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The Dimension of Cycle-Free Orders

H. A. KIERSTEAD* and W. T. TROTTER**
Department of Mathematics, Arizona State University, Tempe, AZ 85287, U.S.A.

and

JUN QIN
Department of Computer Science, Voorhees College, Denmark, SC 29042, U.S.A.

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Abstract. The existence of a four-dimensional cycle-free order is proved. This answers a question of Ma and Spinrad. Two similar problems are also discussed.

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1. Introduction

An ordered set is said to be a *cycle-free order* if its comparability graph is chordal, i.e., its comparability graph does not contain an induced cycle of length greater than three. Many problems which are NP-complete for arbitrary ordered sets can be solved in polynomial time for cycle-free orders. One such example is the jump number problem. Bouchitte and Habib [1] have conjectured that the dimension problem is polynomial for cycle-free orders. Ma and Spinrad [6] suggested that this might be true for the trivial reason that cycle-free orders have dimension at most 3, and they proved that every cycle-free order has dimension at most 4. However, in this paper we prove that their bound is tight, and so one must work harder to determine the complexity of the dimension problem for cycle-free orders.

THEOREM 1. *There exists a cycle-free order P such that* $\dim(P) = 4$.

The theorem is proved in Section 2. In Section 3 we present two related problems concerning the dimension of special classes of ordered sets along with some preliminary results. We shall denote the dimension, width, and height of an ordered set P by $\dim(P)$, $w(P)$, and $h(P)$, respectively.

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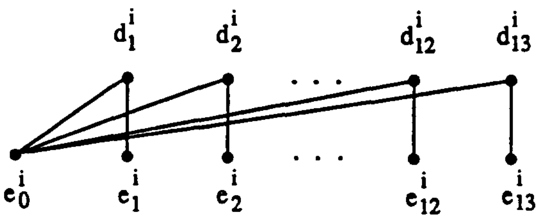


Fig. 2.1. Q_i.

2. Proof of Theorem 3

In order to prove Theorem 3, we shall construct a cycle-free ordered set CF and, using the Pigeon Hole Principle, show that it is 4-dimensional. The ordered set CF consists of $n = 27^{27} + 1$ copies of the sub-ordered sets Q_i's, (Figure 2.1), where i runs from 1 to n, and a big chain C^*:

$$\begin{aligned} &b_1^1 < \cdots < b_1^n < b_2^1 < \cdots < b_2^n \cdots < b_{12}^1 < \cdots < b_{12}^n < b_{13}^1 < \cdots < b_{13}^n < b_0^n \\ &< \cdots < b_0^1 < t_0^1 < \cdots < t_0^n < t_1^1 < \cdots < t_1^n < t_2^1 < \cdots < t_2^n < \cdots < t_{12}^1 \\ &< \cdots < t_{12}^n < t_{13}^1 < \cdots < t_{13}^n. \end{aligned}$$

For each i, j with $1 \leqslant j \leqslant 13$ and $1 \leqslant i \leqslant n$, $b_j^i < e_j^i < d_j^i < t_j^i$, and e_j^i, d_j^i are incomparable with any elements on the big chain C^* between b_j^i and t_j^i. For each i, $b_0^i < e_0^i < t_0^i$ and e_0^i is incomparable with any elements between b_0^i and t_0^i on the big chain C^*. The ordered set CF is shown in Figure 2.3.

Given a vertex x of an ordered set P, $U(x)$ is used to represent the set $\{v \in P: v > x\}$, called the upper set of x. Similarly, the down set of x is $D(x) = \{v \in P: v < x\}$.

PROPOSITION 1. *The ordered set CF is cycle-free.*

Proof. By contrapositive, we assume CF contains a cycle C of length $\geqslant 4$ without chords. So C must be C_k, for some $k \geqslant 0$ (Figure 2.2).

By the construction of the ordered set CF, it is easy to see both $U(d_j^i)$ and $U(e_j^i)$ are chains if $j \neq 0$. Therefore, for any $1 \leqslant q \leqslant k+2$, x_q can be neither d_j^i nor e_j^i for all $i, j > 0$, since $U(x_q)$ is not a chain. If x_q is e_0^i for some i, then y_q and y_{q-1}

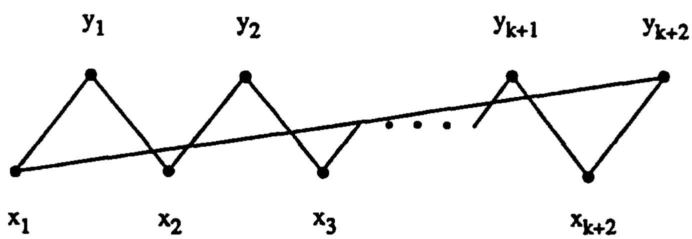


Fig. 2.2. C_k.

