JOURNAL OF COMBINATORIAL THEORY, Series B 56, 278–295 (1992)

Dimensions of Hypergraphs

PETER C. FISHBURN

AT&T Bell Laboratories, Murray Hill, New Jersey 07974

AND

W. T. TROTTER

Bellcore, 445 South Street, Morristown, New Jersey 07962 and Arizona State University, Tempe, Arizona 85287

Communicated by the Editors

Received December 20, 1989

The *dimension* $D(S)$ of a family S of subsets of $\mathbf{n} = \{1, 2, ..., n\}$ is defined as the minimum number of permutations of $\mathbf{n}$ such that every $A \in S$ is an intersection of initial segments of the permutations. Equivalent characterizations of $D(S)$ are given in terms of suitable arrangements, interval dimension, order dimension, and the chromatic number of an associated hypergraph. We also comment on the maximum-sized family of k-element subsets of $\mathbf{n}$ having dimension m, and on the dimension of the family of all k-element subsets of $\mathbf{n}$. The paper concludes with a series of alternative characterizations of $D(S) = 2$ and a list of open problems.

1. INTRODUCTION

We define the dimension D of a finite hypergraph as the minimum number of permutations of its ground set such that every edge of the hypergraph is the intersection of initial segments of the permutations. The paper relates D to other notions of dimensionality and to chromatic numbers of certain graphs and hypergraphs, summarizes prior results that translate into facts about D, proves some new results, and identifies problems for further research. Because many of the theorems for D are not new, a main purpose of our study is to interpret and organize other topics under one elementary concept. Primary connections to the present definition of the dimension of a hypergraph are provided by the theory of k-suitable arrangements initiated in Dushnik [8], the interval dimension of height 1 partially ordered sets from Trotter and Moore [28] and Trotter [25], and the notion of biorder dimension and related work on chromatic

0095-8956/92 $5.00

numbers in Bouchet [2], Cogis [3], and Doignon, Ducamp, and Falmagne [7]. The last of these, referred to henceforth as DDF, and West [30] are particularly rich sources of information on notions of dimensionality of ordered sets and their ties to chromatic numbers.

Let n denote a positive integer. Unless we say otherwise, a finite hypergraph is viewed as a pair $H=(\mathbf{n}, S)$ in which $\mathbf{n}=\{1, 2, ..., n\}$ is the *ground set* and S is a family of subsets of $\mathbf{n}$ called *edges*. If every edge in S is an r-element subset of $\mathbf{n}$ then H is said to be *r-uniform*. A *graph* is a 2-uniform hypergraph.

An *arrangement* or *permutation* of $\mathbf{n}$ is a linear array $\sigma=a_1 a_2 \cdots a_n$ of all n points in $\mathbf{n}$. The *initial segments* of σ are $\varnothing$, $\{a_1\}$, $\{a_1, a_2\}$, ..., $\mathbf{n}$. A set R of permutations of $\mathbf{n}$ *realizes* $H=(\mathbf{n}, S)$ if for every nonempty edge A in S an initial segment s_σ can be chosen for each $\sigma \in R$ so that

$$A=\bigcap_{\sigma \in R} s_\sigma.$$

A moment's reflection shows that R realizes H if and only if, for every $A \in S$ and all $x \in \mathbf{n} \backslash A$, x follows an initial segment s of some $\sigma \in R$ for which $A \subseteq s$. Moreover, if A and B are in the family of subsets obtained as intersections of initial segments of R, then $A \cap B$ is also in this family. Note that the trivial hypergraph $(\mathbf{n}, \varnothing)$ is realized by the empty set of permutations.

DEFINITION. The *dimension* $D(H)$ of $H=(\mathbf{n}, S)$ is the smallest $m \geqslant 0$ for which there exists a set R of m permutations of $\mathbf{n}$ which realizes H.

Clearly $D(\mathbf{n}, \varnothing)=0$, $D(\mathbf{n}, \{\varnothing\})=1$, and, by any n permutations with different last elements, $D \leqslant n$. Placement of elements in $\mathbf{n}$ but not $\bigcup_S A$ at right ends of permutations shows that D is independent of those elements. We therefore write $D(H)$ as $D(S)$: $D(\varnothing)=0$, $D(\{\varnothing\})= D(\{\{1\}, \{1, 2\}\})=1$, and so forth. For convenience *we assume henceforth that* $S \neq \varnothing$.

$D(S)$ has the following straightforward interpretation in terms of dimensionality. Suppose $D(S)=m \geqslant 1$ and $\{\sigma_1, ..., \sigma_m\}$ realizes S. Let L denote the lattice $\mathbf{n}^m$ in $\mathbb{N}^m$. Label the n levels $(1, 2, ..., n)$ of coordinate i of L by the members of σ_i in order, and label each point in L with the intersection of the m initial segments of labels on the coordinates that define the lattice point. Then every nonempty edge in S is the label of some point in L, and this cannot be done with fewer than m coordinates. Figure 1 illustrates the labeling for $\sigma_1=1234$ and $\sigma_2=4231$. With subset braces omitted, $\{\sigma_1, \sigma_2\}$ realizes $\{\varnothing$, 1, 2, 4, 12, 23, 24, 123, 234, 1234$\}$ and every subset thereof.

Various other dimensionality notions for graphs, ordered sets and other relations were defined in Dushnik and Miller [9], Erdős, Harary, and Tutte [10], Trotter and Bogart [27], Trotter, Moore, and Sumner [29],

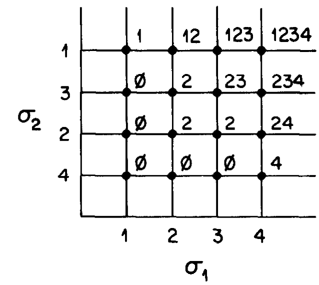

FIGURE 1

Lovász, Nesetril, and Pultr [17], Trotter [26], DDF, Steif [23], West [30], and Cozzens and Roberts [6]. Some of these have no apparent connection to D, but others are intimately connected to it. Of particular importance are the dimension (Dushnik and Miller [9]) and interval dimension (Trotter and Bogart [27]) of a partially ordered set. The latter notion is a special case of biorder dimension developed in Bouchet [2], Cogis [3], and DDF, and we refer readers to DDF and to West [30] for more about this.

