i)$ but cross no rays of the other $\delta(\hat{a}_j)$. Such crossings could occur on either the upper right or lower left of Figure 8. We assume for definiteness that the majority occur on the upper right, discard all a_t not thus involved, trim off the last two remaining a_t on each end to tidy things up, and refer to the set of a_t that remain as S. The value of m for this newly reduced S is no smaller than about 1/4 of its original value.

This brings us to a configuration like the one illustrated in Figure 9. It will be noted that the boundary on the upper right formed by the lowest segments of the initial rays of the $\delta(\hat{a}_t)$ bends rightward as we go higher (convex to $\mathbf{0}$) and that the order of the $\hat{a}_t$ as we go up is inverse to the order of the a_t ray segments on the lower left.

We consider positions for the $\{\text{not } \hat{a}_t\}$ next. One possibility for $\{\text{not } \hat{a}_t\}$ is to have it cut inside or near the vertex of $\{\hat{a}_i\}$ (similar to a way 1 case on Figure 7) with rays extending above the other $\{\hat{a}_t\}$ vertices: see Figure 10. However, this will not work, for if the $\{\text{not } \hat{a}_t\}$ are thus positioned, consider $\mathbf{Z} = \{\text{not}[a_k, \hat{a}_t]\}$. Among other things (see C on Figure 10), $\mathbf{Z}$ includes $\{\hat{a}_j\}$ and $\{\hat{a}_k\}$ and is included in {not

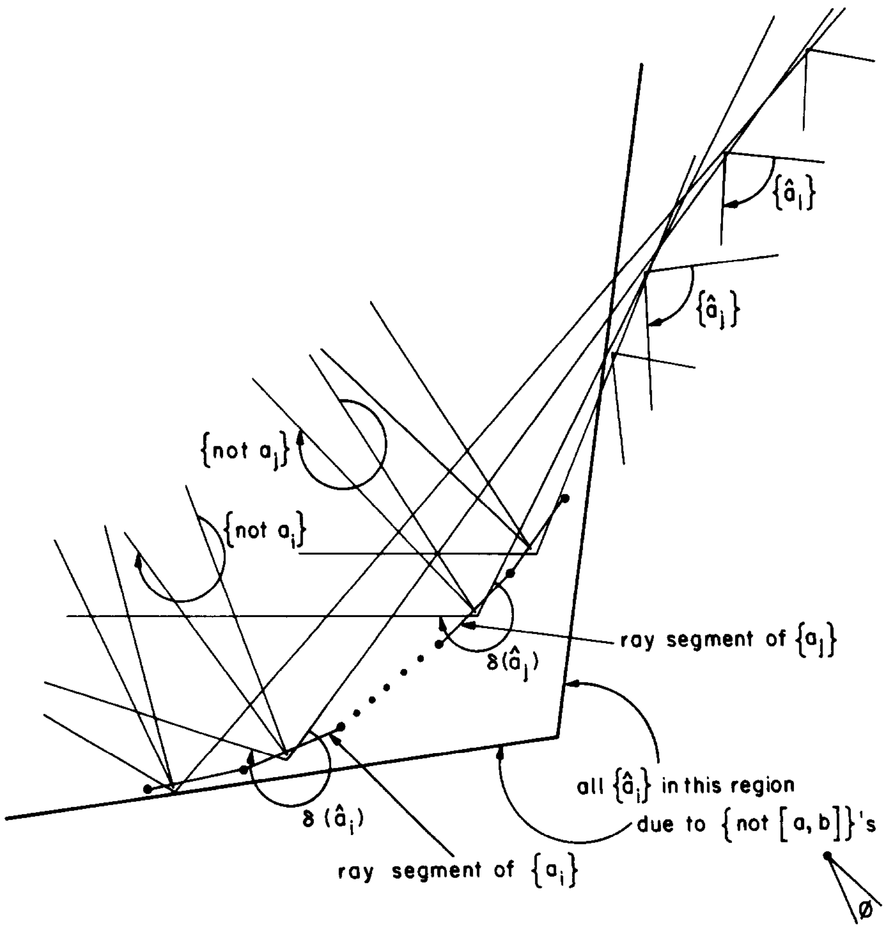

Fig. 9. Positions for $\delta(\hat{a}_t)$ and $\{\hat{a}_t\}$.

