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Large Regular Graphs with No Induced $2K_2$

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Abstract. Let r be a positive integer. Consider r-regular graphs in which no induced subgraph on four vertices is an independent pair of edges. The number v of vertices in such a graph does not exceed $5r/2$; this proves a conjecture of Bermond. More generally, it is conjectured that if $v > 2r$, then the ratio v/r must be a rational number of the form $2 + 1/(2k)$. This is proved for $v/r \geq \frac{21}{10}$. The extremal graphs and many other classes of these graphs are described and characterized.

1. Introduction

A graph G is *H-free* if it has no copy of H as an induced subgraph, where H is a fixed graph. We say that an H-free graph *avoids* H. Let $2K_2$ be the 4-vertex graph consisting of two non-incident edges. We consider the class $\mathbf{G}_r$ of all r-regular $2K_2$-free graphs. We refer to a graph in $\mathbf{G}_r$ as a $\mathbf{G}_r$-graph (or a graph in $\mathbf{G} = \bigcup \mathbf{G}_r$ as a $\mathbf{G}$-graph). Our interest in $\mathbf{G}$-graphs arose from a design problem for interconnection networks: maximize the number of vertices in a hypergraph of diameter 2 in which every edge has size r and every vertex has degree 2 (the edges and vertices of the hypergraph become the vertices and edges of the derived $\mathbf{G}_r$-graph). We refer the reader to the survey paper [1] for a discussion of problems of similar type and an extensive bibliography. The more recent article [2] also explains the origin of the problem.

There are two ways to view this problem: extremally and structurally. When H is forbidden to occur as any subgraph of an n-vertex graph (not only as an induced subgraph), the problem of maximizing the number of edges is a classical problem of extremal graph theory. The appropriate analogue for H-free graphs is to maximize the number of edges in a connected H-free graph subject to a bound on the maximum degree, since when H is not complete a complete graph is H-free. This general problem is shown in [4] to be nontrivial precisely when H is a disjoint

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union of paths. It is solved there for the 4-vertex path, and it is solved in [3] for $H = 2K_2$.

The structural motivation carries on a long tradition of characterizing graphs with various forbidden subgraphs or forbidden induced subgraphs; here we add the requirement of regularity. The reason for requiring the number of vertices to exceed twice the degree is that smaller graphs are relatively dense and likely to avoid $2K_2$. When the number of vertices exceeds $2r$, many structural properties emerge and restrict the possibilities for **G**-graphs, so that we may hope to characterize these. This structural investigation arose from our examination of the following extremal conjecture of J.-C. Bermond:

Conjecture 0. A $\mathbf{G}_r$-graph has at most $5r/2$ vertices.

Our paper begins with a short proof of this conjecture. During the three-year period in which this paper was being refereed, a stronger and more difficult result was proved by Chung, Gyárfás, Tuza, and Trotter [3]; a $2K_2$-free graph with maximum degree at most r has at most $5r^2/4$ edges. Hence the main focus of our paper is the structure of **G**-graphs, including partial proofs of the stronger conjectures given below. We will completely describe all $\mathbf{G}_r$-graphs with at least $21r/10$ vertices. Conjecture 0 follows from the initial steps in this direction; meanwhile, the known examples of $\mathbf{G}_r$-graphs suggest a stronger conjecture:

Conjecture 1. If G is a $\mathbf{G}_r$-graph on v vertices, then r is even and v/r is a rational number of the form $2 + 1/(2k)$.

The following construction by R.L. Graham provides **G**-graphs with the ratios $v/r = 2 + 1/(2k)$.

Example 1. Let $k \geq 1$. Define a graph G_k on $4k + 1$ vertices as follows. The vertices of G_k are the integers $0, 1, \ldots, 4k$, space equally around a circle. A vertex i is joined to each of the $2k$ vertices at distance more than k from it around the circle; i.e., i, j are neighbors when $|i - j| > k \bmod (4k + 1)$. Suppose a, b, c, d (in that cyclic order) induce $2K_2$ in G_k. A short case argument shows that the edges must be ac and bd (crossing), but then the 4 non-edges require the cyclic traversal of a, b, c, d to cover $4k + 1$ positions by traversing at most k positions in each of 4 steps. Hence G_k is a $\mathbf{G}_{2k}$-graph, and $v/r = 2 + 1/(2k)$. Note that G_1 is the 5-cycle C_5. □

The next construction is an easy way to generate additional **G**-graphs.

Example 2. Let G be a $\mathbf{G}_r$-graph on v vertices. Given $p \geq 1$, let G^p denote the graph obtained by replacing each vertex x in G by a set $I(x)$ of p independent vertices. An edge xy in G becomes a complete bipartite graph with partite sets $I(x)$ and $I(y)$ in G^p. Being pr-regular and $2K_2$-free, G^p is a $\mathbf{G}_{pr}$-graph. Since G^p has vp vertices, the vertex/ degree ratio is the same for G^p as for G. We call G^p the *p-fold expansion* of G. This is a special case of what is commonly called the *lexicographic product* $G[H]$, in which each vertex of G is expanded into a copy of H; here H is an independent set of size p. □

Ideally, we would like to characterize **G**-graphs by providing a finite collection of "primitive" classes of **G**-graphs from which all **G**-graphs can be built using a collection of operations such as expansion. We will present several such classes and another operation for building **G**-graphs from smaller ones. The variety of **G**-graphs is surprisingly rich. Nevertheless, all evidence presently available supports a descriptive statement even stronger than Conjecture 1. Indeed, proving Conjecture 2 seems the most likely way to prove Conjecture 1.

Conjecture 2. Every $\mathbf{G}_r$-graph on $2r + q$ vertices is the q-fold expansion of a $\mathbf{G}_{r/q}$-graph on $2r/q + 1$ vertices.

The example of C_5^p shows that Conjecture 0 is best possible for all even r. We will develop many structural properties of **G**-graphs that enable us to describe all $\mathbf{G}_r$-graphs with at least $21r/10$ vertices. This yields a proof that C_5^p is the unique extremal graph (for r even) and a partial proof of Conjectures 1 and 2:

Theorem 1. *If a $\mathbf{G}_r$-graph G has $v = 2r + q \geq 21r/10$ vertices, then $v/r \in \{2 + 1/(2k)$: $1 \leq k \leq 5\}$, and G is the q-fold expansion of a $\mathbf{G}_{r/q}$-graph on $2r/q + 1$ vertices.*

Indeed, our structural results culminate at Theorem 19 with a proof that the **G**-graphs with $v/r \geq \frac{21}{10}$ are expansions of exactly 7 basic graphs.

We adopt several notational conveniences. Let $V = V(G)$ and $E = E(G)$ denote the vertex and edge sets of a finite simple graph G. If U, W are disjoint subsets of V, let $e(U)$ be the number of edges with both ends in U, and let $e(U, W)$ be the number of edges joining U and W. For vertices $x, y \in V$, let $x \leftrightarrow y$ denote adjacency, and let $x \| y$ denote nonadjacency. We choose this notation due to its easy extension to sets of vertices. We write $x \leftrightarrow A$ when $x \leftrightarrow a$ for all $a \in A$ and analogously define $x \| A$, $A \leftrightarrow B$, and $A \| B$.