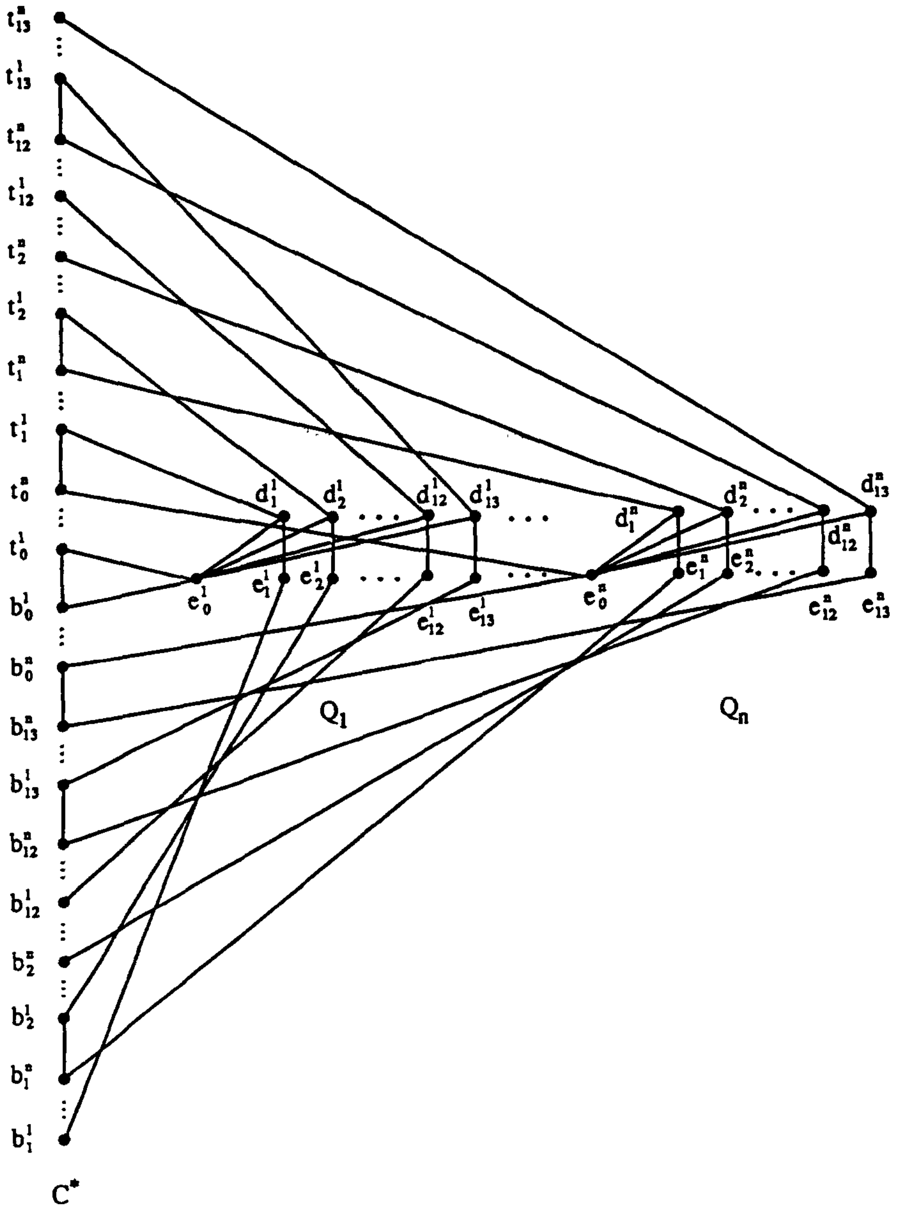


Fig. 2.3. The cycle-free ordered set CF.