We recall that a finite irreflexive partially ordered set, or *poset*, is a pair $(X, <_0)$ in which $<_0$ is an irreflexive and transitive binary relation on a nonempty finite set X. A *linear extension* of $<_0$ is a linear (strong, strict, total) order $<_*$ on X that includes $<_0$. The *dimension* $d(X, <_0)$ of poset $(X, <_0)$ is the minimum number of linear extensions of $<_0$ whose intersection equals $<_0$. Alternatively (Hiraguchi [15], Ore [18]), $d(X, <_0)$ is the smallest number of one–one mappings $f_1, ..., f_m$ from X into $\mathbb{R}$ such that, for all $x, y \in X$,

$$x <_0 y \Leftrightarrow f_i(x) < f_i(y) \qquad \text{for} \quad i = 1, ..., m.$$

The *interval dimension* $I(X, <_0)$ of $(X, <_0)$ is the minimum number of mappings $F_1, ..., F_m$ from X into closed real intervals such that, for all $x, y \in X$,

$$x <_0 y \Leftrightarrow \sup F_i(x) < \inf F_i(y) \qquad \text{for} \quad i = 1, ..., m.$$

In general, $I \leqslant d$.

Consider $d(S, \subset)$, the order dimension of S ordered by proper inclusion. It is easily seen that $d(S, \subset) = 1 \Leftrightarrow D(S) = 1$, but, when S is the family of all $(n-1)$-element subsets of $\mathbf{n}$ and $n \geqslant 2$, $d(S, \subset) = 2$ and $D(S) = n$. Indeed, we always have $d(S, \subset) \leqslant D(S)$, so $D(S)$ is related to the dimension of the inclusion poset $(S, \subset)$. But D also depends on the specific contents of the edges in S, as recognized in Trotter [26]. In particular, if

$(S, \subset)$ and $(T, \subset)$ are order isomorphic then $d(S, \subset) = d(T, \subset)$, but $D(S)$ and $D(T)$ need not be equal. The simplest example is

$$S = \{1, 2, 3\} \qquad \text{and} \qquad T = \{12, 13, 23\}$$

with $(S, \subset) \cong (T, \subset)$, $d(S, \subset) = d(T, \subset) = 2$, $D(S) = 2$ by $\{\sigma_1 = 123, \sigma_2 = 321\}$, and $D(T) = 3$.

An outline of the paper follows.

Section 2 summarizes elementary facts about D, concluding with a not so elementary proof that a graph on n points and $D = 3$ can have $3(n-2)$ edges but no more.

Section 3 says more about d and I. We define the *membership poset* $P(H)$ of $H = (\mathbf{n}, S)$ by

$$P(H) = (\mathbf{n} \cup S, \in\}.$$

Figure 2 shows P for $H = (4, \{\varnothing, \{1\}, \{2\}, \{2, 3\}, \{1, 3, 4\}\})$. In Section 3 we define another poset $P^*(H)$ whose elements are nonempty subsets of $\mathbf{n} \cup S$ and observe that

$$d(S, \subset) \leqslant I(P(H)) = d(P^*(H)) = D(S) \leqslant d(P(H)) \leqslant D(S) + 1.$$

Thus D is identical to d and to I for suitably defined derived posets, and $d(S, \subset) \leqslant D(S) \leqslant d(P(H))$ so that the dimension of a hypergraph is bounded by the order dimensions of its inclusion and membership posets. Proofs in Cogis [3], Trotter [25], and DDF establish most of these results.

Let $S_{k,n}$ denote the set of all k-element subsets of $\mathbf{n}$. We note here that, when $1 \leqslant k \leqslant n-1$,

$$D(\{A \subseteq \mathbf{n} : |A| \leqslant k\}) = D(S_{k,n})$$

since it is transparent that if $\{\sigma_1, ..., \sigma_m\}$ realizes $S_{k,n}$ then it also realizes $S_{k-1,n}, ..., S_{1,n}$. $D(S_{k,n})$ was first studied in Dushnik [8] and denoted by $N(n, k+1)$. Dushnik defined it as the minimum number of arrangements of

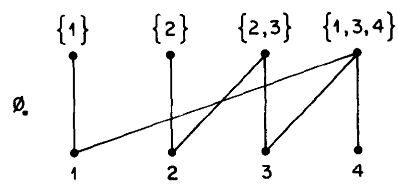

diagram of P(H)

FIGURE 2

n that are $(k+1)$-*suitable* in the sense that, for every $(k+1)$-set A and every $i \in A$, i follows the other k elements of A in at least one arrangement. His interest in suitability stemmed from his earlier work on the order dimension d; in particular, he noted that if $1<k<n$ then $D(S_{k,n})$ equals $d(S_{1,n} \cup S_{k,n}, \subset)$.

Dushnik's [8] paper led to other work on suitability and related things in Spencer [21, 22], Trotter [24], Cozzens and Roberts [5], and Füredi and Kahn [12]. We review selected results of these papers in Section 4, including $D(S_{2,n}) \geqslant 1+\log_2 \log_2 n$ and $D(S_{k,n}) \leqslant (k+1)^2 [1+\log(n/(k+1))]$.

Section 5 relates $D(S)$ to chromatic numbers. Given a hypergraph $H=(\mathbf{n}, S)$, define its *antimembership set* to be

$$V=\{(x, A) : A \in S, x \in \mathbf{n} \setminus A\}.$$

When $V \neq \varnothing$, we define the *inversion graph* $G(H)$ as the pair (V, E) where, for all (x, A), $(y, B) \in V$,

$$\{(x, A), (y, B)\} \in E \qquad \text{if} \quad x \in B \quad \text{and} \quad y \in A.$$

We also define a hypergraph $\mathscr{H}(H)$ with ground set V whose two-element edges are the edges in E. With $\chi(G)$ the usual chromatic number of a graph, and $\chi(\mathscr{H})$ the minimum number of colors that can be assigned to vertices in V so that no edge of $\mathscr{H}$ is monochromatic, results in Cogis [4] and DDF are used to verify $D(S)=\chi(\mathscr{H}(H))$ along with $D(S)=2 \Leftrightarrow \chi(G(H))=2 \Leftrightarrow \chi(\mathscr{H}(H))=2$. We note also that $\chi(G(H))<D(S)$ sometimes holds for larger values of D (Trotter [26]).

Dushnik and Miller [9] and Baker, Fishburn, and Roberts [1] established several equivalent characterizations of posets for which $d=2$: see also Fishburn [11]. Section 6 does the same thing for hypergraphs for which $D=2$. Our equivalences include results of preceding sections along with interval representability for S as defined in Trotter and Moore [28].

Section 7 concludes the paper with a brief summary and interesting open problems.

2. Basic Facts

Throughout this section, S and T denote nonempty families of subsets of **n**. This is needed for parts of our first lemma.