Let xy denote the edge between x and y when $x \leftrightarrow y$. For $x \in V(G)$, let $N(x) = \{y: x \leftrightarrow y\}$ denote the *neighbor set* of x. The *degree* of x is $d(x) = |N(x)|$, with the degree of a regular graph being the common degree of its vertices. Let $\bar{N}(x) = \{y: x \| y\}$ denote the *non-neighbor set* of x; this includes x. It is convenient to define $N(S) = \bigcap_{u \in S} N(u)$, so that $x \in N(S)$ and $x \leftrightarrow S$ are equivalent (this differs from the more common usage of $N(S)$ for $\bigcup_{u \in S} N(u)$). Similarly, let $\bar{N}(S) = \bigcap_{u \in S} \bar{N}(u)$, and define $N(S \mid T) = N(S) \cap \bar{N}(T)$. We extend the degree notation analogously: $d(S) = |N(S)|$ and $d(S \mid T) = |N(S \mid T)|$. We drop set brackets where no confusion arises; for example, $N(ab \mid uz) = N(a) \cap N(b) \cap \bar{N}(u) \cap \bar{N}(z)$ and $S - x = S - \{x\}$. Finally, motivated by Conjecture 2 and the operation of expansion, we say that vertices with identical neighborhoods are *equivalent*, and we use $\langle u \rangle = \{x \in V: N(x) = N(u)\}$ to denote the *equivalence class* of u.

Due to the frequency and variety of its use, we do not explicitly state the condition that a **G**-graph is $2K_2$-free when we invoke it. Instead, we use stereotypic statements, indicated by the use of "&" and the verb "force". We may say "$a \leftrightarrow b$ & $c \leftrightarrow d$ force $a \leftrightarrow d$" when we know $a \| c$, $b \| c$, and $b \| d$, or "$a \leftrightarrow b$ & $c \leftrightarrow d$ force $d \in N(a) \cup N(b)$" when we know $c \| \{a, b\}$, or "$a \leftrightarrow b$ forces U independent" when we know $U \subset \bar{N}(ab)$. This convention becomes particularly useful when we apply it to sets of vertices, as in "$w \leftrightarrow z$ & $y \leftrightarrow S$ force $w \leftrightarrow S$."

A *triangle* in a graph is a pairwise-adjacent triple of vertices or the subgraph they induce. In section 2 we characterize the triangle-free G-graphs. For other G-graphs, section 3 proves the existence of a "dominating triangle," meaning a triangle having a neighbor of every vertex. The results in Section 3 suffice to prove Conjecture 0. Section 4 considers the structural consequences of edges not in triangles. Sections 5–6 show that all **G**-graphs have edges not on triangles and obtain other structural properties. In sections 7–9, these are applied to bound the size of G-graphs of various types.

2. Triangle-free G-Graphs

We begin with two elementary observations.

Lemma 1. *Let I be an independent set of vertices in a $2K_2$-free graph. For any pair x, y of non-adjacent vertices, the sets $N(x) \cap I$ and $N(y) \cap I$ are ordered by inclusion.*

Lemma 2. *For any ordered pair I_1, I_2 of independent sets in a $2K_2$-free graph, either there exists $x \in I_1$ with $x \leftrightarrow I_2$, or there exists $y \in I_2$ with $y \| I_1$.*

These lemmas allow us to dispose of triangle-free **G**-graphs.

Theorem 2. *If G is a triangle-free* **G***-graph, then r is even and $G = C_5^{r/2}$.*

Proof. Choose an arbitrary edge ab in G. Let $A = N(a) - b$ and $B = N(b) - a$. To avoid triangles, A and B must be disjoint independent sets of size $r - 1$. Let $U = \bar{N}(ab)$. If U is empty, then G has only $2r$ vertices and is not a **G**-graph.

Hence $U \neq \varnothing$; now $a \leftrightarrow b$ forces U independent. Given $u, u' \in U$, we claim $N(u) = N(u')$; suppose not. Since $N(u) \cup N(u') \subseteq A \cup B$ and $d(u) = d(u') = r$, Lemma 1 allows us to assume $N(u') \cap A \subset N(u) \cap A$ and $N(u) \cap B \subset N(u') \cap B$. Since $|A| = |B| = r - 1$, u and u' have neighbors in each of A, B, so there exist $x \in N(auu')$ and $y \in N(bu'|u)$. Now $x \leftrightarrow u \;\&\; b \leftrightarrow y$ force $x \leftrightarrow y$, which makes $xu'y$ a triangle. This contradiction yields $N(u) = N(u')$.

By the preceding paragraph, any $x \in A \cup B$ is adjacent to all or none of U. Let $S_1 = A \cap N(U)$, $S_2 = A - S_1$, $T_1 = B \cap N(U)$, and $T_2 = B - T_1$. Avoiding triangles requires $S_1 \| T_1$. Since $S_2 \| U$, the neighbors of $x \in S_2$ are restricted to $N(b)$, and then r-regularity forces $N(x) = N(b)$. Similarly, $N(y) = N(a)$ for $y \in T_2$. With these observations, $V(G)$ has been partitioned into five independent sets $(\langle a \rangle, S_1, U, T_1, \langle b \rangle)$ such that vertices are adjacent if and only if they belong to cyclically consecutive sets. Regularity then forces each set to have $r/2$ vertices. Therefore r is even and G is the $r/2$-fold expansion of a 5-cycle. □

In view of Theorem 2, we henceforth consider only **G**-graphs containing triangles.

3. Dominating Triangles

We say that a triangle in G is a *dominating triangle* if every vertex of G is adjacent to at least one vertex of the triangle. We want to show that every **G**-graph with a triangle has a dominating triangle. First we need a lemma.

Lemma 3. *Let I be an independent set in a* **G**-*graph G, and let S be an arbitrary set of vertices. If $x \leftrightarrow S$ and $I \| (S \cup x)$, then $N(S \cup I)$ is nonempty.*

Proof. Since $x \leftrightarrow S$, $N(S)$ is nonempty. Choose $y \in N(S)$ to maximize $|N(y) \cap I|$. If there exists $z \in (I - N(y))$, choose $w \in N(z|y)$, which exists because $z \| S$ and G is regular. Now $w \leftrightarrow z \,\&\, y \leftrightarrow S$ force $w \leftrightarrow S$. Furthermore, since $w \| y$ and $z \in (I \cap N(w|y))$, Lemma 1 implies that w has more neighbors in I than y. This contradicts the maximality of $N(y) \cap I$, so we conclude $y \leftrightarrow I$. □

Lemma 4. *If a, b belong to a triangle in a* **G**-*graph, then there is a vertex c such that abc is a dominating triangle.*

Proof. Choose a vertex $x \in N(ab)$, and let $T = \{a, b, x\}$. If T is not a dominating triangle, let $S = \{a, b\} \cup N(x|ab)$. Since $\bar{N}(ab)$ is independent, we have $\bar{N}(T)$ independent and $\bar{N}(T) \| (S \cup x)$. Applying Lemma 3 with $I = \bar{N}(T)$ yields a point c with $c \leftrightarrow (S \cup \bar{N}(T))$. This includes $c \leftrightarrow \bar{N}(ab)$, so abc is a dominating triangle. □

Let $T = abc$ be a dominating triangle. To simplify notation, we use capital letters to denote the sets of vertices not in T whose adjacencies in T are the corresponding lower-case letters. For example, set $A = N(a|bc)$, $BC = N(bc|a) - a$, $ABC = N(T)$, etc. We also henceforth express v as $2r + q$ with $q > 0$, and we let $\alpha_1 = |A| + |B| + |C|$, $\alpha_2 = |AB| + |AC| + |BC|$, and $\alpha_3 = |ABC|$. Various relationships among these sets follow easily.