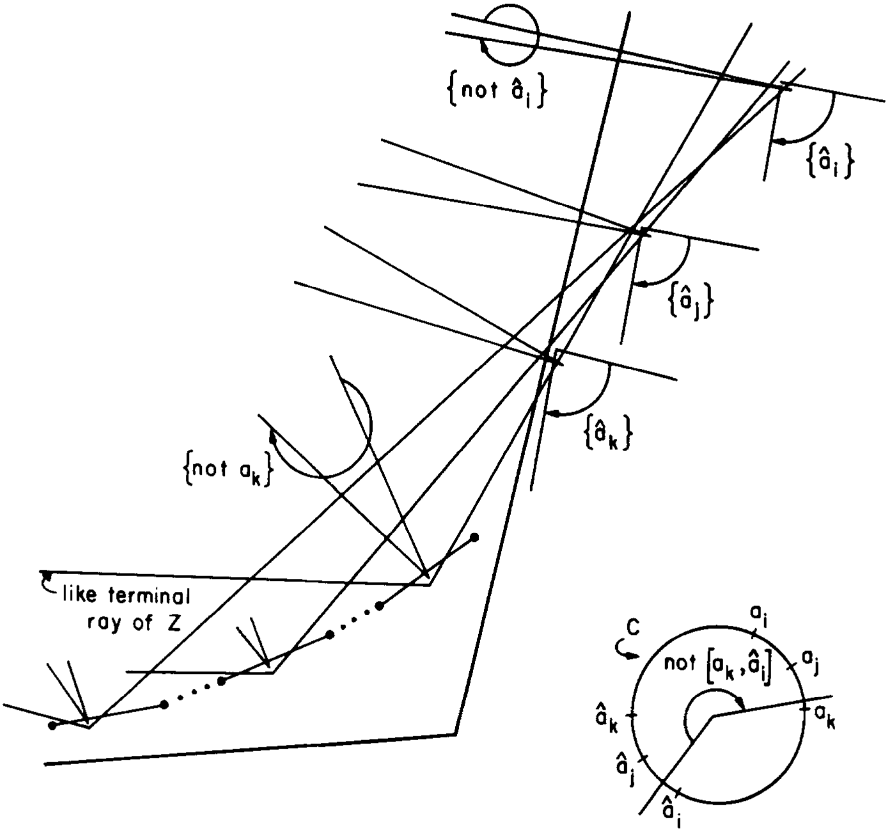

Fig. 10. Disallowed positions for the {not $\hat{a}_i$}.

a_k} and {not $\hat{a}_i$}. Then **Z**'s terminal ray goes leftward beneath {not a_k} like the terminal ray of $\delta(\hat{a}_k)$. Its initial ray must then pass through or to the right of the vertices of {not a_k} and {not $\hat{a}_i$} and through or to the left of the vertices of {$\hat{a}_j$} and {$\hat{a}_k$}. But this is impossible since if the initial ray passes to the left of {$\hat{a}_k$} it intersects the initial ray of $\delta(\hat{a}_j)$ prior to the vertex of {$\hat{a}_j$} and therefore remains above that ray, which implies that it cannot be wholly below the complement of {not $\hat{a}_i$}.

We conclude that at most two a_i could have {not $\hat{a}_i$} positioned as in Figure 10, and with no real loss in generality assume henceforth that a way 1 position is never used for the {not $\hat{a}_i$}. It follows that way 2 must be used for these $(n-1)$-sets relative to their {$\hat{a}_i$}, either on initial rays (off to the right) or on terminal rays (downward). It is easily seen that the first possibility is futile, so we consider {not $\hat{a}_i$} cuts of terminal {$\hat{a}_i$} rays.

Figure 11 illustrates the latter situation for $a < b < c$ in S. Terminal rays of the {$\hat{a}_i$} must be tilted as shown to accommodate penetrations by the corresponding {not $\hat{a}_i$}, and the $\delta(a_i)$, shown with dashed rays, must be positioned in a manner

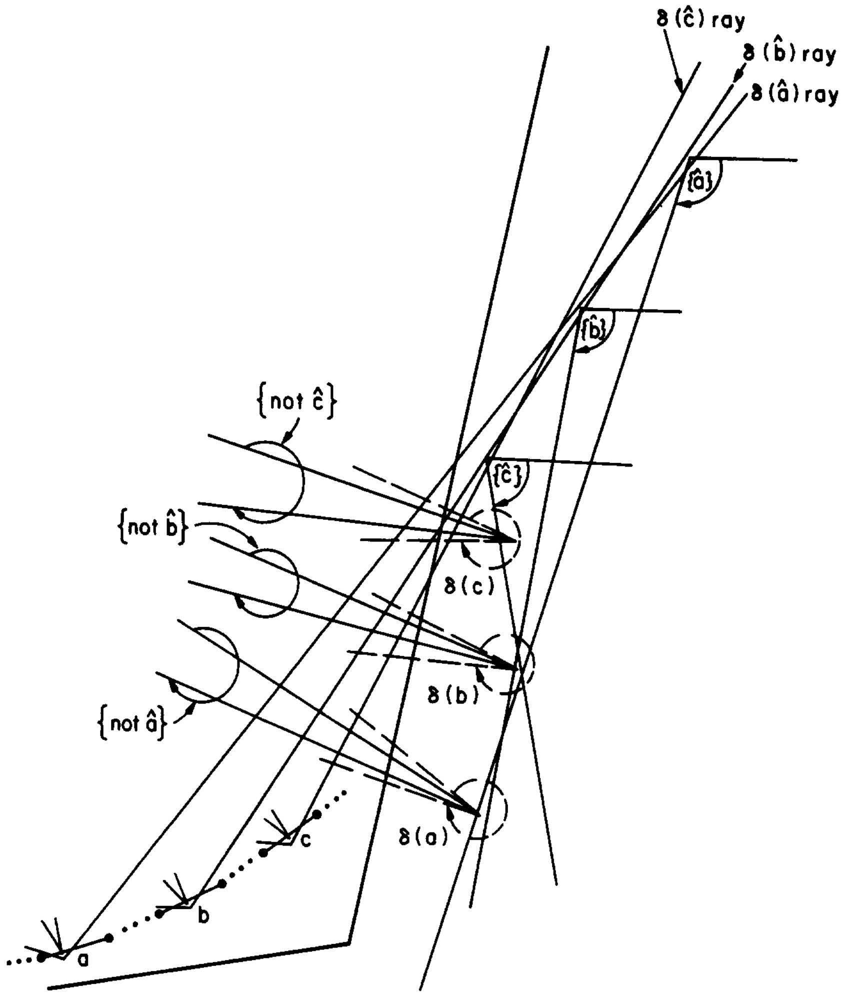

Fig. 11. New positions for the $\{\text{not } \hat{a}_i\}$ and $\delta(a_i)$.

similar to the $\delta(\hat{a}_i)$ in the lower left. Each $\delta(a_i)$ is included in its $\{\text{not } \hat{a}_i\}$ and includes all singletons except $\{\hat{a}_i\}$ and $\{a_i\}$.