Lemma 1. *For all S and T, and all $B \subseteq \mathbf{n}$:*

a. $1 \leqslant D(S) \leqslant n$.

b. $\min\{|S| : D(S)=m\}=m$ *for* $1 \leqslant m \leqslant n$.

c. $S \subset T \Rightarrow D(S) \leqslant D(T)$.

d. $S = \{A \cup B : A \in T\} \Rightarrow D(S) \leqslant D(T)$.

e. $S = \{A \backslash B : A \in T\} \Rightarrow D(S) \leqslant D(T)$.

f. $\max\{D(S), D(T)\} \leqslant D(S \cup T) \leqslant D(S) + D(T)$.

Proof. Part (a) was noted in the Introduction. For (b), $D(S) = m$ when S is the family of $(m-1)$-element subsets of $\mathbf{m}$, and if $|S| < m$ then $D(S) < m$. Part (c) follows from the definition of D. For (d), shift elements in B to the left ends of permutations that realize T. For (e), shift elements in B to the right ends of permutations that realize T. Part (c) implies the first inequality in (f), and the second follows from the fact that $S \cup T$ is realized by the union of a set of permutations that realizes S and a set of permutations that realizes T. ∎

We know of no simple strengthening of the upper bound in Lemma 1(f). For example, it is not always true that $D(S \cup T) \leqslant D(S) + D(T) - D(S \cap T)$.

LEMMA 2. *$D(S) = 1 \Leftrightarrow S$ is linearly ordered by $\subset$. $D(S) = n \Leftrightarrow S$ contains every $(n-1)$-element subset of* $\mathbf{n}$.

Proof. If $\subset$ linearly orders S, $\mathbf{n}$ can be arranged so that every $A \in S$ is an initial segment of the arrangement, so $D(S) = 1$. If $\subset$ does not linearly order S, sets $A, B \in S$ for which $A \backslash B$ and $B \backslash A$ are nonempty force $D(S) \geqslant 2$.

As already noted, $D(S) = n$ if all $(n-1)$-sets are in S. Suppose one of these, say $\{2, ..., n\}$, is missing from S. Then any set of $n-1$ permutations that end in 12, 13, ..., $1n$ realizes S. ∎

In the rest of this section we consider how many k-sets S can contain when $D(S)$ is fixed at $m \leqslant n$. Let

$$\alpha_m(k, n) = \max\{|S| : D(S) = m, |A| = k \text{ for all } A \in S\}$$

for $1 \leqslant k \leqslant n$ and $1 \leqslant m \leqslant n$. Clearly $\alpha_1(k, n) = 1$. The next step is

Lemma 3. *$\alpha_2(k, n) = n - k + 1$, and these values are realized for any $1 \leqslant k \leqslant n-1$ by a permutation of* $\mathbf{n}$ *and its reverse.*

Proof. For the final assertion, observe that $\{\sigma_1 = 12 \cdots n, \sigma_2 = n \cdots 21\}$ realizes $S = \{[i, j] : i \leqslant j\}$, where $[i, j] = \{k \in \mathbb{Z} : i \leqslant k \leqslant j\}$. Therefore $\alpha_2(2k, n) \geqslant n - k + 1$.

To show that $\alpha_2(k, n) \leqslant n - k + 1$, fix k, $2 \leqslant k < n$, and let σ_1 and σ_2 be any two permutations of $\mathbf{n}$. When $|A| = k$ and A is the intersection of initial segments of σ_1 and σ_2, let $p(A)$ and $q(A)$ be the maximum positions of an

A element in σ_1 and σ_2 respectively. When thus defined, the union of the elements in σ_1 that follow position $p(A)$ and the elements in σ_2 that follow position $q(A)$ is $\mathbf{n}\setminus A$. It follows that if A and B are distinct k-sets that are intersections of initial segments, then

$$\{k \leqslant p(A) < p(B);\, k \leqslant q(B) < q(A)\} \qquad \text{or}$$
$$\{k \leqslant p(B) < p(A);\, k \leqslant q(A) < q(B)\}.$$

Therefore, if $A_1, A_2, ..., A_r$ are distinct k-sets that are intersections of initial segments, we have say $\{k \leqslant p(A_1) < p(A_2) < \cdots < p(A_r) \leqslant n;\ k \leqslant q(A_r) < \cdots < q(A_2) < q(A_1) \leqslant n\}$. Hence $r \leqslant n-k+1$. ∎

The general determination of $\alpha_m(k, n)$ appears very difficult. We prove one further result which shows that a graph $(k=2)$ for which $D=3$ can have $3(n-2)$ edges, but no more.

THEOREM 1. $\alpha_3(2, n) = 3(n-2)$ *for* $n \geqslant 3$.

Remark. We know of two proofs of Theorem 1. The first, given below, proceeds from basic facts about the inner structures of three permutations without appealing to other results. The second, which illustrates further connections between our definition of hypergraph dimension and related concepts, uses a (difficult) theorem in Schnyder [20] which says that a graph $G = (\mathbf{n}, S)$, with $|A| = 2$ for all $A \in S$, is planar if and only if $d(P(G)) \leqslant 3$. With $P(G)$ as described for Fig. 2, it follows from our definitions that a set of three permutations of $\mathbf{n}$ realizes S if and only if $d(P(G)) \leqslant 3$. By Schnyder's theorem, this happens if and only if G is planar. Then, by Euler's theorem which says that the maximum number of edges in a planar graph is $3n-6$, we conclude that $\alpha_3(2, n) = 3n-6$.

Proof. The set of permutations $\{\sigma_1, \sigma_2, \sigma_3\}$ with

$$\sigma_1 = 123 \cdots n-1\ \ n$$
$$\sigma_2 = 1\ \ n\ \ n-1 \cdots 3\ \ 2$$
$$\sigma_3 = 2\ \ n\ \ n-1 \cdots 3\ \ 1$$

realizes $S = \{ij : i<j, \text{ and } i \in \{1, 2\} \text{ or } j = i+1\}$, so $\alpha_3(2, n) \geqslant (n-1) + (n-2) + (n-3) = 3(n-2)$.

To prove that $\alpha_3(2, n) \leqslant 3(n-2)$, let σ_1, σ_2, and σ_3 denote any three permutations of $\mathbf{n}$. For three such permutations, we say that a 2-set $\{i, j\}$ is *good* if $\{i, j\}$ is the intersection of initial segments, and is *bad* otherwise. Note that $\{i, j\}$ is bad if and only if there is an $x \in \mathbf{n}\setminus\{i, j\}$ such that x precedes the rightmost of i and j in all three permutations. We manipulate permutations so as to maximize the number of good pairs or, equivalently, minimize the number of bad pairs. Using induction on n, it will be shown that the number of good pairs never exceeds $3(n-2)$ or, equivalently, the number of bad pairs is never less than $\binom{n-3}{2} = \binom{n}{2} - 3(n-2)$.