Lemma 5. *The following statements hold for a dominating triangle abc in a* **G**-*graph, with permutations of A, B, C freely applicable.*

1. $q = |C| - d(ab) = |B| - d(ac) = |A| - d(bc)$.
2. $\alpha_2 + 2\alpha_3 < r - 3$.
3. $|A| \geq 2$, *A is independent, and $x \in A$ implies* $|N(x) \cap (B \cup C)| \geq 3$.
4. *If $x \in A$ and $x \| B$, then $N(x) \cap C \neq \varnothing$ and $B \leftrightarrow N(x) \cap C$.*

Proof. (1): Comparing $N(ab)$ and $V(G)$ yields $2r = v + d(ab) - |C|$ (etc.). (2): Since T is dominating, $v - 3 = \alpha_1 + \alpha_2 + \alpha_3$, and the edges incident to T are counted by $3r - 6 = \alpha_1 + 2\alpha_2 + 3\alpha_3$. Hence $\alpha_2 + 2\alpha_3 = r - 3 - q$. (3): By (1), $|A| \geq q + 1$. $A \subseteq \bar{N}(bc)$ implies $N(x) \subseteq (N(b) \cup N(c))$, so A is independent. If $N(x) \cap (B \cup C) \leq 2$, then x has at least $r - 3$ neighbors in $AB \cup AC \cup BC \cup ABC$. The resulting $\alpha_2 + \alpha_3 \geq r - 3$ contradicts (2). (4): By (3), $N(x) \cap C \neq \varnothing$. Now $B \leftrightarrow b \,\&\, x \leftrightarrow N(x) \cap C$ force $B \leftrightarrow N(x) \cap C$. □

When I_1 and I_2 are independent sets in a **G**-graph, we say that $\{I_1, I_2\}$ are *linked* (*by xy*) if there exist vertices $x \in I_1$ and $y \in I_2$ with $x \leftrightarrow I_2$ and $y \leftrightarrow I_1$. When $I_1 \leftrightarrow I_2$, we say I_1 and I_2 are *totally linked*. Lemma 2 says that if a pair of independent sets is not linked, then one of them must have a vertex totally independent of the other. We use this to show that at least one of the pairs $\{A, B\}$, $\{B, C\}$, $\{A, C\}$ generated by a dominating triangle abc must be linked; in particular, if there is an unlinked pair, then another pair is totally linked.

Lemma 6. *Let abc be a dominating triangle in a* **G**-*graph, and suppose $x \in A$ satisfies $x \| B$. Then $B \leftrightarrow C$ and $x \leftrightarrow C$. Equivalently, if abc is a dominating triangle for which $B \leftrightarrow C$ is false, then every vertex of A has a neighbor in each of B and C.*

Proof. It suffices to show that $y \leftrightarrow C$ for some $y \in B$, since this means $x \leftrightarrow a \,\&\, y \leftrightarrow C$ force $x \leftrightarrow C$, and then $B \leftrightarrow b \,\&\, x \leftrightarrow C$ force $B \leftrightarrow C$. By Lemma 5.4, x has neighbors in C totally adjacent to B. If $\{B, C\}$ is not linked, then we have some $u \in C$ with $u \| B$ (Lemma 2). If $\{A, C\}$ also is not linked, then there exists $w \in A \cup C$ with $w \| A \cup C$, in which case w has neighbors in B that violate $x \| B$ or $u \| B$ (by Lemma 5.4).

Hence the assumption that $\{B, C\}$ is not linked implies that $\{A, C\}$ is linked by some edge wz. We contradict this by showing it leads to $2v \leq 4r$. This follows from the fact that every vertex of G now has at least two neighbors in $\{a, c, w, z\}$. We consider vertices by their adjacencies in abc; first $N(ac) \leftrightarrow a, c$ and $a \cup C \leftrightarrow c, w$ and $c \cup A \leftrightarrow a, z$. Also, we have $AB \cup BC \subset N(a) \cup N(c)$ by definition, and $AB \cup BC \leftrightarrow b \,\&\, w \leftrightarrow z$ force $AB \cup BC \subset N(w) \cup N(z)$. Finally, $B \leftrightarrow b \,\&\, x \leftrightarrow z$ force $B \leftrightarrow z$, and $B \leftrightarrow b \,\&\, u \leftrightarrow w$ force $B \leftrightarrow w$, so $B \leftrightarrow w, z$. □

Lemma 6 is the last tool needed to prove Conjecture 0. The counting technique used in the proof appears again in later sections to bound the size of special classes of $\mathbf{G}_r$-graphs.

Theorem 0. *A $\mathbf{G}_r$-graph has at most $5r/2$ vertices, and the only $\mathbf{G}_r$-graph with $5r/2$ vertices is $C_5^{r/2}$.*

Proof. For triangle-free graphs, this is Theorem 2. Otherwise, we may assume that abc is a dominating triangle with AB linked by xy. Note that $AB \cup ABC \leftrightarrow a, b$; also $A \leftrightarrow a, y$ and $B \leftrightarrow b, x$. Finally, $x \leftrightarrow y \,\&\, c \leftrightarrow N(c)$ force $N(c) \subseteq N(x) \cup N(y)$. Hence every vertex has at least two neighbors in $\{a, b, c, x, y\}$. Since a, b have three such neighbors, this implies $2v < 5r$. □

4. General Structure of G-Graphs

Lemma 6 also yields a classification of dominating triangles. We say that a dominating triangle abc is a *Type i triangle* for $i \in \{1, 2, 3\}$ if exactly i of the pairs $\{A, B\}$, $\{A, C\}$, $\{B, C\}$ are linked. Lemma 6 implies that Type 1 and Type 2 triangles have a totally linked pair. The graphs G_k constructed in Example 1 have Type 3 triangles when $k \geq 2$. In particular, the triangle abc formed by the vertices $a = k + 1$, $b = 3k$ and $c = 0$ is a Type 3 triangle. To see this, observe that $A = \{3k + 1, \ldots, 4k\}$, $B = \{1, \ldots, k\}$, $C = \{2k, 2k + 1\}$, and $\{k, 2k\} \leftrightarrow A$, $\{3k + 1, 2k + 1\} \leftrightarrow B$, $\{1, 4k\} \leftrightarrow C$.

We next present a construction due partly to D.B. Shmoys that yields **G**-graphs with Type 2 triangles.