Figure 11 does not show crossings of the $\{a_i\}$ with their $\delta(a_i)$. Such a crossing might occur in one of three ways:

1. the left ray segments of the terminal rays of the $\{\hat{a}_i\}$ are positioned down below what is now shown on Figure 11; the $\{\text{not } \hat{a}_i\}$ and $\delta(a_i)$ penetrations are down there also and their rays extend leftward as before; the initial (upper) ray of $\delta(a_i)$ is cut by a ray of $\{a_i\}$;
2. the left ray segments of the terminal rays of the $\{\hat{a}_i\}$ are up to the right, somewhat as shown in Figure 11; the terminal (lower) ray of $\delta(a_i)$ is cut by the terminal ray of $\{a_i\}$, which goes leftward from the vertex of $\{a_i\}$; that vertex

is on a leftward extension of the $\{a_i\}$ segment (shown in the figure) which is on the initial ray of $\{a_i\}$;

3. a picture like Figure 11 applies with an initial ray for $\{a_i\}$ cutting upward between the vertices of $\{\text{not } \hat{a}_i\}$ and $\delta(a_i)$.

It is easily checked that the first two of these three ways are infeasible: we omit the details. (At most one a_i can be handled by those ways.)

This leaves the third way, which is shown in Figure 12 after re-positioning regions slightly from Figure 11. The initial rays of $\{a\}$, $\{b\}$ and $\{c\}$ are dashed. The middle one, for $\{b\}$, cuts between the vertices of $\{\text{not } \hat{b}\}$ and $\delta(b)$, and is to the right of $\delta(a)$, $\{\text{not } \hat{a}\}$, $\delta(c)$ and $\{\text{not } \hat{c}\}$ for the necessary inclusions. The terminal rays of $\{a\}$, $\{b\}$ and $\{c\}$ are extensions of our original segments on the lower left.

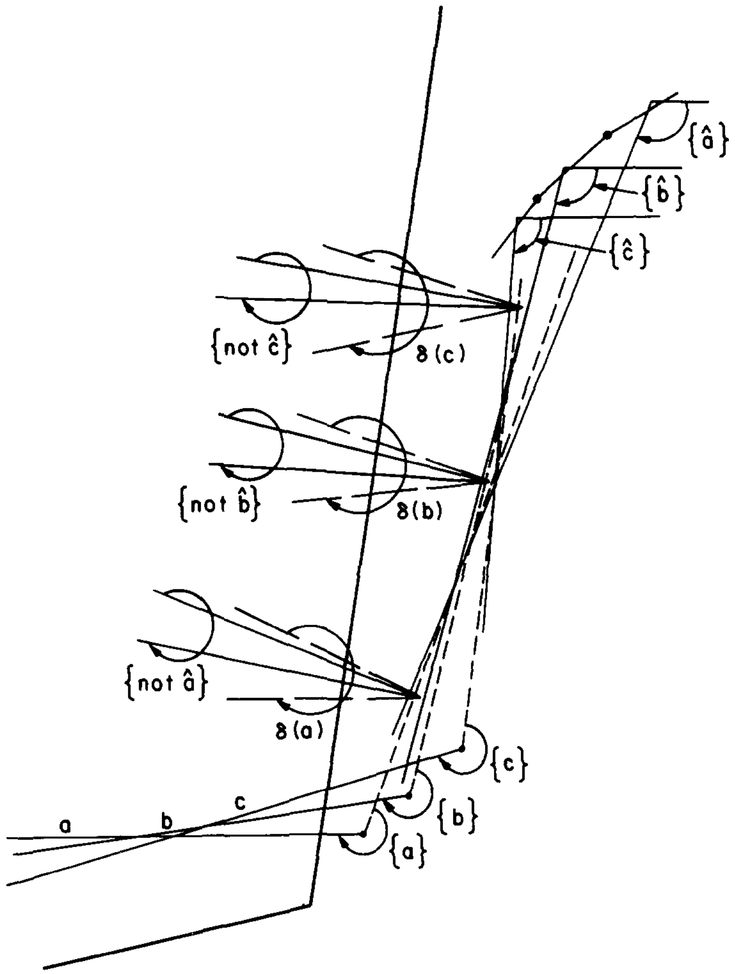

Fig. 12. Preparation for a contradiction.

Because of the relative slopes of those segments and the dashed initial rays, the vertices of $\{a\}$, $\{b\}$ and $\{c\}$ must have the pattern illustrated on Figure 12.

We conclude the proof of Lemma 8 by showing that it is impossible to position $\{a, c\}$ from X_2 in the necessary way when a picture like Figure 12 obtains. Since $\{a, c\}$ includes $\{a\}$ and $\{c\}$ it must be a big angular region whose rays are above and to the left of those of $\{a\}$ and $\{c\}$. And $\{a, c\}$ must cut $\{b\}$, $\{\hat{b}\}$, $\{\hat{c}\}$ and $\{\hat{a}\}$ as well as all other singletons besides $\{a\}$ and $\{c\}$. This forces the vertex of $\{a, c\}$ into either the little triangular area beneath b on the lower left or into the dashed triangular area that contains the vertex of $\delta(b)$.