Given $n \geqslant 3$, suppose σ_1 ends with 1. Then if σ_2 and σ_3 are modified by moving 1 into their first positions with the orders on the other elements unchanged, it is easily seen that every good pair stays good. Consequently, we assume henceforth with no loss of generality that

$$\sigma_1 = (23) \cdots 1$$
$$\sigma_2 = (13) \cdots 2$$
$$\sigma_3 = (12) \cdots 3,$$

where parentheses indicate that the order of the two enclosed elements is immaterial so far as good and bad pairs are concerned. Such a triple of permutations on $\mathbf{n}$ is a *special triple*. For any special triple with $n \leqslant 4$, every pair is good, so that $\alpha_3(2, n) = \binom{n}{2} = 3n-6$ if $n \leqslant 4$, as desired.

When U is a set of permutations on $\mathbf{n}$, we use U_i to denote the set of permutations with $n-1$ elements obtained by deleting i from each permutation in U. Note that bad pairs in U_i are also bad pairs in U. Also, given permutations U, we say that (i, j) is a *dominant pair* or that j *dominates* i if i precedes j in each permutation of U. In a special triple, note that a dominant pair (i, j) must have $i, j \in \mathbf{n} \setminus \{1, 2, 3\}$. For a set of permutations U, the property of having no dominant pair is called *non-dominance*. Finally, $\{i, j\}$ is bad if and only if some element k precedes the rightmost of i and j in all permutations, in which case we call k a *spoiler* of $\{i, j\}$.

For $n \geqslant 5$, we proceed by induction, assuming that triples on $n' < n$ elements have at least $\binom{n'-3}{2}$ bad pairs. We restrict the structure of an *optimal* triple, meaning one with the minimum number of bad pairs. We already know that it must be special. If j dominates i in a triple U, then all pairs that contain j and not i are bad. There are $n-2$ such pairs. Since U_j also contributes at least $\binom{n-4}{2}$ bad pairs to U, there are at least $\binom{n-3}{2}+2$ bad pairs in U. Hence we may assume non-dominance for an optimal triple.

Let U be a special triple with no dominant pair. If the penultimate elements are not all distinct, then we may assume that 4 is penultimate in σ_1 and σ_2. By non-dominance, 4 must be third in σ_3. Now the final $n-4$ elements in σ_3 form bad pairs with element 4, all having the fourth element of σ_3 as a spoiler. Adding these to the bad pairs in U_4 yields at least $\binom{n-3}{2}$ bad pairs, as desired.

Hence we may assume that the penultimate elements in the three permutations are distinct, which means we have completed the proof unless

$n \geqslant 6$. For ease of reference, let these elements be 4, 5, 6 for $\sigma_1, \sigma_2, \sigma_3$, respectively. If U_{123} is also a special triple, then U looks like

$$\sigma_1 = (23)[56] \cdots 41$$
$$\sigma_2 = (13)[64] \cdots 52$$
$$\sigma_3 = (12)[45] \cdots 63$$

with a choice of ordering within each pair of brackets. No matter how these choices are made, there must be at least three bad pairs with $i \in \{1, 2, 3\}$ and $j \in \{4, 5, 6\}$. For example, for the choices as written above, 4 spoils $\{1, 5\}$, 5 spoils $\{2, 6\}$, and 6 spoils $\{3, 4\}$. Also, if $x > 6$, then 4 spoils $\{1, x\}$, 5 spoils $\{2, x\}$, and 6 spoils $\{3, x\}$. Adding these to the bad pairs guaranteed from U_{123} yields at least $3 + 3(n-6) + \binom{n-6}{2} = \binom{n-3}{2}$ bad pairs in U.

Hence it suffices to show that the number of bad pairs is in fact minimized when U_{123} is special. We prove this by making transpositions of adjacent elements to push non-penultimate occurrences of 4, 5, 6 toward the left without increasing the number of bad pairs. If U_{123} is not special, then we may assume by symmetry that 5, 6 do not occupy positions three and four in σ_1. One of these, which we may assume is 5, must be immediately preceded in σ_1 by some $x > 6$. Since x also precedes 5 in σ_2, non-dominance implies that x follows 5 in σ_3.

We claim that replacing $x5$ by $5x$ in σ_1 to obtain a new triple U' does not increase the number of bad pairs. If this is false, then some good pair must become bad, which can only be $\{x, j\}$ for some j and has 5 as a unique spoiler in U'. In view of σ_2, we must have $j = 2$. It now suffices to show that there is some bad pair in U that is good in U'. Let y be the first element following x in σ_3 that precedes x in σ_1; this is well-defined, since the element 3 has these properties. The element x spoils $\{5, y\}$ in U, but in U' it does not. Hence $\{5, y\}$ is the desired pair unless some other element z spoils $\{5, y\}$ in both U and U'. This implies that z precedes x in σ_1 and y in σ_3. However, the fact that 5 is the unique spoiler of $\{x, 2\}$ in U implies that z follows x in σ_3. Together, these statements contradict the choice of y, so there is no such z, and $\{5, y\}$ turns from bad to good to prevent an increase in the number of bad pairs. ∎

3. Associated Posets

Let $S(i) = \{A \in S : i \in A\}$ and $<_S = \{(i, A) : i \in A \in S\}$ with $P(H) = (\mathbf{n} \cup S, <_S)$. We now define $P^*(H)$ for $H = (\mathbf{n}, S)$ to accompany the inclusion poset $(S, \subset)$ and the membership poset $P(H)$.

First, let $Q(H) = (\mathbf{n} \cup S, \lesssim)$ where, for $i, j \in \mathbf{n}$ and $A, B \in S$,

$$\begin{aligned} \lesssim\, = {} & <_S \cup \{(i, j) : S(j) \subseteq S(i)\} \cup \{(A, B) : A \subseteq B\} \\ & \cup \{(A, i) : A \times S(i) \subseteq\, <_S\}. \end{aligned}$$

This follows DDF [7, pp. 80–81], which notes that $Q(H)$ is the *maximal* quasi-order (reflexive, transitive) on $\mathbf{n} \cup S$ for which $\lesssim \cap\, (\mathbf{n} \times S) = <_S$. Because $Q(H)$ is a quasi-order, its symmetric part $\sim$, defined by $x \sim y$ if $x \lesssim y$ and $y \lesssim x$, is an equivalence relation on $\mathbf{n} \cup S$. Using a reduction procedure in Roberts [19, p. 45], we define the poset

$$P^*(H) = ((\mathbf{n} \cup S)/\sim,\ <_1),$$

where $(\mathbf{n} \cup S)/\sim$ is the set of equivalence classes of $\mathbf{n} \cup S$ under $\sim$, and for all such classes X and Y,

$$X <_1 Y \qquad \text{if } a \lesssim b \text{ and not } (b \lesssim a) \text{ for some (hence for all) } a \in X \text{ and } b \in Y.$$

See Fig. 3 for examples.