Example 3. Let m be a positive integer, and set $r = m^2 + m$. Construct H_m as follows. Form $V(H_m)$ from disjoint sets Q_1, Q_2, Q_3, Q_4, Q_5 of sizes m, m^2, m^2, 1, m, respectively, and let Q_1, Q_2, Q_3 be independent sets and Q_5 be a clique. Put $Q_i \leftrightarrow Q_{i+1}$ (cyclically). Finally, we add some edges between Q_3 and Q_5. Label the vertices in Q_5 as $u_1, \ldots, u_m$, and partition Q_3 into m blocks $U_1, \ldots, U_m$ with m vertices each. Then put $u_i \| U_i$ and $u_i \leftrightarrow U_j$ for $i \neq j$. It is straightforward to check that H_m is r-regular and $2K_2$-free.

Note that H_1 is the 5-cycle. For $m \geq 2$, choose $a = u_1$, $c = u_2$, and $b \in Q_1$. Then

abc is a dominating triangle, with $A = U_2$, $B = Q_2$, and $C = U_1$. Since $A \leftrightarrow C \leftrightarrow B$ and $A \| B$, abc is a Type 2 triangle. In fact, H_m has $\binom{m}{2}(m^2 + m)$ dominating triangles $\left(\text{and } \binom{m}{2} \text{ non-dominating triangles}\right)$, all of which are Type 2 and lead to the same structural decomposition of H_m. In this decomposition, the sizes of A, B, C, AB, AC, BC, ABC are m, m^2, m, 0, $m^2 - m$, 0, $m - 2$, respectively. Although $\langle a \rangle = \langle c \rangle = 1$, the set $\langle b \rangle$ consists of b and $m - 1$ vertices of AC (the remainder of Q_1 in the description above). The set AC also contains one special vertex z (the vertex of Q_4 in the description above) such that $z \leftrightarrow \bar{N}(b) - z$ and $\bar{N}(b) - z$ is independent. □

Our examples of **G**-graphs with Type 1 triangles arise naturally from the structure we prove that such graphs must have, so we postpone presentation of such a graph until Section 7. Meanwhile, we note that the adjacency structure in the subgraph induced by Q_3 and Q_5 in H_m will always occur under suitable conditions. However, applications of this lemma will require considerable knowledge about the structure of **G**-graphs. Hence we postpone it until Lemma 16 at the end of Section 6, even though it applies for arbitrary **G**-graphs with appropriate subsets, because its applications will come very late.

In the remainder of this section, we prove an important technical result about edges of **G**-graphs not on triangles. As in the proof of Theorem 2, we seek to show that common non-neighbors of adjacent vertices not in a triangle have identical neighborhoods. The proof is much more difficult than the equality of neighborhoods in Theorem 2; it depends on the regularity of **G**-graphs via the counting of vertex neighborhoods. We first isolate a remark that applies to all edges and will be useful separately.

Remark 1. *If ab is an edge of a **G**-graph, then $\bar{N}(ab)$ is an independent set of size $d(ab) + q$.*

Proof. $a \leftrightarrow b$ forces independence, and $|N(uw)| = d(u) + d(w) + |\bar{N}(uw)| - v = |\bar{N}(uw)| - q$.

Theorem 3. *If ab is an edge of a **G**-graph belonging to no triangle, then $\bar{N}(ab)$ is an equivalence class (of size q).*

Proof. The set $U = \bar{N}(ab)$ is independent, and no vertex outside U can have the same neighborhood as a vertex in U. Thus it suffices to show $N(u_1) = N(u_2)$ for arbitrary distinct vertices $u_1, u_2 \in U$. This requires several auxiliary sets and facts about their adjacencies. Let

$$Q = \langle a \rangle \qquad S = N(a) - R \qquad S_0 = N(u_2) \cap S \qquad S_1 = S - S_0$$
$$R = \langle b \rangle \qquad T = N(b) - Q \qquad T_0 = N(u_2) \cap T \qquad T_1 = T - T_0$$

Being equivalence classes, Q, R are independent sets. Since $\langle a \rangle \subseteq N(b)$ and $\langle b \rangle \subseteq N(a)$, the sets Q, R, S, T partition $V - U$. By construction, $S \leftrightarrow Q \leftrightarrow R \leftrightarrow T$.

Also, $S_0 \leftrightarrow u_2 \,\&\, b \leftrightarrow T_1$ force $S_0 \leftrightarrow T_1$; similarly $T_0 \leftrightarrow S_1$. The edges described thus far appear in Fig. 1.

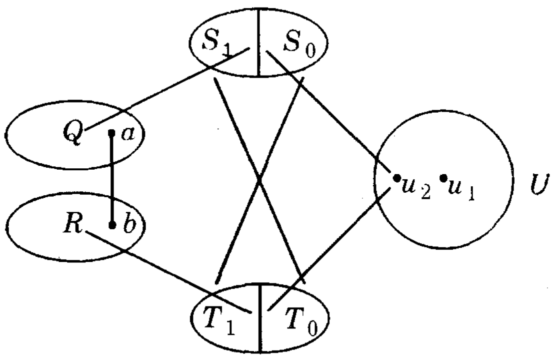

Fig. 1. Sketch of a **G**-graph with an edge on no triangle

We claim that S_1 and T_1 are independent sets. It suffices to obtain a contradiction from assuming an edge xy in S_1. First $x \leftrightarrow y \,\&\, b \leftrightarrow T_1$ force $T_1 \subset N(x) \cup N(y)$, and $x \leftrightarrow y \,\&\, u_2 \leftrightarrow S_0$ force $S_0 \subset N(x) \cup N(y)$. Also $\{x, y\} \leftrightarrow (Q \cup T_0)$, as noted above. We have counted T_1, S_0 at least once and Q, T_0 twice, the total being at most $2r - 2$. However, $Q \cup T_0 \cup T_1 = N(b)$ and $S_0 \cup T_0 \cup Q = N(u_2) \cup Q$, which yields the contradiction $2r + |Q| \leq 2r - 2$.

We next claim that if S_1 or T_1 is nonempty, then there exist $c \in S_1$ and $d \in T_1$ with $c \| T_1$ and $d \| S_1$. For each $x \in S_1$, there exists $y \in T_1$ with $x \| y$; otherwise $N(x) \supset N(b)$. Similarly, for each $y \in T_1$ there exists $x \in S_1$ with $x \| y$. Now the claim follows from Lemma 2.

For the remainder of the proof, we suppose that $N(u_1) \neq N(u_2)$ and seek a contradiction. We claim first that this forces $u_1 \leftrightarrow (S_1 \cup T_1)$. We may assume $u_1 \leftrightarrow x$ for some $x \in S_1$, which also implies that T_1 is nonempty. Now $u_1 \leftrightarrow x \,\&\, b \leftrightarrow d$ force $u_1 \leftrightarrow d$, after which $u_1 \leftrightarrow d \,\&\, a \leftrightarrow S_1$ force $u_1 \leftrightarrow S_1$ (including $u_1 \leftrightarrow c$), and then $u_1 \leftrightarrow c \,\&\, b \leftrightarrow T_1$ force $u_1 \leftrightarrow T_1$.

