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A Note on Removable Pairs

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ABSTRACT

A long standing conjecture in the dimension theory for finite ordered sets asserts that every ordered set (of at least three points) contains a pair whose removal decreases the dimension at most one. Two stronger conjectures have been made:

(1) If (x, y) is a critical pair, then $dim(P) \le 1 + dim(P - \{x, y\})$.

(2) For every $x \in P$, there exists $y \in P - \{x\}$ so that $dim(P) \le 1 + dim(P - \{x, y\})$.

K. Reuter has disproved conjecture 1 by constructing a four–dimensional poset P containing a critical pair (x, y) so that $dim(P - \{x, y\}) = 2$. In this note, we construct for every $n \ge 5$ an n-dimensional poset P_n containing a critical pair (x, y) so that $dim(P_n - \{x, y\}) = n - 2$. Point y is a maximal point of P_n.

1. Preliminaries

Recall that the *dimension* of a finite ordered set P in the least positive integer t so that there exist t linear extensions $L_1, L_2, ..., L_t$ so that $P = L_1 \cap L_2 \cap ... \cap L_t$. An incomparable pair (x, y) is called a *critical* pair if any point less than x is less than y and any point greater than y is greater than x. The dimension of P is the least t for which there exist t linear extensions of P so that for every critical pair (x, y), there is at least one i for which $y < x$ in L_i. We refer the reader to the survey article [3] by D. Kelly and W.T. Trotter and the chapters [6], [7] by Trotter for additional background information on dimension theory.

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2. Removable Pairs

The following conjecture is one of the best known open problems in dimension theory and is a featured problem in ORDER. We believe the first reference to the conjecture is [1].

Conjecture 0 *If P is an ordered set having at least three points, then P contains a distinct pair (x, y) so that* $dim(P) \leq 1 + dim(P - \{x, y\})$.

A pair $x, y \in P$ for which $dim(P) \leq 1 + dim(P - \{x, y\})$ is called a *1–removable* pair, so that Conjecture 0 asserts that every poset contains a 1–removable pair.

The first reference to the following conjecture is apparently [5].

Conjecture 1 *Every critical pair is 1–removable.*

In [2], D. Kelly made the following conjecture which is also stronger than Conjecture 0.

Conjecture 2 *For every* $x \in P$*, there is a point* $y \in P - \{x\}$ *so that x, y is a 1-removable pair.*

K. Reuter [4] has disproved Conjecture 1 by constructing the ordered set shown in Figure 1. This ordered set P has dimension 4, (x, y) is a critical pair, and $dim(P - \{x, y\}) = 2$. Note that y is a maximal point.

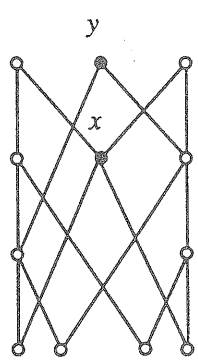

Figure 1

The purpose of this note is to show that Reuter's example is not an isolated phenomenon. To accomplish this, we will establish the following result.

Theorem *For every n*
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Proof For n = 4, we h
set of P_n contains 4n –
$\{c_i : 1 \leq i \leq n-2\} \cup \{$
$j \leq n-2$ and $i \neq j$, we h
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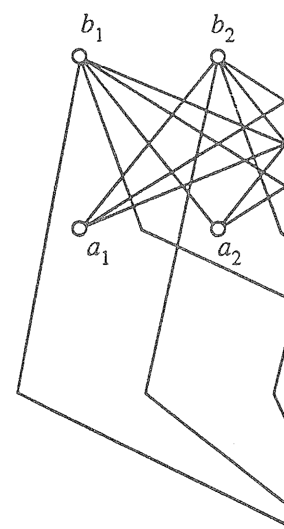

We first show that d
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Theorem *For every $n \geq 4$, there exists an n–dimensional ordered set P_n containing a critical pair (x, y) so that y is a maximal element in P_n, but (x, y) is not 1-removable, i.e., $dim(P - \{x, y\}) = n - 2$.*

Proof For $n = 4$, we have Reuter's example shown in Figure 1. For $n \geq 5$, the point set of P_n contains $4n - 4$ points labelled $\{a_i : 1 \leq i \leq n-2\} \cup \{b_i : 1 \leq i \leq n-2\} \cup \{c_i : 1 \leq i \leq n-2\} \cup \{d_i : 1 \leq i \leq n-2\} \cup \{x, y, z, w\}$. For all i, j with $1 \leq i, j \leq n-2$ and $i \neq j$, we have the cover relations $a_i <: b_j$ and $c_i <: d_j$. For each i with $1 \leq i \leq n-2$, we have $a_i <: y$, $c_i <: y$, $c_i < x$, $z <: b_i$, $w <: b_i$, $w < d_i$, $x <: b_i$, and $z < d_i$. We also have $w <: y$. We illustrate this definition with a diagram for P_n when $n = 5$.