This brings us to a fundamental result.

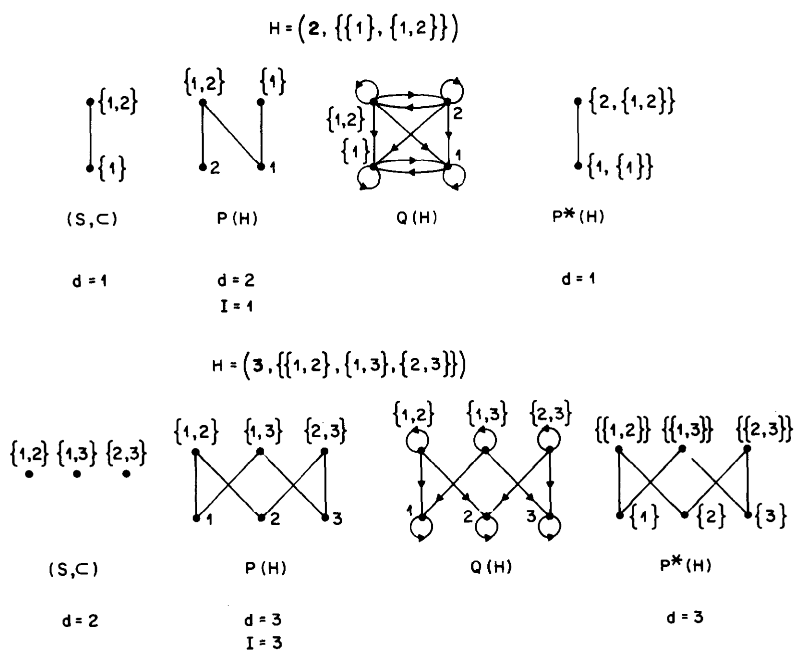

FIGURE 3

THEOREM 2. *For all hypergraphs* $H = (\mathbf{n}, S)$,

$$d(S, \subset) \leqslant I(P(H)) = d(P^*(H)) = D(S) \leqslant d(P(H)) \leqslant D(S) + 1.$$

Proof. The relations $D(S) = I(P(H)) \leqslant d(P(H)) \leqslant I(P(H)) + 1$ are proved in Trotter [25], and the relations $d(P^*(H)) = I(P(H)) \leqslant d(P(H)) \leqslant I(P(H)) + 1$ are contained in Cogis [3] and DDF. For DDF, Propositions 2.9 and 4.1 plus remarks on pages 90 and 99 verify $d(P^*(H)) = I(P(H))$.

This leaves only $d(S, \subset) \leqslant D(S)$. Suppose $D(S) = m \geqslant 1$ and $\{\sigma_1, ..., \sigma_m\}$ realizes S. For each $i \in \mathbf{m}$ define $f_i: S \to \{1, ..., n\}$ when $\sigma_i = a_1 a_2 \cdots a_n$ by

$$f_i(A) = \max\{j : a_j \in A\}.$$

(If $\varnothing \in S$, we ignore it with no loss in generality.) Suppose $A \subset B$ for $A, B \in S$. Then $f_i(A) \leqslant f_i(B)$ for all i and, since each $b \in B \backslash A$ follows the final element of A in some σ_i, $f_i(A) < f_i(B)$ for some i. On the other hand, if $A, B \in S$, $A \not\subset B$ and $B \not\subset A$, then $B \backslash A$ and $A \backslash B$ are nonempty and we have $f_i(A) < f_i(B)$ and $f_j(B) < f_j(A)$ for some $i, j \in \mathbf{m}$. Hence $d(S, \subset) \leqslant m$ by the Hiraguchi–Ore characterization of d. ∎

Figure 3 illustrates one case with $d(P(H)) = D(S)$ and another with $d(P(H)) = D(S) + 1$. DDF [7, p. 101] remark (as interpreted through Theorem 2) that no simple criterion is known to differentiate these two cases, and as far as we are aware the question remains open. In their terminology, what we refer to as $D(S)$, or as the interval dimension of the membership poset $P(H)$, is the "bidimension" of the membership poset.

4. COMPLETE UNIFORM HYPERGRAPHS

Section 2 asked how small (Lemma 1(b)) or large (Lemma 3, Theorem 1) S or a uniform part of S can be when D is fixed. We now reverse this to ask about D when the hypergraph is specified. A simple example is $S = \{2, ..., n, 12, 13, ..., 1n\}$. This has $D = 2$ if $n = 3$ and $D = 3$ if $n \geqslant 4$ (cf. proof of Theorem 1).

We summarize here what is known about the dimensions of complete uniform hypergraphs, using the equivalence between $D(S_{k,n})$ and Dushnik suitability explained earlier. To avoid awkward notation we write $D(k, n)$ for $D(S_{k,n})$ or, in Dushnik's notation, for $N(n, k+1)$, $1 \leqslant k \leqslant n-1$. Clearly, $D(k, n)$ is nondecreasing in n. Exact values are noted first.

THEOREM 3. $D(1, n) = 2$ *for all* $n \geqslant 2$. *If* $n \geqslant 4$, $2 \leqslant j \leqslant \sqrt{n}$ *and*

$$\left\lfloor \frac{n + j^2}{j} \right\rfloor \leqslant k + 1 < \left\lfloor \frac{n + (j-1)^2}{j-1} \right\rfloor,$$

then $D(k-1, n) = n - j + 1$. *In addition, for all* $m \geqslant 2$,

$$D(2m-2, m^2+m-1) = D(2m-2, m^2+m) = m^2$$
$$D(2m-3, m^2+t) = m^2 - m \qquad \textit{for} \quad t \in \{-2, -1, 0, 1\}.$$

All results of Theorem 3 were proved in Dushnik [8] except for $t \in \{0, 1\}$ in the last line, which is proved in Trotter [24].

The second sentence of the theorem specifies D for all k between about $2\sqrt{n}$ and $n-1$. Trotter [24] has a table of all D values for $n \leqslant 14$, including $D(2, 12) = 4$, $D(2, 13) = D(2, 14) = 5$, $D(3, 6) = D(3, 10) = 6$, $D(3, 11) = D(3, 14) = 7$, $D(4, 10) = 8$, and $D(4, 11) = D(4, 14) = 9$.