Hence $N(u_1) \cup N(u_2) = S \cup T$, and we can make the description symmetric in u_1 and u_2 by refining the partition so that $S_1 = S \cap \bar{N}(u_2)$, $S_2 = S \cap \bar{N}(u_1)$, and $S_3 = S \cap N(u_1 u_2)$; similarly for T. By symmetry in u_1, u_2, all of S_1, S_2, T_1, T_2 are nonempty. This symmetry also implies $S_2 \leftrightarrow (T_1 \cup T_3)$ and $T_2 \leftrightarrow (S_1 \cup S_3)$. Finally, $S_1 \leftrightarrow u_1 \,\&\, u_2 \leftrightarrow S_2$ force $S_1 \leftrightarrow S_2$; similarly $T_1 \leftrightarrow T_2$.

Now we count vertex neighborhoods. Given $x \in S_1$ and $y \in T_1$, we have Q, T_3, T_2, $S_2 \subset N(x)$ and R, S_3, S_2, $T_2 \subset N(y)$. Since $N(x)$, $N(y)$ also contain u_1, u_2, the sizes of these eight sets sum to less than $2r$. Using $2r = |N(a)| + |N(b)| = |Q| + |R| + |S| + |T|$ and canceling like terms, we obtain $|S_1| + |T_1| > |S_2| + |T_2|$. However, doing this with x, y in S_2, T_2 instead of S_1, T_1 yields $|S_2| + |T_2| > |S_1| + |T_1|$. This contradiction proves $N(u_1) = N(u_2)$. □

Theorem 3 will help significantly if we can prove that every **G**-graph has an edge not on a triangle. This is one aim of the next section.

5. Critical Triangles

We begin by selecting a special dominating triangle. If *abc* is a dominating triangle in a **G**-graph G, let $\mu(abc) = \min\{|A|, |B|, |C|\}$, and let $\mu(G) = \min\{\mu(abc)$: *abc* is a dominating triangle of $G\}$. A dominating triangle *abc* is a *critical triangle* in G if $\mu(abc) = \mu(G)$. For convenience, we say that a critical triangle *abc* is *c-critical* if $\mu(G) = \mu(abc) = |C|$, and that a **G**-graph is a *Type i* **G**-graph if it has a Type *i* critical triangle. The illustration of a **G**-graph in Fig. 2 will aid visualization for the arguments of the next several sections.

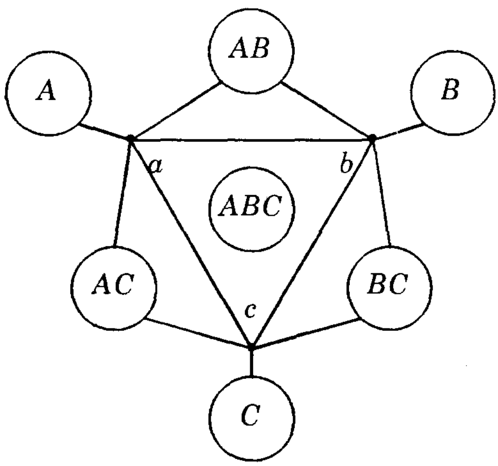

Fig. 2. Sketch of a **G**-graph with dominating triangle *abc*

We next make some easy but important remarks about dominating triangles, collected here to emphasize the current notational conventions.

Lemma 7. *If abc be a dominating triangle in a* **G***-graph, then*
1. *If* $x \leftrightarrow y$ *with* $x \in A$, $y \in B$, *then* $N(c) \subset N(x) \cup N(y)$.
2. *If xyz is a triangle with* $x \in A$, $y \in B$, $z \in C$, *then xyz is dominating.*
3. *If* $\{A, B\}$ *is linked by xy, then* $N(z|xy) \subseteq AB \cup \{c\}$.
4. *If abc is c-critical and* $\{A, B\}$ *is linked by xy, then xy belongs to no triangle.*
5. $\langle a \rangle \subseteq BC \cup a$, $\langle b \rangle \subseteq AC \cup b$, $\langle c \rangle \subseteq AB \cup c$.

Proof. (1): $x \leftrightarrow y \,\&\, c \leftrightarrow N(c)$ force $N(c) \subset N(x) \cup N(y)$. (2): Since *abc* is dominating, (1) applies to each edge of *xyz*. (3): Follows from $x \leftrightarrow A$, $y \leftrightarrow B$, and (1). (4): If *xy* belongs to a triangle, it belongs to a dominating triangle *xyz* (Lemma 4). Now (3) and Lemma 5.1 imply $d(z|xy) \le d(ab) < |C|$, contradicting the criticality of *abc*. (5): The specified set contains all vertices whose adjacencies in $\{a, b, c\}$ agree with the specified vertex. □

Lemma 8. *If abc is a c-critical triangle in G, then* $C \leftrightarrow AB$.

Proof. Suppose there exist $u \in AB$ and $z \in C$ with $u \| z$. First, consider the case where $z \leftrightarrow A$ or $z \leftrightarrow B$; we may assume $z \leftrightarrow B$. Choose some $y \in B$, and let $A' = A \cap \bar{N}(y)$. We eliminate this case by showing that uyb is a dominating triangle with $\mu(uyb) < \mu(abc)$. First $u \leftrightarrow a \,\&\, z \leftrightarrow B$ force $u \leftrightarrow B$. We have $y \leftrightarrow A - A'$ by definition. Also $z \leftrightarrow y \,\&\, a \leftrightarrow A'$ force $z \leftrightarrow A'$, and then $u \leftrightarrow b \,\&\, z \leftrightarrow A'$ force $u \leftrightarrow A'$. Finally, $u \leftrightarrow y$ forces $N(c) \subseteq N(u) \cup N(y)$. Hence uyb is dominating and $d(b|uy) \leq d(ab) < |C|$ (Lemma 5.1).

Therefore, z must be nonadjacent to some $x' \in A$ and $y' \in B$. Now $c \leftrightarrow z$ forces $x' \| y'$. The absence of edges among $\{x', y', z\}$ implies that no pair of $\{A, B, C\}$ is totally linked; hence abc is a Type 3 triangle. This means $\{A, B\}$ is linked by some edge xy. But now $x \leftrightarrow y' \,\&\, c \leftrightarrow z$ force $x \leftrightarrow z$, and $y \leftrightarrow x' \,\&\, c \leftrightarrow z$ force $y \leftrightarrow z$. This makes xyz a triangle, which is forbidden by Lemma 7.4. □

To guarantee edges not on triangles, it suffices to show that $\{A, B\}$ is linked when abc is c-critical, since every edge linking them then belongs to no triangle (by Lemma 7.4). This is easy to show for c-critical triangles of Types 2 and 3, and proving it for Type 1 will be our main task in the remainder of the section.

Lemma 9. *If abc is a Type 2 or Type 3 c-critical triangle, then $\{A, B\}$ is linked.*

Proof. If abc is Type 3, we are done. If it is Type 2 and $\{A, B\}$ is not linked, then by Lemma 2 and symmetry of A and B we may choose $x \in A$ with $x \| B$. Since $\{A, C\}$ must be linked, there exists $z \in C$ with $z \leftrightarrow A$. Consider $\bar{N}(c) = A \cup B \cup AB \cup c$. We have $z \leftrightarrow A \cup c$ by choice, $z \leftrightarrow B$ by Lemma 6, and $z \leftrightarrow AB$ by Lemma 8, so $N(c) \cup N(z) = V$. This yields the contradiction $v \leq 2r$. □

When we consider triangles abc with $\{A, B\}$ linked by an edge xy, there is a natural *xy-partition* of C; its definition and fundamental properties follow next.