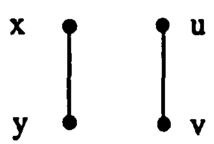
(cyclically) can not both be in the big chain C^*, because $y_q \parallel y_{q-1}$, and hence at least one of y_q and y_{q-1}, say y_q, must be d^i_j for some j. However, $D(d^i_j) - \{e^i_j\}$ is a chain and hence x_{q+1} (cyclically) has to be e^i_j, contradicting what we observed above. Therefore, x_q can only be from the big chain C^* for all q, violating the fact that $\{x_1, \ldots, x_{k+2}\}$ is an antichain. ∎

Next, we are going to show that any 3 linear extensions of CF are not enough to realize it and so dim(CF) $=4$. Given a linear extension L of CF, a vertex v from one of the Q_i's is said to be up or down in L, if v is larger or less than any vertices incomparable with v in the big chain C^*, respectively. Also, v is said to be neither in L, if v is neither up nor down in L.

Fix any realizer $\{L_1, L_2, L_3\}$ of CF. We color each vertex v in one of the Q_i's by a triple (a_1, a_2, a_3), where $a_1, a_2, a_3 \in \{-1, 0, 1\}$, such that $a_i = -1$ if v is down in L_i, $a_i = 0$ if v is neither up nor down in L_i, and $a_i = 1$ if v is up in L_i. So, $3^3 = 27$ colors are used. Then we assign to each Q_i a color which is a sequence $\{c(e^i_0), \ldots, c(e^i_{13}), c(d^i_1), \ldots, c(d^i_{13})\}$ of colors, where $c(v)$ denotes the color a vertex v receives. In this way, we colored these Q_i's with 27^{27} colors. But there are $(27^{27}+1)$ Q_i's contained in our ordered set CF, and hence at least two Q_i's receive the same color, say Q_1 and Q_2. From the coloring we did, it follows that, in each linear extension L_i, d^1_j and e^1_k are up, down or neither if and only if d^2_j and e^2_k are, respectively, where $1 \leqslant j \leqslant 13$ and $0 \leqslant k \leqslant 13$.

Let $A_{i,k} = \{(e^1_j, d^1_j)$: e^1_j is up in L_i, d^1_j is down in L_k, and $j > 0\}$, for $i, k = 1, 2, 3$. Since, for any j, it is impossible that e^1_j is up and d^1_j is down in the same linear extension of CF, $A_{i,i} = \emptyset$ for $i = 1, 2, 3$. From the fact that, for each $j = 1, 2, \ldots, 13$, e^1_j is up in some linear extension L_i and d^1_j is down in some linear extension L_k, it follows that $\{(e^1_j, d^1_j)$: $j = 1, \ldots, 13\} = A_{1,2} \cup A_{1,3} \cup A_{2,1} \cup A_{2,3} \cup A_{3,1} \cup A_{3,2}$. Thus, $|A_{1,2}| + |A_{1,3}| + |A_{2,1}| + |A_{2,3}| + |A_{3,1}| + |A_{3,2}| \geqslant 13$, and hence at least one $A_{i,k}$, say $A_{1,2}$, has size at least 3. Without loss of generality, we may assume that $\{(e^1_j, d^1_j): j = 1, 2, 3\} \subseteq A_{1,2}$. Hence e^1_j is up in L_1 and d^1_j is down in L_2, for $j = 1, 2, 3$. Since Q_2 has the same color as Q_1, e^2_j is up in L_1 and d^2_j is down in L_2, for $j = 1, 2, 3$, too. As a consequence, we obtained the following sub-ordered set Q^* of CF with the properties we mentioned above (Figure 2.4).

We claim that $\{L_1, L_2, L_3\}$ cannot realize all incomparable pairs contained in Q^* shown in Figure 2.4. First, we make the following observation, given $2K_2$:



if $v < x$ and $y < u$ in all linear extensions, then we need two more linear extensions to realize this $2K_2$, since it is impossible that $u < y$ and $x < v$ in one linear extension.

Since $e^1_1, e^1_2, e^1_3, e^2_1, e^2_2, e^2_3$ are all up in L_1, we are forced to have the following order in L_1, $e^1_0, e^2_0 < e^1_1 < d^1_1 < e^2_1 < d^2_1 < e^1_2 < d^1_2 < e^2_2 < d^2_2 < e^1_3 < d^1_3 < e^2_3 < d^2_3$. And in L_2, since $d^1_1, d^1_2, d^1_3, d^2_1, d^2_2, d^2_3$ are all down, which forces e^1_0 and e^2_0 down, we have $e^2_0, d^2_1, d^2_2, d^2_3 < e^1_0$ in L_2, as shown in Figure 2.5, where the relations between any two vertices not joined by a line segment are not known yet.