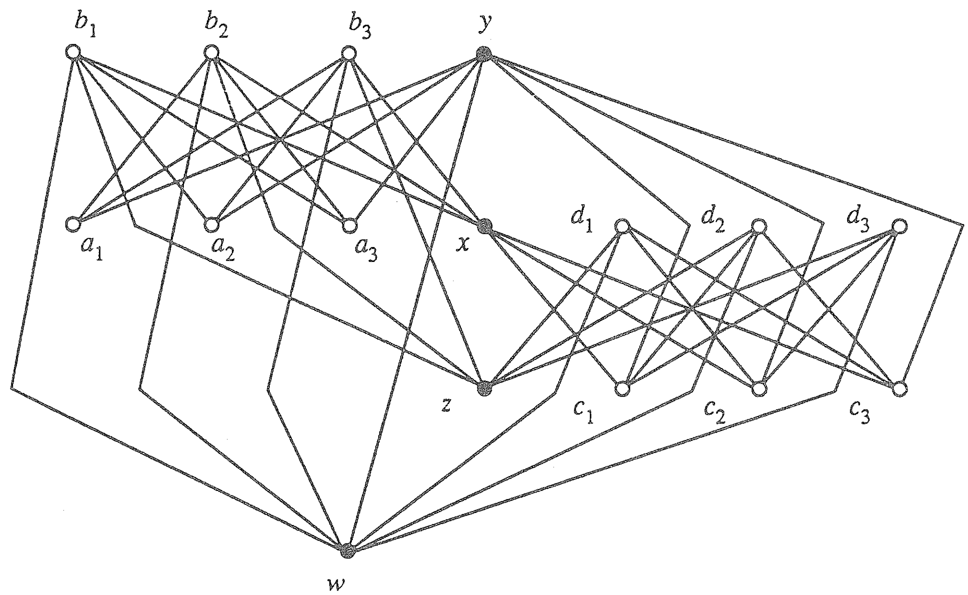

Figure 2

We first show that $\dim(P_n) \geq n$. To the contrary, suppose $\dim(P_n) \leq n-1$, and let $L_1, L_2, \ldots, L_{n-1}$ be linear extensions whose intersection is P_n. Without loss of generality, we may assume that $b_i < a_i$ in L_i for $i = 1, 2, \ldots, n-2$. Thus we must have $x > y$ in L_{n-1} and $z > y$ in L_{n-1}. However, this implies that for each $i = 1, 2, \ldots, n-2$, there exists a unique $j_i \in \{1, 2, \ldots, n-2\}$ so that $c_i > d_i$ in L_{j_i}. Hence

$w > x$ in L_{n-1} also. But this implies that $w > x > y$ in L_{n-1} which is impossible since $w < y$ in P_n. The contradiction completes the proof that $\dim(P_n) \geq n$.

We observe that y is maximal element of P_n and that (x, y) is a critical pair. We now show that $\dim(P_n - \{x, y\}) \leq n - 2$. To accomplish this, consider the poset $Q_n = P_n - \{x, y\}$. In Q_n, we observe that z and w have duplicated holdings so that $\dim(Q_n - \{z\}) = \dim(Q_n)$. Let $Q'_n = Q_n - \{z\}$. We show that Q'_n has $n-2$ linear extensions which intersect to give Q'_n. Let $A = \{a_1, a_2, ..., a_{n-2}\}$, $B = \{b_1, b_2, ..., b_{n-2}\}$, $C = \{c_1, c_2, ..., c_{n-2}\}$ and $D = \{d_1, d_2, ..., d_{n-2}\}$.

For $i = 1, 2$, L_i is any linear extension of Q'_n so that $A - \{a_i\} < C - \{c_i\} < w < d_i < c_i < b_i < a_i < B - \{b_i\} < D - \{d_i\}$. For $i = 3, 4, ..., n-2$, L_i is any linear extension of Q'_n so that $w < C - \{c_i\} < d_i < c_i < D - \{d_i\} < A - \{a_i\} < b_i < B - \{b_i\}$. It is easy to see that any family constructed by these rules forms a realizer. With this observation, our proof is complete. ❑

We pause to note that the construction given in the preceding family for $\{P_n : n \geq 5\}$ does not work for $n = 4$. In this case, $\dim P_n = 4$, but $\dim(P_n - \{x, y\}) = 3$. Thus to handle the case $n = 4$, we need a special example, and Reuter's construction suffices.

3. Concluding Remarks

We view the results of this note as providing additional evidence as to the difficulty of Conjecture 0, but we are unable to decide whether our theorem argues for or against the conjecture. It is easy to see that the examples satisfy Conjecture 2, so at least this stronger form of the original conjecutre remains open.

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1. Introduction

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1 This paper is based on a p
H. S. Wilf at the Universi