Lemma 10. *Suppose that abc is a dominating triangle for which $\{A, B\}$ is linked by xy, and let $C_1 = C \cap N(x|y)$, $C_2 = C \cap N(y|x)$, and $C_3 = C \cap N(xy)$.*
1. *C_1, C_2, C_3 partition C.*
2. *$C_1 \leftrightarrow A$ and $C_2 \leftrightarrow B$.*
3. *If abc is c-critical, then $C_3 = \varnothing$.*
4. *If abc is Type 1, or if abc is Type 3 and c-critical, then C_1 and C_2 are nonempty.*

Proof. (1): Lemma 7.1. (2): $C_1 \leftrightarrow c \,\&\, y \leftrightarrow A$ force $C_1 \leftrightarrow A$; similarly for $C_2 \leftrightarrow B$. (3): Lemma 7.4. (4): If abc is Type 3 and c-critical, this follows from (3) and $\{A, C\}$, $\{B, C\}$ linked. If abc is Type 1, then C_1 empty implies $y \leftrightarrow C$, in which case $\{B, C\}$ is also linked unless there exists $y' \in B$ with $y' \| C$, which implies $A \leftrightarrow C$ (Lemma 6) and contradicts the fact that abc is Type 1 (similarly for C_2). □

The next result uses the xy-partition to establish fundamental adjacencies in **G**-graphs. It will be applied in this section to show that $\{A, B\}$ is the unique linked pair in a c-critical Type 1 triangle abc. It will be used again later for both Type 1 and Type 3 graphs. The two contexts are combined here because the argument is almost the same. The graphs H_m of Example 3 show that the conclusion of Lemma 11 (and the nonemptiness of C_1 and C_2 in Lemma 10) does not hold for Type 2 graphs; in fact, $B \| AC$ and $C_1 = \varnothing$ in those graphs.

Lemma 11. *Suppose abc is a dominating triangle for which $\{A, B\}$ is linked. If abc is Type 1, or if abc is Type 3 and c-critical, then $B \leftrightarrow AC$ and $A \leftrightarrow BC$.*

Proof. We use the xy-partition of C. Also, let $S = BC \cap N(x)$, $S' = BC - S$, $T = AC \cap N(y)$, $T' = AC - T$. We reduce the task to proving $S' = T' = \varnothing$. If abc is Type 1 this suffices because $A \leftrightarrow B$ and we can select any edge between them as xy. Under the other hypothesis, abc c-critical implies $y \| S$ (Lemma 7.4), and then $S \leftrightarrow c$ & $y \leftrightarrow A$ force $S \leftrightarrow A$ (similarly, $T \leftrightarrow B$).

Whether S', T' are empty or not, $S' \leftrightarrow c$ & $x \leftrightarrow B$ force $S' \leftrightarrow B$, also $S' \leftrightarrow b$ & $x \leftrightarrow C_1 \cup C_3$ force $S' \leftrightarrow C_1 \cup C_3$, and $S' \| \{a, x\}$ & $a \leftrightarrow x$ force $S' \| C_2$. Similarly, $T' \leftrightarrow A \cup C_2 \cup C_3$ and $T' \| C_1$. Finally, $S' \leftrightarrow C_1$ & $T' \leftrightarrow C_2$ force $S' \leftrightarrow T'$, since C_1 and C_2 are nonempty (Lemma 10).

If S', T' are not both empty, we may select $u \in S'$. We prove that uyb is a dominating triangle and use this to prove $T' \neq \varnothing$. We know that $y \leftrightarrow A$, C_1, C_3, T, b and $u \leftrightarrow C_2$, C_3, T', b; hence $\bar{N}(b) \subseteq N(u) \cup N(y)$. Let $U = N(u|yb)$ and $Y = N(y|bu)$. If $T' = \varnothing$, then $U \subseteq C_2 \cup C_3$. Since $a \in N(b|uy)$ and $a \| C_2 \cup C_3$, Lemma 6 then implies $U \leftrightarrow Y$. Since C is independent, we conclude $Y \cap C = \varnothing$. Since C_2 is nonempty, this contradicts $C_2 \subseteq Y$, which follows from $C_2 \| b, u$.

Hence we can also select $w \in T'$. It now suffices to show that each vertex of G is adjacent to at least two of $\{u, w, y, z\}$, which yields the contradiction $2v \leq 4r$. We have already observed that $u \leftrightarrow B$, C_1, C_3, T', b, c and $y \leftrightarrow A$, C_2, C_3, T, S', b; similarly $w \leftrightarrow A$, C_2, C_3, S', a, c and $x \leftrightarrow B$, C_1, C_3, S, T', a. This proves the claim except for vertices in $S \cup T \cup N(ab)$. For $N(ab)$, $u \leftrightarrow y$ & $N(ab) \leftrightarrow a$ force $N(ab) \subset N(u) \cup N(y)$, and $w \leftrightarrow x$ & $N(ab) \leftrightarrow b$ force $N(ab) \subset N(w) \cup N(x)$.

Finally, we have $S \subset N(z)$ and $T \subset N(y)$ and need another neighbor for vertices in $S \cup T$; by symmetry, we need only consider $v \in S$. Recall that $T' \| C_1 \| S'$ and $T' \leftrightarrow C_2 \| S'$. If $v \| w$, then $c \leftrightarrow w$ & $w \| C_1$ force $v \| C_1$, and $v \leftrightarrow b$ & $w \leftrightarrow C_2$ force $v \leftrightarrow C_2$. If $v \| u$, then $v \leftrightarrow C_2$ & $u \leftrightarrow C_1$ force $v \leftrightarrow C_1$. This incompatibility between v and the (nonempty) set C_1 prohibits $v \| w, u$, which completes the proof. □

For the next result about Type 1 triangles, we need the following lemma.

Lemma 12. *If I and S are disjoint vertex sets in a **G**-graph, I is independent, and $\bar{N}(S) \cap I = \varnothing$, then there is a clique in S containing a neighbor of every vertex in I.*

Proof. By hypothesis, every vertex of I has a neighbor in S. Let K be a minimal subset of S containing a neighbor of every vertex in I. If any vertices of K were nonadjacent, their neighborhoods in I would be ordered by inclusion (Lemma 1), which would violate the minimality of K. □

Now we can guarantee two special vertices with respect to Type 1 triangles.

Lemma 13. *If abc is a Type 1 triangle and $\{A, B\}$ is the unique linked pair, then AC and BC each contain a vertex adjacent to all of C.*

Proof. Let $C_A = C \cap N(A) - N(B)$ and $C_B = C \cap N(B) - N(A)$, with $C^* = C \cap N(B) \cap N(A)$. Since $C \leftrightarrow c$ & $A \leftrightarrow B$, these three sets partition C. Let $C_{A'} = C \cap \bar{N}(B)$ and $C_{B'} = C \cap \bar{N}(A)$; note that $C_{A'} \subseteq C_A$ and $C_{B'} \subseteq C_B$. Since every vertex of $C - C_{B'}$ has a neighbor in A, there exists $x \in A$ with $x \leftrightarrow (C - C_{B'})$ (Lemma 2). Similarly, there exists $y \in B$ with $y \leftrightarrow (C - C_{A'})$. If $C_{B'} = \varnothing$, then $x \leftrightarrow C$, in which case $\{A, C\}$ is also linked unless there exists $x' \in B$ with $x' \| C$, which implies $B \leftrightarrow C$ (Lemma 6) and contradicts the fact that abc is Type 1. Hence $C_{B'}$, $C_{A'}$ are nonempty.