THEOREM 4. $D(k-1, n) \leqslant \min\{k2^k \log_2 \log_2 n, k^2(1 + \log(n/k))\}$ *for* $1 \leqslant k \leqslant n-1$. *In addition, for all* $n \geqslant 3$,

$$1 + \log_2 \log_2 n \leqslant D(2, n) \leqslant \log_2 \log_2 n + \tfrac{1}{2} \log_2 \log_2 \log_2 n + 3.16.$$

The first term of the min expression appears in Spencer [22] and the second appears in Füredi and Kahn [12]. The inequalities for $D(2, n)$ appear in Trotter [24], based partly on Spencer [22]. The fact that $D(2, n) \to \infty$ was first noted in Dushnik [8]. In the terminology of our paper it says that graphs can have arbitrarily large dimensions.

5. CHROMATICS IN D

We have been assuming that $S \neq \varnothing$. This section assumes also that $S \neq \{\mathbf{n}\}$ so that the anticontainment set V is not empty.

The inversion graph $G(H)$ on V is a specialization of the hypergraph $\mathscr{H}(H) = (V, \mathscr{S})$ whose edges $\mathscr{A} \in \mathscr{S}$ are subsets we propose to call strong cycles in V. A *strong r-cycle* in V is a subset $\mathscr{A}$ of $r \geqslant 2$ vertices in V that can be arranged as $(x_1, A_1), (x_2, A_2), \ldots, (x_r, A_r)$ so that

$$x_i \in A_{i+1} \Big\backslash \bigcup_{j \neq i+1} A_j, \qquad i = 1, \ldots, r-1$$
$$x_r \in A_1 \Big\backslash \bigcup_{j \neq 1} A_j.$$

Let $\mathscr{S}_r$ be the set of strong r-cycles in V, and let $\mathscr{S} = \mathscr{S}_2 \cup \mathscr{S}_3 \cup \cdots$. The edge set of $G(H)$ is $\mathscr{S}_2$.

It is easily seen that $\mathscr{S}$ is empty if and only if $\subset$ linearly orders S, and in this case $\chi(\mathscr{H}) = 1$. Otherwise, $\chi(\mathscr{H}) \geqslant 2$, with χ independent of vertices in $\mathscr{H}$ that are in no strong cycle. Because of this we write $\chi(\mathscr{S})$ for the chromatic number of $\mathscr{H}(H)$ with the understanding that $\chi(\varnothing)=1$. The chromatic number of $G(H)$ is $\chi(\mathscr{S}_2)$.

By our Theorem 2, the basic theorem for D in terms of χ is proved as Proposition 3.2 in DDF.

THEOREM 5. *For all hypergraphs* $H=(\mathbf{n}, S)$,

$$D(S)=\chi(\mathscr{S}).$$

For the hypergraph H in the lower half of Fig. 3, we have

$$V=\{(1, 23), (2, 13), (3, 12)\}$$

$$\mathscr{S}=\mathscr{S}_2=\{\{(1, 23), (2, 13)\}, \{(1, 23), (3, 12)\}, \{(2, 13), (3, 12)\}\}$$

so $\mathscr{H}=G$ forms a triangle with $\chi(\mathscr{S})=3$. Note that the vertices do not form a strong 3-cycle since, for example, 1 is in both $\{1, 3\}$ and $\{1, 2\}$. On the other hand, V is a *weak* 3-cycle, which is any triple of vertices that can be arranged as (x, A), (y, B), (z, C) with $x\in B$, $y\in C$, $z\in A$. Proposition 3.4 in DDF implies that $D(S)=\chi(\mathscr{S}_2)$ if there is no weak 3-cycle in V. Here, as elsewhere, we use the fact by Theorem 2, and Proposition 2.9 in DDF, that our $D(S)$ us tantamount to the DDF bidimension of the membership poset.

Another important result, from Cogis [3] and proved also as Proposition 5.2 in DDF, is

THEOREM 6. $D(S)=2\Leftrightarrow\chi(\mathscr{S}_2)=2$.

The DDF proof is based on a characterization in Dushnik and Miller [9] of posets with $d=2$ in terms of their incomparability graphs and on the characerization of comparability graphs in Ghouilà-Houri [13] and Gilmore and Hoffman [14].

Although $\chi(\mathscr{S}_2)\leqslant\chi(\mathscr{S})=D(S)$, with equality if $D=2$, it was first observed in Trotter [26] that $\chi(\mathscr{S}_2)<D(S)$ is possible when $\chi(\mathscr{S}_2)\geqslant 4$. His example appears also in DDF and in the next proof.

THEOREM 7. *For each* $k\geqslant 4$ *there is an* $H=(\mathbf{n}, S)$ *with* $\chi(\mathscr{S}_2)=k$ *and* $D(S)\geqslant 3(k-1)/2$.

Proof. Fix $m\geqslant 3$, take $n=3m$, and let $A=\{1, ..., m\}$, $B=\{m+1, ..., 2m\}$, and $C=\{2m+1, ..., 3m\}$. For each $i\in\mathbf{n}$ define edge E_i for H by

$$E_i=(A\cup B)\setminus\{i\} \quad \text{if } i\in A$$

$$E_i=(B\cup C)\setminus\{i\} \quad \text{if } i\in B$$

$$E_i=(C\cup A)\setminus\{i\} \quad \text{if } i\in C.$$

Let $S = \{E_i\}$. It is easily seen that a permutation of $\mathbf{n}$ has at most two i for which i follows all elements in E_i. Therefore $D(S) \geqslant |S|/2 = 2m/2$. (It is not hard to verify that $D(S) = \lceil 3m/2 \rceil$.)

We specify the vertex set V of G in six disjoint pieces,

$$V_A = \{(i, E_i) : i \in A\}, \qquad V_1 = \{(j, E_i) : j \in C, i \in A\},$$

$$V_B = \{(i, E_i) : i \in B\}, \qquad V_2 = \{(j, E_i) : j \in A, i \in B\},$$

$$V_C = \{(i, E_i) : i \in C\}, \qquad V_3 = \{(j, E_i) : j \in B, i \in C\},$$

so $|V_A| = |V_B| = |V_C| = m$ and $|V_1| = |V_2| = |V_3| = m^2$. Recall that G has an edge between (x, E_i) and (y, E_j) if and only if $x \in E_j$ and $y \in E_i$. It is easily checked that

(1) there are no G edges in V_i, $i = 1, 2, 3$;

(2) $x \in V_A$ [resp. V_B, V_C] is joined to every other point in V_A [resp. V_B, V_C] and to every point in V_3 [resp. V_1, V_2], and x belongs to no other edges.