Suppose $\{a, c\}$ cuts $\{b\}$ at the little triangular area below b. Then the initial ray of $\{a, c\}$ goes upward at a slope at least as great as the slope of $\{c\}$'s initial

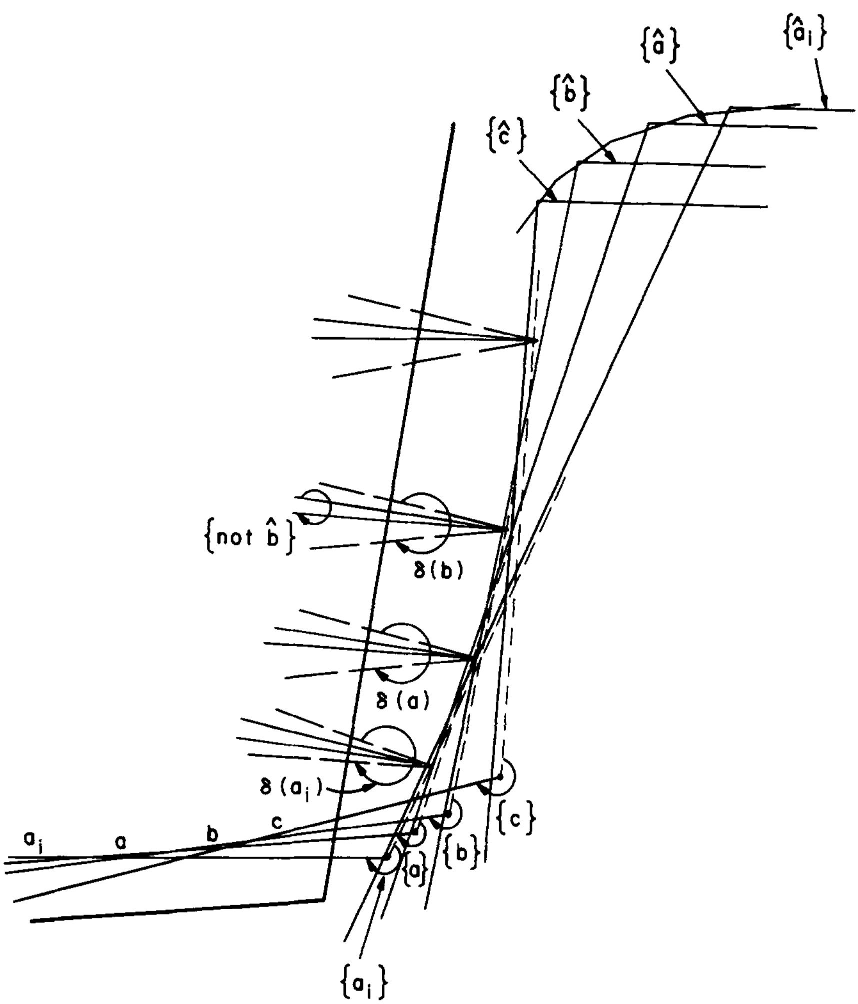

Fig. 13. The contradiction.

(dashed) ray. But since the slope of $\{c\}$'s initial ray exceeds the slope of $\{\hat{b}\}$'s terminal ray, we get $\{\hat{b}\} \subset \{a, c\}$ for a contradiction.

We are therefore forced to put the vertex of $\{a, c\}$ near to the vertex of $\delta(b)$. With slight changes in the initial rays of $\{a\}$ and $\{c\}$, this can be done satisfactorily in a picture like Figure 12. However, it cannot be done satisfactorily if we consider another a_t from S that precedes a ($a_t < a$), and such a_t's are available when a, b and c are chosen initially above the first few members of S. Figure 13 shows what happens with this addition. The initial ray of $\{a\}$ cuts $\delta(a)$ and goes to the right of $\delta(a_t)$; the terminal ray of $\{\hat{a}_t\}$, coming down from the top right, goes to the right of $\delta(a)$ and cuts $\delta(a_t)$. Hence the initial ray of $\{a\}$ lies to the left of the terminal ray of $\{\hat{a}_t\}$ from somewhere below the vertex of $\delta(a)$ upward. But then an otherwise feasible position for the vertex of $\{a, c\}$ near the vertex of $\delta(b)$ forces $\{\hat{a}_t\} \subset \{a, c\}$ for our final contradiction.

5. More Bounds

We conclude the proof of Theorem 2 and the desired contradiction to the supposition behind Theorem 3 by considering inclusion types of natural semisets as defined prior to Theorem 3.

LEMMA 9. *The number of elements in a type* (2) *or type* (3) *inclusion uniform natural semiset S is bounded above.*

Proof. It was already noted (and is obvious) that a type (3) S can have only way 2 placements of $\{\text{not } a_i\}$ relative to the singletons. The same thing is easily seen for a type (2) S, whether concave to $\mathbf{0}$ or convex to $\mathbf{0}$, in view of Lemma 6 and $\{a_j\} \subset \{\text{not } a_i\}$ for $j \neq i$. Lemma 8 completes the proof. □

We now consider type (1) inclusion uniformity, first with S concave to $\mathbf{0}$ and then with S convex to $\mathbf{0}$.

LEMMA 10. *The number of elements in a type* (1) *inclusion uniform semiset S that is concave to* $\mathbf{0}$ *is bounded above.*

Proof. Suppose to the contrary, and let $m = |S|$ be large relative to the upper bound on way 2 for Lemma 8. We can then assume that way 1 obtains for all $\{\text{not } a_i\}$ versus the $\{a_i\}$ since the few a_i that exhibit way 2 can be deleted from S without changing its status as a type (1) inclusion uniform natural semiset.