If $z \| AC \cup A = N(a|b)$ for some $z \in C$, then C being independent implies $N(z) \subseteq N(b) - a$, which violates regularity. Since $C_{B'} \| A$, every vertex of $C_{B'}$ thus has a neighbor in AC, and then Lemma 12 guarantees that AC contains a minimal complete graph K whose vertices together dominate $C_{B'}$. Most of this proof involves showing $K \leftrightarrow C_A \cup C^*$, from which the desired vertices will emerge easily at the end.

Let $Q = N(C_A \cup C^*) \cap K$ and $U = K - Q$, with $m = |Q|$ and $n = |U|$. We claim that $U \leftrightarrow A$. Otherwise, consider $u \in U$ with $u \notin N(A)$. By the choice of K, u has a neighbor $x \in C_{B'}$. Now $u \leftrightarrow x \,\&\, A \leftrightarrow (C_A \cup C^*)$ force $u \leftrightarrow (C_A \cup C^*)$, contradicting $u \in U$.

We aim to show $n = 0$. Otherwise, we count $2(n + m)$ vertex neighborhoods and obtain a total count of at least $v(n + m)$, contradicting $v > 2r$. The easy case is $m = 0$, $n > 0$. Here the $2n$ neighborhoods are $N(x)$, $n - 1$ copies of $N(c)$, and $N(u)$ for each $u \in K$. First $K \leftrightarrow B$ (Lemma 11) and $K \leftrightarrow A \cup a \cup c$ imply that vertices of $B \cup A \cup a \cup c$ are counted at least n times, and in fact $x \leftrightarrow B$ implies that B is counted $n + 1$ times. Vertices in $N(c)$ are counted $n - 1$ times from c; for the additional incidence in $N(c|a)$, we have $z \leftrightarrow (C - C_{B'}) \cup BC$ (Lemma 10) and every vertex of $C_{B'}$ adjacent to at least one vertex of K. The remainder of $N(c)$ is $N(ac)$; since $|B| > |N(ac)|$ (Lemma 5.1), the excess count on B remedies the possible deficiency on $N(ac)$. The remaining vertices are $w \in AB$, where $w \leftrightarrow b \,\&$ edges of K force w adjacent to at least $n - 1$ vertices of K. Now $w \leftrightarrow b \,\&\, K \leftrightarrow z$ force $w \leftrightarrow K$ or $w \leftrightarrow z$, which remedies the deficiency.

Hence we may assume $m > 0$. Here the $2(m + n)$ neighborhoods are $N(a)$, n copies of $N(c)$, $m - 1$ copies of $N(y)$, and $N(u)$ for each $u \in K$. Since $A \leftrightarrow U \cup y \cup a$, A is counted at least $n + m$ times. Also, $AC \leftrightarrow B$ (Lemma 11) implies that $B \cup AC \cup c \cup b$ is counted $n + m$ times. If $w \in N(b)$, then $w \leftrightarrow b \,\&$ edges of K force w adjacent to at least $n + m - 1$ vertices of K; a provides the additional neighbor when $w \in N(ab)$, and c provides n additional neighbors when $w \in BC$.

This leaves C, which is counted n times from c. For $C_A \cup C^*$, we find the m additional neighbors in Q. For C_B, we count $m - 1$ for $N(y)$. For $C_{B'}$, we are guaranteed a neighbor in K, but for $C_B - C_{B'}$ we may have a deficiency of 1. The deficiency is eliminated if $u \| x$ for some $u \in Q$, because then $u \notin \bar{N}(C_{B'}) \,\&\, x \leftrightarrow (C - C_{B'})$ force $u \leftrightarrow (C - C_{B'})$. Hence we may assume $K \leftrightarrow x$. Now we remedy the deficiency by proving that $T = N(ab|x)$ is as large as $C_B - C_{B'}$ and has excess count. Since $K \leftrightarrow x \,\&\, b \leftrightarrow T$ force $K \leftrightarrow T$, T is counted at least $m + n + 1$ times. By Lemma 11 and the choice of T and x, respectively, $x \leftrightarrow N(b|a) \cup (N(ab) - T) \cup (C - C_{B'})$. Hence $r \geq d(b|a) + d(ab) - |T| + |C - C_{B'}|$, or $|T| \geq |C - C_{B'}|$.

We have now assigned a count of at least $m + n$ to each vertex, except that vertices of BC have been counted $m + 2n - 1$ times, for c and their neighbors in K. This is at least $m + n$ if $n > 0$, and if $K \leftrightarrow BC$ we have an additional neighbor in K. Hence we have proved that $n = 0$ (i.e. $K \leftrightarrow C_A \cup C^*$) and that $u \| w$ for some $u \in K$, $w \in BC$. Now $w \leftrightarrow b \,\&\, u \leftrightarrow (C_A \cup C^*)$ force $w \leftrightarrow (C_A \cup C^*)$. This means that w itself is a minimal clique in BC that dominates $C_{A'}$. By the argument symmetric to that above, we conclude that $w \leftrightarrow C_B \cup C^*$. Hence $w \leftrightarrow C$. Now $u \leftrightarrow a \,\&\, w \leftrightarrow C$ force $u \leftrightarrow C$, and u, w are the desired vertices. □

Finally, we reach the objective of this section, which by Lemma 7.4 yields edges not on triangles.

Theorem 4. *If abc is a c-critical triangle, then*
1. $\{A, B\}$ *is linked by an edge xy.*
2. $\langle c\rangle = \bar{N}(xy)$, *and* $|\langle c\rangle| = q$.
3. $|C| \geq 2q + |ABC|$.

Proof. (1): Immediate by Lemma 9 unless *abc* is Type 1, in which case we may assume that $\{B, C\}$ is its unique linked pair. Lemma 13 then guarantees $u \in AC$ with $u \leftrightarrow A$. Since $u \leftrightarrow B$ (Lemma 11), *acu* is a dominating triangle. Since $u \leftrightarrow A$, we have $N(a|cu) \subseteq AB$, implying $|N(a|cu)| \leq |AB| < |C|$ (Lemma 5.1) and contradicting the c-criticality of *abc*. (2): From (1), Lemma 7.4, and Theorem 3. (3): From (2) and Remark 1, applied to $C = \bar{N}(ab)$. □

Since $\{A, B\}$ is linked whenever *abc* is a c-critical triangle, Type 2 **G**-graphs lack the symmetry of a and b in their c-critical triangles, so our approach to characterizing them differs from that for Type 1 and Type 3 **G**-graphs. We postpone the discussion of Type 2 **G**-graphs to Section 9.

6. Structure of Type 1 and Type 3 G-Graphs

We can now sketch out the structure of Type 1 and 3 **G**-graphs. We assume *abc* is a c-critical triangle, so $\{A, B\}$ is linked by an edge xy (Lemma 13). Let $C_1 = C \cap N(x|y)$ and $C_2 = N(y|x)$ be the xy-partition of C (Lemma 10). The next theorem follows readily from earlier results.