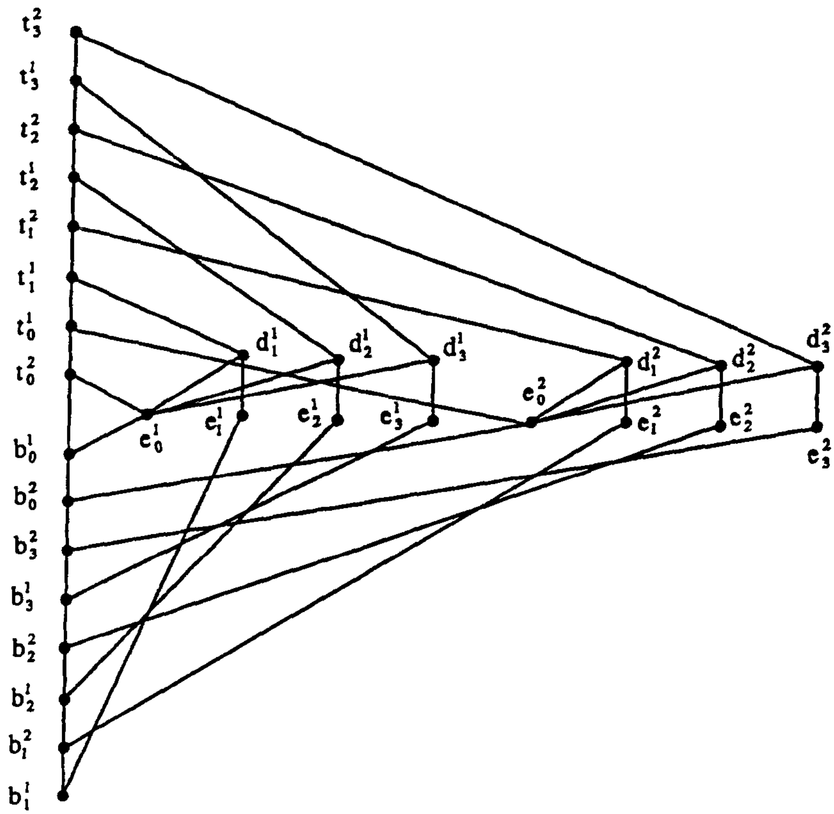


Fig. 2.4. Q^*.

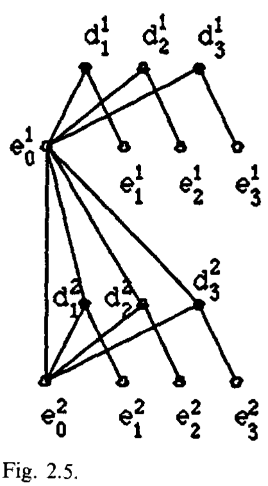
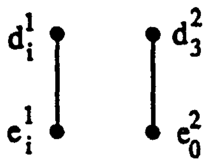


Fig. 2.5.

Since $e_1^1, e_2^1, e_3^1 < d_3^2$ in L_1, and $e_0^2 < d_1^1, d_2^1, d_3^1$ in both L_1 and L_2, it follows from our observation above applied to the following $2k_2$'s:



for $i = 1, 2, 3$, that $d_3^2 < e_1^1, e_2^1, e_3^1$ in L_2. So, e_1^1, e_2^1, e_3^1 are not down in L_2. By the coloring, e_1^2, e_2^2, e_3^2 are not down in L_2, either. Hence they all must be down in L_3. Therefore, in L_3, we are forced to have $e_1^1 < e_1^2 < e_2^1 < e_2^2 < e_3^1 < e_3^2$, the same order as they are in L_1. It follows that, in $L_2, e_2^1 < e_1^2$ and so $e_2^1 < e_1^2 < d_1^2 < e_0^1 < d_3^1$. In L_1 and L_3, however, $e_2^1 < e_3^1 < d_3^1$, too. Hence, the incomparable pair (e_2^1, d_3^1) cannot be realized by $\{L_1, L_2, L_3\}$, a contradiction. We conclude our proof. ■

3. Problems

An ordered set P is said to be an *interval order* if its points can be represented by closed intervals of the real line R so that any two intervals I and J satisfy $I < J$ in P iff the right end point of I is less than the left end point of J. A *circular arc graph* is a graph whose vertices can be represented by arcs of a fixed circle so that any two arcs I and J satisfy $I \sim J$ iff $I \cap J \neq \emptyset$. An ordered set is a *circular-arc order* if its comparability graph is a circular-arc graph. Four distinct vertices a_1, a_2, b_1, and b_2 are said to form a $2K_2$ if $a_1 < b_1, a_2 < b_2$, and a_1 and a_2 are incomparable to b_2 and b_1, respectively. Fishburn's well known characterization [3] of interval orders states that an ordered set is an interval order iff it does not contain $2K_2$ as a suborder.