Because of (2), G has an abundance of complete subgraphs on $m+1$ vertices, so $m+1$ colors are needed to color G. And $m+1$ colors suffice. Denote them by $c_1, c_2, \ldots, c_{m+1}$. Assign c_i to each vertex in V_i, $i = 1, 2, 3$. Then assign a different one of $c_1, c_2, c_4, \ldots, c_{m+1}$ to each of the m vertices in V_A, assign $c_2, c_3, \ldots, c_{m+1}$ to the vertices in V_B, and assign $c_1, c_3, \ldots, c_{m+1}$ to the vertices in V_C. It follows from (1) and (2) that for every edge in G the two vertices have different colors. Therefore $\chi(\mathscr{S}_2) = m+1$.

Theorem 7 follows with $k = m+1$. ∎

We do not know at present whether it is possible to have $\chi(\mathscr{S}_2) = 3$ and $D(S) \geqslant 4$. Another apparently open question is whether for fixed $\chi(\mathscr{S}_2)$ it is possible to have arbitrarily large $D(S)$. The proof of Theorem 7 shows only that $D(S) - \chi(\mathscr{S}_2)$ can become large as χ increases.

6. Characterizations of $D = 2$

We make one further definition before listing characterizations of $D = 2$. A family S of sets is *interval representable* if there is a mapping F from $\bigcup_S A$ into closed real intervals such that, for all $x \in \bigcup_S A$ and all $A \in S$,

$$x \in A \Leftrightarrow F(x) \subseteq \left[\inf \bigcup_{y \in A} F(y), \sup \bigcup_{y \in A} F(y)\right].$$

This definition and the next theorem are taken from Trotter and Moore [28].

THEOREM 8. *$D(S) \leqslant 2 \Leftrightarrow S$ is interval representable.*

Theorems 2, 6, and 8 together yield

THEOREM 9. *Suppose $H = (\mathbf{n}, S)$ and $\subset$ does not linearly order S. Then the following are mutually equivalent:*

a. $D(S) = 2$.

b. $I(P(H)) = 2$.

c. $d(P^*(H)) = 2$.

d. $\chi(\mathscr{S}_2) = 2$.

e. *S is interval representable.*

Part (c) yields further characterizations by way of the $d = 2$ characterizations in Dushnik and Miller [9] or in Theorem 5.9 in Fishburn [11]. And, by a well-known result, (d) holds if and only if the inversion graph $G(H)$ has no odd cycle.

Although (d) may provide the easiest route for testing $D(S) \leqslant 2$, one can also characterize $D(S) \leqslant 2$ by families of minimal forbidden configurations. For part (c) the relevant family is the set of 3-irreducible posets described in Trotter and Moore [28] and Kelly [16], with $d(P) \leqslant 2$ if and only if no induced subposet of P is order isomorphic to a 3-irreducible poset. For (b) the relevant family is the set of 3-interval irreducible posets of height 1 in Trotter and Moore [28] or Trotter [25]. For (a) it is the family of forbidden hypergraphs that correspond to the 3-interval irreducible posets of height 1 by the H and $P(H)$ association. These are identified in Trotter [25] and Trotter and Moore [28]. Table 3 in the latter paper lists the forbidden hypergraphs up to editing and duals (see next paragraph).

To illustrate, we have determined that there are precisely eight minimal hypergraphs for $n = 4$ that violate $D \leqslant 2$. Up to permutations of elements, they are

$$\{12, 13, 23\}, \{12, 13, 234\}, \{12, 134, 234\}, \{124, 134, 234\},$$
$$\{12, 23, 34, 14\}, \{2, 13, 14, 123, 124\}, \{2, 3, 4, 12, 13, 14\},$$
$$\{2, 3, 12, 13, 14, 123\}.$$

When $n = 4$, $D(S) \leqslant 2$ if and only if no $T \subseteq S$ is isomorphic (by permutation) to one of these eight. By *editing* we mean the removal of elements from one or more edges in a way that does not change whether T is forbidden. For example, when 4 is removed from the second, third, or fourth listed set, we obtain the forbidden $\{12, 13, 23\}$. Sets that edit down to smallest forbidden sets are not listed in Table 3 of Trotter and Moore [28]. The fifth and sixth of our eight sets appear in Table 3 as $\mathscr{C}_4$ and $\mathscr{W}_1$, but the final two do not. They are the duals of $\mathscr{G}_1$ and $\mathscr{G}_2$, respectively. To obtain the dual of H, we take the diagram of $P(H)$, remove the labels, relabel the *top* points 1, ..., $|S|$, and then label each bottom point by the set of top points that cover it. For example, $\mathscr{G}_1$ in Table 3 is $\{135, 12, 34, 56\}$. Its dual is $\{2, 3, 4, 12, 13, 14\}$, which is the seventh set in the preceding list for $n=4$.

It may also be of interest to identify maximal S for a given n that have $D=2$. Omitting $\varnothing$ and $\mathbf{n}$ for convenience, the maximal S for $n=3, 4, 5$ are:

$$n=3\colon \{1, 2, 3, 12, 13\}$$

$$n=4\colon \{1, 2, 4, 12, 13, 14, 123, 134\}$$

$$\{1, 2, 3, 4, 12, 23, 34, 123, 234\}$$

$$n=5\colon \{1, 2, 3, 4, 5, 12, 23, 34, 45, 123, 234, 345, 1234, 2345\}$$

$$\{1, 2, 3, 4, 12, 23, 34, 123, 234, 235, 1235, 2345\}$$

$$\{1, 2, 3, 12, 13, 14, 15, 123, 124, 145, 1234, 1245\}$$

$$\{1, 2, 3, 12, 13, 14, 15, 124, 135, 145, 1245, 1345\}$$

$$\{1, 2, 3, 12, 13, 14, 124, 125, 134, 1234, 1245\}.$$

Thus, for $n=5$, $D(S) \leqslant 2$ if and only if S is a subset of one of the preceding five sets under a permutation on $\{1, 2, 3, 4, 5\}$.

Let $\mathscr{M}_n$ denote the set of maximal S with ground set $\mathbf{n}$ for which $D(S)=2$, with the understanding that no two sets in $\mathscr{M}_n$ are isomorphic under relabeling of elements. With $\varnothing$ and $\mathbf{n}$ included, it follows from Lemma 2 that max $|S|$ for M_n is $\binom{n+1}{2}+1$. Other aspects of $\mathscr{M}_n$ remain to be studied. These include the number of sets in $\mathscr{M}_n$ (1, 2, and 5 for $n=3$, 4, and 5, respectively), the smallest $|S|$ for $\mathscr{M}_n$, and the distribution of $|S|$ over $\mathscr{M}_n$.