Figure 14 illustrates S concave to $\mathbf{0}$ with way 1 penetrations of the $\{a_i\}$ by the $\{\text{not } a_i\}$. We show seven consecutive a_i from S, the middle five of which are denoted by a through e. The position of $\{\text{not}[a, e]\}$ is prescribed by its inclusion in each of $\{\text{not } a\}$ through $\{\text{not } e\}$ and its inclusion of the singletons other than $\{a\}$ through $\{e\}$. Other $\{\text{not}[a_i, a_j]\}$ have similar positions.

Possible positions for the $\{\hat{a}_i\}$ and the $\delta(\hat{a}_i)$ are dictated by considerations similar to those for Figure 9. Since $\{\hat{a}_i\} \subset \{\text{not}[a_j, a_k]\}$ for all $a_j < a_k$ in S, the $\{\hat{a}_i\}$ must lie

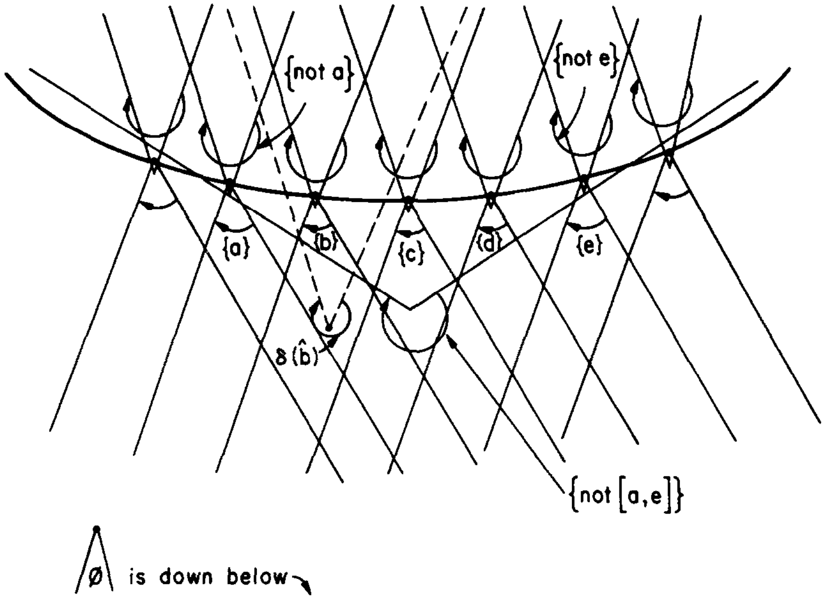

Fig. 14. S concave to $\mathbf{0}$, type (1) inclusion uniformity, way 1 for $\{a_i\}$ versus $\{\text{not } a_i\}$.

below the rays of regions like $\{\text{not}[a, e]\}$. $\delta(\hat{a}_i)$ is included in $\{\text{not } a_i\}$, includes all singletons other than $\{a_i\}$ and $\{\hat{a}_i\}$, and cuts these two. One might imagine $\delta(a_i)$ down on the lower right as a big region whose vertex cuts the initial ray of $\{a_i\}$, with ray tiltings of other $\{a_j\}$ to accommodate a number of $\delta(a_j)$ in this fashion, but this is infeasible since then there would be points in $\delta(a_i)$ [in the complement of $\{\text{not } a_i\}$ up top] that are not in $\{\text{not } a_i\}$. Hence each $\delta(a_i)$ must be a big region whose rays go upward outside the rays of $\{a_i\}$ and which does not cut any $\{a_j\}$ for $j \neq i$. Moreover, because $\delta(a_i)$ cannot properly include $\{\hat{a}_i\}$, either its vertex must cut below rays of all $\{\text{not}[a_j, a_k]\}$, or else one or the rays of $\delta(a_i)$ must cross under the rays of the $\{\text{not}[a_j, a_k]\}$ off to the right or the left. Since m is supposed to be large, one of these possibilities must occur in profusion. We consider them further in detail.

Suppose a large number of $\delta(\hat{a}_i)$ vertices go below the $\{\text{not}[a_j, a_k]\}$ in the manner of $\delta(\hat{b})$ on Figure 14, and that $\delta(\hat{a}_i) \approx_0 \{\hat{a}_i\}$ is satisfied by $\{\hat{a}_i\}$ cutting $\delta(\hat{a}_i)$ near its vertex. Then all cuts of the $\{\hat{a}_i\}$ by their $\{\text{not } \hat{a}_i\}$ occur off to the right or off to the left outside the region taken up by the $\{a_i\}$. Assume for definiteness that the majority of the latter cuts occur off to the right. If enough of the vertices of the $\{\hat{a}_i\}$ thus involved are not also off to the right, then a way 2 pattern occurs for these $\{\hat{a}_i\}$ versus their $\{\text{not } \hat{a}_i\}$ and we get a contradiction by the proof of Lemma 8. We therefore assume that the $\{\hat{a}_i\}$ vertices are off to the right in a way that allows way 1 penetrations of the $\{\hat{a}_i\}$ by their $\{\text{not } \hat{a}_i\}$. The two general patterns that could obtain are shown in Figure 15.