PROBLEM 1 (Trotter [8]). *Determine the maximum dimension of an interval order of width w.*

Using the following lemma of Rabinovitch [7] we can give an easy upper bound.

LEMMA 1 (Rabinovitch [7]). *Let P be an ordered set. Given two disjoint subsets A, B of P, suppose that there are no such vertices $a_1, a_2 \in A$ and $b_1, b_2 \in B$ that form a $2K_2$. Then there exists a linear extension L of P such that for any $x \in A$ and $y \in B$, if x and y are incomparable in P, then $x > y$ in L.* ■

When the conclusion of the Lemma is satisfied we say that the linear extension L puts A above B. Note that if either A or B is a chain, or if P is an interval order, then the condition of Lemma 1 is satisfied, and so some linear extension L puts A above B.

PROPOSITION 2. *Suppose P is an interval order. Then* $\dim(P) \leqslant 2\log_2(w(P))$.

Proof. Suppose P is an interval order with width $w(P) = n$. By Dilworth's Theorem, P can be partitioned into n chains $C_1, \ldots, C_n$. Let $t = \lceil \log_2 n \rceil$ and $S_1, \ldots, S_{2^t}$ be the enumeration of all subsets of $\{1, 2, \ldots, t\}$. A realizer of P is constructed as follows. Given $k \in \{1, 2, \ldots, t\}$, let $A_k = \bigcup_{k \in S_i} C_i$ and $B_k = P - A_k$. Then, using Lemma 1 and the following note, choose two linear extensions L_k and L'_k of P which put A_k above B_k and B_k above A_k, respectively.

Now we need to prove that $\{L_1, \ldots, L_t, L'_1, \ldots, L'_t\}$ is a realizer of P. Given any two incomparable points x and y, $x \in C_i$ and $y \in C_j$ for some $i \neq j$. Without loss of generality, assume that $S_i - S_j \neq \emptyset$. Pick a number $k \in S_i - S_j$. It is easy to see $x > y$ in L_k and $x < y$ in L'_k, since $x \in A_k$ and $y \in B_k$. Our claim is proven and so $\dim(P) \leqslant 2t$. ∎

Recently Füredi, Hajnal, Rödl, and Trotter [4] have shown that for any interval order P, $\dim(P) \leqslant \log_2 \log_2(h(P)) + (1/2 + o(1)) \log_2 \log_2 \log_2(h(P))$. Thus if the upper bound of Proposition 1 is optimal, the interval order P that witnesses this must satisfy $h(P) = \Omega(2^{w(P)})$.

PROBLEM 2. *Determine the maximum dimension of a circular arc order.*

It is easy to see that crowns (ordered sets on at least six vertices whose comparability graph is a cycle) are three dimensional circular arc orders. In one of the earliest papers on the dimension of ordered sets, Dushnik and Miller [2] proved that the dimension of an ordered set P is less than or equal to 2 if and only if the complement of P is a comparability graph. This fact leads immediately to the following upper bound.

PROPOSITION 3. *Suppose P is a circular-arc order. Then* $\dim(P) \leqslant 4$.

Proof. Suppose that P is a circular-arc order. Then there is a 1–1 map assigning to each $x \in P$ an arc A_X on a fixed circle C. Pick a point v on the circle C. Clearly, the set $X = \{x \in P : v \in A_X\}$ forms a chain of P, and $P - X$ forms a sub-ordered set whose comparability graph G is an interval graph. Since the complement of G and hence the complement of $P - X$ are a comparability graph, $P - X$ has dimension $\leqslant 2$. Let L_1 and L_2 be two linear extensions of P that realize $P - X$. Using Lemma 1 and the following note, we can choose two more linear extensions L_3 and L_4 of P such that L_3 puts Y above $P - Y$ and L_4 puts $P - Y$ above Y. An easy check shows that $\{L_1, L_2, L_3, L_4\}$ is a realizer of P and so $\dim(P) \leqslant 4$. ∎

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