7. Discussion

Our aim has been to introduce an intuitively straightforward definition of the dimension D of a finite hypergraph and to explore its connections to other concepts of graphs and ordered sets. We have identified equivalents of D in terms of interval dimension, order dimension, and the chromatic number of an associated hypergraph. When $D=2$, the last of these reduces to the chromatic number of the inversion graph, and an interval representability equivalent also holds. The correspondence between D and the study of suitable arrangements was discussed.

It is apparent that our central idea is not new in a technical sense, but we hope that its conception as dimension will be appealing and useful. Previous works to which we are heavily indebted include Dushnik [8], Trotter and Moore [28], Cogis [3], Trotter [25], and DDF, i.e. Doignon, Ducamp, and Falmagne [7].

We conclude with a summary of open problems suggested by our study.

1. Determine $\alpha_m(2, n)$, the maximum number of edges in a graph on n elements that has $D=m$, for $4 \leqslant m < n$, or obtain good bounds on this maximum. Do likewise for $\alpha_m(k, n)$ when $k \geqslant 3$.

2. Determine exact values of $D(k, n)$ for relevant (k, n) pairs that are not covered by Theorem 3 or the paragraph preceding Theorem 4. Augment or improve the bounds in Theorem 4.

3. Describe aspects of $\mathscr{M}_n$, the set of maximal S with $D(S)=2$ on ground set $\mathbf{n}$, as discussed at the end of the preceding section. Is it always true that $|S| < |T|$ for $S \in \mathscr{M}_n$ and $T \in \mathscr{M}_{n+1}$?

4. In reference to Theorem 2 specify conditions, the simpler the better, that distinguish $d(P(H)) = D(S)$ from $d(P(H)) = D(S)+1$.

5. Is it possible to have $\chi(G(H)) = 3$ and $D(S) > 3$?

6. Is there a constant c such that $D(S)/\chi(G(H)) < c$ is always true? If so, what is the smallest such c? If not, is it true that $D(S)/\chi(G(H))$ can be arbitrarily large when $\chi(G(H))$ is bounded above?

Acknowledgment

We are indebted to a referee for numerous improvements in the paper, including a more compact version of our original proof of Theorem 1.

References

1. K. A. Baker, P. C. Fishburn, and F. S. Roberts, Partial orders of dimension 2, *Networks* **2** (1971), 11–28.
2. A. Bouchet, "Étude combinatoire des ordonnés finis. Applications," Thèse, Université Scientifique et Médicale de Grenoble, 1971.
3. O. Cogis, "La dimension Ferrers des graphes orientés," Thèse, Université de Pierre et Marie Curie, Paris, 1980.
4. O. Cogis, A characterization of digraphs with Ferrers dimension 2, mimeographed, 1984.
5. M. B. Cozzens and F. S. Roberts, On k-suitable sets of arrangements and the boxicity of a graph, *J. Combin. Inform. System Sci.* **9** (1984), 14–24.
6. M. B. Cozzens and F. S. Roberts, On dimensional properties of graphs, *Graphs Combin.* **5** (1989), 29–46.
7. J.-P. Doignon, A. Ducamp, and J.-C. Falmagne, On realizable biorders and the biorder dimension of a relation, *J. Math. Psych.* **28** (1984), 73–109.

8. B. DUSHNIK, Concerning a certain set of arrangements, *Proc. Amer. Math. Soc.* **1** (1950), 788–796.
9. B. DUSHNIK AND E. W. MILLER, Partially ordered sets, *Amer. J. Math.* **63** (1941), 600–610.
10. P. ERDŐS, F. HARARY, AND W. T. TUTTE, On the dimension of a graph, *Mathematika* **12** (1965), 118–122.
11. P. C. FISHBURN, "Interval Orders and Interval Graphs," Wiley, New York, 1985.
12. Z. FÜREDI AND J. KAHN, On the dimensions of ordered sets of bounded degree, *Order* **3** (1986), 15–20.
13. A. GHOUILÀ-HOURI, Caractérisation des graphes non orientés dont on peut orienter les arrêtes de manière à obtenir le graphe d'une relation d'ordre, *C. R. Acad. Sci. Paris* **254** (1962), 1370–1371.
14. P. C. GILMORE AND A. J. HOFFMAN, A characterization of comparability graphs and of interval graphs, *Canad. J. Math.* **16** (1964), 539–548.
15. T. HIRAGUCHI, On the dimension of orders, *Sci. Rep. Kanazawa Univ.* **4** (1955), 1–20.
16. D. KELLY, The 3-irreducible partially ordered sets, *Canad. J. Math.* **29** (1977), 367–383.
17. L. LOVÁSZ, J. NESETRIL, AND A. PULTR, On a product dimension of graphs, *J. Combin. Theory Ser. B* **29** (1980), 47–67.
18. O. ORE, "Theory of Graphs," Amer. Math. Soc., Providence, RI, 1962.
19. F. S. ROBERTS, "Measurement Theory with Applications to Decisionmaking, Utility, and the Social Sciences," Addison–Wesley, Reading, MA, 1979.
20. W. SCHNYDER, Planar graphs and poset dimension, *Order* **5** (1989), 323–343.
21. J. SPENCER, "Probabilistic Methods in Combinatorics," Ph.D. thesis, Harvard University, 1970.
22. J. SPENCER, Minimal scrambling sets of simple orders, *Acta Math. Acad. Sci. Hungary* **22** (1971), 349–353.
23. J. E. STEIF, The frame dimension and the complete overlap dimension of a graph, *J. Graph Theory* **9** (1985), 285–299.
24. W. T. TROTTER, Some combinatorial problems for permutations, *in* "Proceedings, Eighth Southeastern Conference on Combinatorics, Graph Theory, and Computing, Louisiana State University, Baton Rouge, 1977," pp. 619–632.
25. W. T. TROTTER, Stacks and splits of partially ordered sets, *Discrete Math.* **35** (1981), 229–256.
26. W. T. TROTTER, Graphs and partially ordered sets, *in* "More Selected Topics in Graph Theory" (R. J. Wilson and L. W. Beineke, Eds.), Academic Press, New York, 1982.
27. W. T. TROTTER AND K. P. BOGART, On the complexity of posets, *Discrete Math.* **16** (1976), 71–82.
28. W. T. TROTTER AND J. I. MOORE, Characterization problems for graphs, partially ordered sets, lattices, and families of sets, *Discrete Math.* **16** (1976), 361–381.
29. W. T. TROTTER, J. I. MOORE, AND D. P. SUMNER, The dimension of a comparability graph, *Proc. Amer. Math. Soc.* **60** (1976), 35–38.
30. D. B. WEST, Parameters of partial orders and graphs: Packing, covering and representation, *in* "Graphs and Order" (I. Rival, Ed.), Reidel, Dordrecht, 1985.

582b/56/2-11