The lower pattern in Figure 15 with big $\{\hat{a}_i\}$ forces way 2 (cf. the proof of

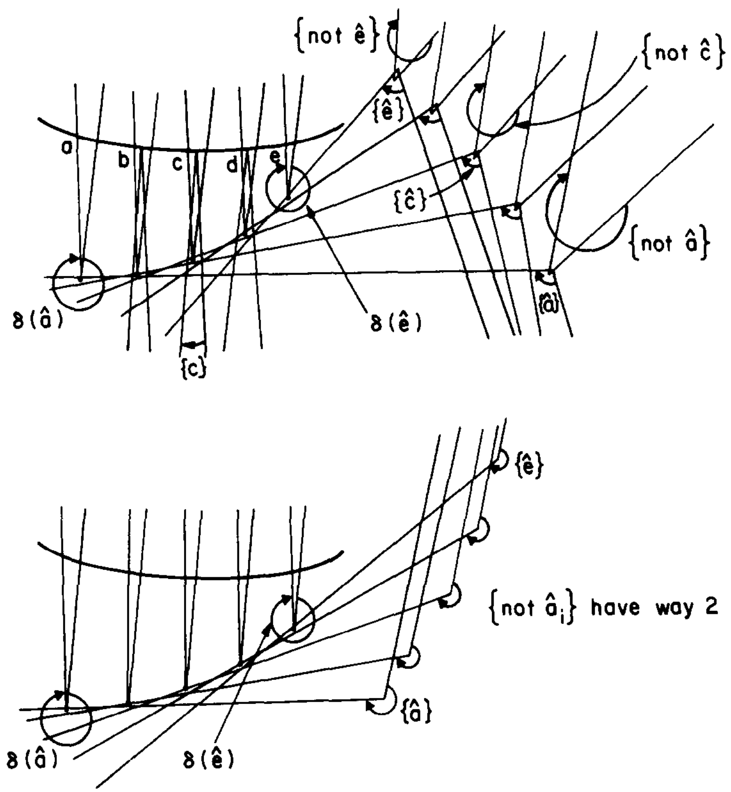

Fig. 15. Crossings of $\delta(\hat{a}_i)$ and $\{\hat{a}_i\}$ beneath $\{a_i\}$ vertices.

Lemma 9) for the $\{\text{not } \hat{a}_i\}$ penetrations, so we focus on the upper pattern. As drawn, it is easily checked that there is no feasible way to position the $\delta(a_i)$ regions. For upper-pattern feasibility we need to have the vertices of the $\{\hat{a}_i\}$ going upward to the right, as shown in Figure 16, and necessarily convex to **0**. The problem in this case is similar to the problem of Figure 10: $\mathbf{Z} = \{\text{not}[d, \hat{a}]\}$, which includes $\{a\}$, $\{b\}$, $\{c\}$, $\{\hat{b}\}$, $\{\hat{c}\}$ and $\{\hat{d}\}$, and is included in $\{\text{not } a\}$ and $\{\text{not } \hat{a}\}$, has no feasible position. For example, to include $\{c\}$ and be included in $\{\text{not } d\}$, its terminal ray goes beneath the vertex of $\{\text{not } d\}$ and moves off to the left above the vertices of $\{c\}, \{b\}, \ldots$. Its initial ray then must go up to the right above the vertices of $\{\hat{d}\}$, $\{\hat{c}\}$ and $\{\hat{b}\}$ and in the process cross the initial ray of $\delta(b)$ before the vertex of $\{\hat{b}\}$ and remain above that ray thereafter. But then the initial ray of $\mathbf{Z}$ cannot have the complement of $\{\text{not } \hat{a}\}$ wholly above it, for a contradiction.

We conclude that, with m suitably large, most of the crossings for $\delta(\hat{a}_i) \approx_0 \{\hat{a}_i\}$ occur on the upper right or upper left of Figure 14. With no loss of generality (or take the majority case) we assume that those crossings occur off to the right. This gives a picture like Figure 9 viewed from the lower right corner with the obvious difference for the already positioned singletons in the lower left part. By tracing the proof of Lemma 8 from its first mention of Figure 9 through Figure 11, it is easily

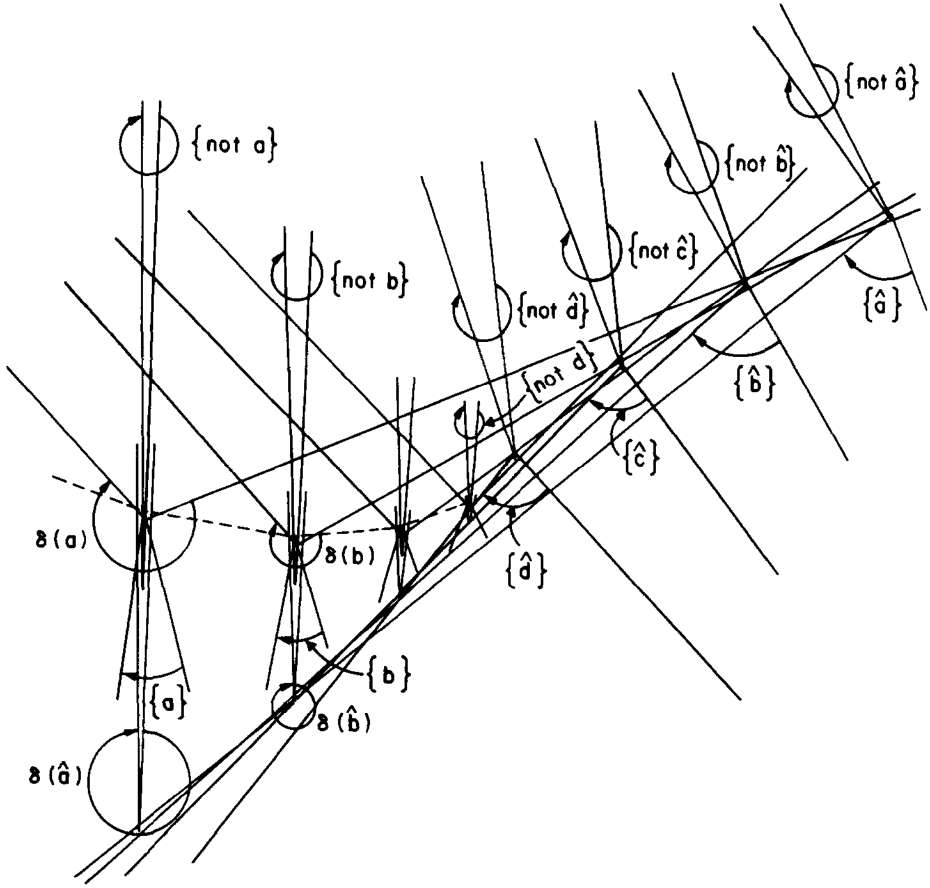

Fig. 16. Preparation for another contradiction.

checked that we arrive at a configuration like Figure 11. But then we have a way 2 situation with the $\{\hat{a}_i\}$ and their $\{\text{not } \hat{a}_i\}$, and this is precisely the situation limited by Lemma 8. Hence $|S|$ for Lemma 10 is bounded above. □

We conclude with the final possibility.

LEMMA 11. *The number of elements in a type* (1) *inclusion uniform natural semiset S that is convex to* $\mathbf{0}$ *is bounded above.*

Proof. Suppose to the contrary. In this case the value of $m = |S|$ must be somewhat larger than in the proof of Lemma 10 for reasons explained shortly.

Much of the analysis for S convex to $\mathbf{0}$ is similar to the analysis for Lemma 10. This is true in particular if the vertices of the $\delta(\hat{a}_i)$ lie near the vertices of the $\{a_i\}$. Then, even though the concave pattern of Figure 14, or the lower left part of Figure 10, or the dashed-line part of Figure 16 is convex rather than concave, the contradictions obtained previously still apply. For example, a concave to convex change in the dashed-line pattern of Figure 16 does not avoid the problem of placing $\mathbf{Z}$ described in the penultimate paragraph of the preceding proof. We omit repetitious details of this part of the proof of Lemma 11.

There is, however, an alternative in placing the $\delta(\hat{a}_i)$ for type (1) when S is convex to $\mathbf{0}$ that is unavailable when S is concave to $\mathbf{0}$. This consists of having one ray of $\delta(\hat{a}_i)$ pass beneath the rays of $\{\text{not } a_i\}$ while cutting through the peak of $\{a_i\}$ and passing above all other $\{a_j\}$. This then leaves the vertices of the $\delta(\hat{a}_i)$ free to be placed off to the left or the right, with latitude for the positions of the other rays. In fact, because of the need for the $\{\text{not}[a_j, a_k]\}$ to include the $\{\hat{a}_i\}$ and the need for $\delta(\hat{a}_i)$ to cut $\{\hat{a}_i\}$, the $\delta(\hat{a}_i)$ must be patterned to accommodate the $\{\hat{a}_i\}$, which in turn must not penetrate the complements of the $\{\text{not}[a_j, a_k]\}$ regions. Figure 17 shows one way to do this. Here the $\delta(\hat{a}_i)$ bend down to the right, but they could also bend up to the right. We do not illustrate the latter case on Figure 17 but remark that its analysis is similar to what follows.

Figure 17 is drawn to suggest that it can accommodate most of the regions considered earlier. Each $\{\text{not}[a_i, a_j]\}$ on the upper left has terminal ray above $\{a_{i-1}\}$ and beneath $\{\text{not } a_i\}, \ldots, \{\text{not } a_j\}$, then its initial ray cuts upward: the heavy lines there indicate that all singletons not in the a_i part of C must be to the right of those lines. A pattern symmetric to this is shown on the upper right for the $\hat{a}_i$. The position of $\delta(a_i)$ is similar to $\delta(\hat{a}_i)$, except that $\delta(a_i)$ cuts $\{\text{not } a_i\}$ and goes under the rays and vertex of $\{\text{not } \hat{a}_i\}$ instead of the other way around. It is easily checked that $\{a_i, \hat{a}_j\}$ in X_2 cannot be little (else it will include unwanted singletons) and that these X_2 regions can only be accommodated by the crossing pattern in the lower part of the figure formed from initial rays of the $\{a_i\}$ and terminal rays of the $\{\hat{a}_i\}$. We could take the vertex of $\{a_i, \hat{a}_j\}$ as the intersection point of $r_1(\{a_i\})$ and $r_2(\{\hat{a}_j\})$ with rays coincident with those rays but in the opposite directions. The position of $\{a_i, \hat{a}_j\}$ could differ slightly from this, but it must cut rays of all other singletons and be a big region with vertex down in the crossing area and upward rays: see $\{b, \hat{c}\}$. Figure 17 also accommodates X_2 from the same side as well as $\mathbf{Z}$ regions used for previous contradictions.

Since Figure 17 or a similar diagram does not appear to give an easy contradiction, we extend our analysis. Nothing said earlier forces S's points to lie on nearly a semicircular arc of C. Indeed, with another division by 2 in Section 3, we can assume that S involves less than a quarter of C's perimeter. This is pictured in Figure 18 where, with $\beta = \beta(\{a_m, \hat{a}_1\})$ as in Section 2, we draw dashed lines parallel to the rays of β and consider the arc of C between them in the direction of β's vertex. Assuming as before that θ_0 is very small and the vertices of the $\gamma(i)$ are just barely outside of C, it follows that all $h \in \mathbf{n}$ on that arc between its end points from $\mathbf{n}$ satisfy $\beta(\{a_m, \hat{a}_1\}) \subset \gamma(u)$ or, in terms of continuing notation, that

$$\{a_m, \hat{a}_1\} \subset \delta(h).$$

One such $\delta(h)$ is pictured on Figure 17.

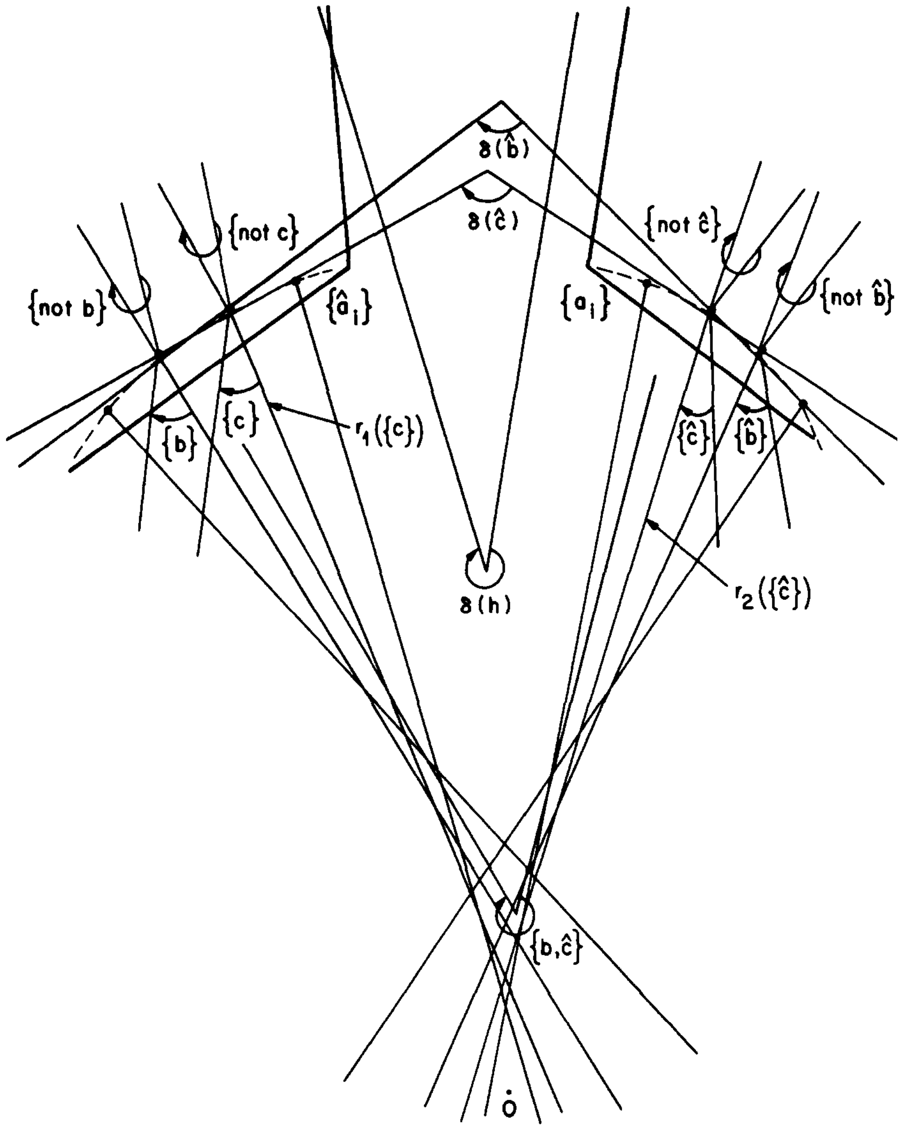

Fig. 17. Approximately feasible positions.

Our analysis for Figure 17 and the fact that all $\{a_i\}$ and $\{\hat{a}_i\}$ are included in $\delta(h)$ implies that it is a big angular region above the $r_1(\{a_i\})$ and the $r_2(\{\hat{a}_i\})$ and that its rays are as close to the vertical as the rays of $\{a_m, \hat{a}_1\}$, which in turn are at least as steep as $r_1(\{a_m\})$, the rightmost r_1 ray on the left, and $r_2(\{\hat{a}_1\})$, the leftmost r_2 ray on the right. But then the rays of $\delta(h)$ are steeper than those of $\{a_2, \hat{a}_{m-1}\}$, thus forcing $\{a_2, \hat{a}_{m-1}\} \subset \delta(h)$. However, this is a contradiction since $\beta(\{a_2, \hat{a}_{m-1}\})$ has its vertex on the opposite side of C from $\beta(\{a_m, \hat{a}_1\})$ where, as seen on Figure 18, we get $\beta(\{a_2, \hat{a}_{m-1}\}) \not\subset \gamma(h)$. □

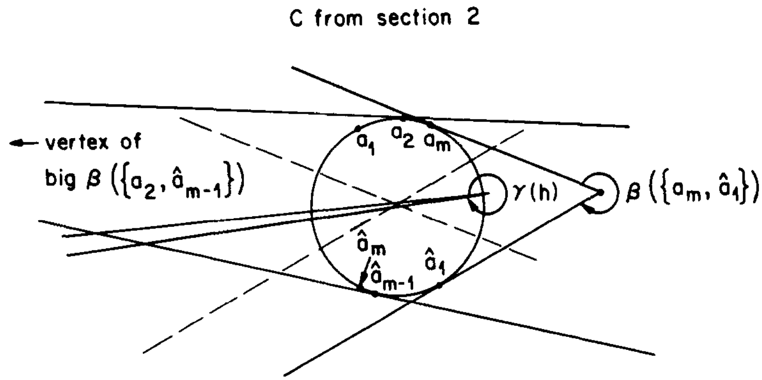

Fig. 18. $\beta(\{a_m, \hat{a}_1\}) \subset \gamma(h)$; $\beta(\{a_2, \hat{a}_{m-1}\}) \not\subset \gamma(h)$.

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