Theorem 5. *If abc is a c-critical Type* 1 *or Type* 3 *triangle and* $\{A, B\}$ *is linked by xy, then*
1. $C_1 = \bar{N}(by)$ *and* $C_2 = \bar{N}(ax)$.
2. *The edges by and ax belong to no triangles.*
3. C_1 *and* C_2 *are equivalence classes of size q.*
4. $ABC = \varnothing$.

Proof. (1): From $y \leftrightarrow A$, $x \leftrightarrow B$, and Lemma 11. (2): From (1), the criticality of *abc*, and the fact that C_1, C_2 are both nonempty (Lemma 10.4) and therefore smaller than C. (3): From (1) and (2) by Theorem 3. (4): From (3) and Theorem 4.3. □

Note that Theorem 5.3 and 5.4 agree with Lemma 5.1 and Theorem 4.2, i.e. $2q = |C_1| + |C_2| = |C| = 1 + |AB| + |ABC| + q$. To study the structure of other vertex subsets not self-symmetric in A and B, we introduce more detailed notation. In addition to C_1, C_2, this notation applies whenever we discuss a c-critical Type 1 or Type 3 triangle *abc* (through Section 8). Define $A_1 = A \cap N(B)$, $B_1 = B \cap N(A)$,

$S = AC \cap N(C)$, $S' = AC \cap \bar{N}(C)$, $T = BC \cap N(C)$, $T' = BC \cap \bar{N}(C)$. Note that $x \in A_1$ and $y \in B_1$ and that we already know $C_1 \leftrightarrow A$ and $C_2 \leftrightarrow B$ (Lemma 10.2).

Theorem 6. *If abc is a c-critical Type* 1 *or Type* 3 *triangle and* $\{A, B\}$ *is linked by xy, then*

1. $A_1 \| N(a)$ *and* $B_1 \| N(b)$.
2. $\langle c \rangle = AB \cup c$.
3. $N(C_1) \cap B = B - B_1$ *and* $N(C_2) \cap A = A - A_1$.
4. $S \leftrightarrow T'$ *and* $T \leftrightarrow S'$.
5. *If* $u \in S$, *then* $|BC - N(u)| = 2q + d(ua|b)$ (*and* $w \in T$ *implies* $d(ac|bw) = 2q + d(wb|a)$).
6. $AC = S \cup S'$ *and* $BC = T \cup T'$.
7. S' *and* T' *are independent sets, with* $\langle b \rangle \subseteq S' \cup b$ *and* $\langle a \rangle \subseteq T' \cup a$.

Proof. We verify the first in each symmetric pair of statements. (1): From Theorem 5.2. (2): $\langle c \rangle \subseteq AB \cup c \subseteq \bar{N}(xy) = \langle c \rangle$ by Lemma 7.5, (1), and Theorem 4.2, respectively. (3): Theorem 5.1 implies $B_1 \| C_1$. For $w \in B - B_1$, we can choose $u \in A$ with $u \| w$, and then $w \leftrightarrow b$ & $u \leftrightarrow C_1$ force $w \leftrightarrow C_1$. (4): $S \leftrightarrow C_1$ & $b \leftrightarrow T'$ force $S \leftrightarrow T'$. (5): $u \leftrightarrow B$ (Lemma 11) implies $BC - N(u) = \bar{N}(ua)$, which by Remark 1 has size $q + d(ua)$. Also, $ABC = \varnothing$ (Theorem 5.4) implies $N(uab) = \langle c \rangle$, which has size q (Theorem 4.2).

(6): Since C_1 and C_2 are equivalence classes (Theorem 5.3), it suffices to show $u \leftrightarrow C_1$ if and only if $u \leftrightarrow C_2$ for $u \in AC$. For sufficiency, $u \leftrightarrow C_2$ & $x \leftrightarrow C_1$ (by (1)) force $u \leftrightarrow C_1$. For necessity, (5) guarantees a vertex $w \in BC - N(u)$. Now $w \leftrightarrow b$ & $u \leftrightarrow C_1$ force $w \leftrightarrow C_1$, next $w \leftrightarrow C_1$ & $y \leftrightarrow C_2$ (by (1)) force $w \leftrightarrow C_2$, and finally $u \leftrightarrow a$ & $w \leftrightarrow C_2$ force $u \leftrightarrow C_2$. (7): By (1), $A_1 \leftrightarrow C_1$ forces S' independent. Using Lemma 7.5, (6), and $C \leftrightarrow S$, we have $\langle b \rangle \subseteq S' \cup b$. □

For Type 1 **G**-graphs, $A \leftrightarrow B$ makes some statements trivial; in particular, $A - A_1 = B - B_1 = \varnothing$ for a Type 1 c-critical triangle. We employ this simplification in the next section. However, there are still arguments that we can apply to Type 1 and Type 3 triangles simultaneously. For S', T' and $A - A_1$, $B - B_1$, we need a lemma that will partition a pair of independent sets into subsets with identical neighborhoods among these vertices. This will be a simple extension of Lemma 2.

When A and B are independent sets in a **G**-graph and the vertices of A have k distinct neighborhoods in B, we define the *B-partition* of A to be the partition of A into sets $A_1, \ldots, A_k$ such that $N(u) \cap B \supset N(w) \cap B$ for $u \in A_i$ and $w \in A_j$ with $i < j$. We refer to the B-partition of A and the A-partition of B as the *mutual partitions* of $\{A, B\}$.

Lemma 14. *Let A and B be independent sets in a* **G**-*graph, and let* $A_1, \ldots, A_k$ *and* $B_1, \ldots, B_l$ *be the mutual partitions of* $\{A, B\}$. *If A and B are linked, then* $k = l$, $i + j \le k + 1$ *implies* $A_i \leftrightarrow B_j$, *and* $i + j > k + 1$ *implies* $A_i \| B_j$. *If A* [*B*] *has a vertex independent of B* [*A*], *then the same conclusion holds with k replaced by* $k - 1$ [*l replaced by* $l - 1$] (*or both*).

Proof. Index the vertices of $A = \{x_i\}$ and $B = \{y_j\}$ in decreasing order of number of neighbors in the other set. Since these neighborhoods are ordered by inclusion (Lemma 1), the nonzero positions of the resulting adjacency matrix form the Ferrers diagram for a partition of an integer, the integer being the number of edges between A and B. When $\{A, B\}$ is linked, the number of distinct $|N(x_i) \cap B|$ (row sizes) and the number of distinct $|N(y_j) \cap A|$ (column sizes) is equal; it is the number of "corner dots" (end of a row and a column) in the partition. The dots of the partition encode the adjacencies, which yields the statement about the adjacency of A_i and B_j.

If A has a vertex independent of B, then $A_k = A \cap \bar{N}(B)$. Now $A - A_k$ and B are linked independent sets, and we can apply the previous result for $k - 1$ and l. If also B has a vertex independence of A, then $B_l = B \cap \bar{N}(A)$, and we can delete B_l and apply the main result for $k - 1$ and $l - 1$. □

This lemma suggests several partitions, again applicable through Section 8. Let $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ be the mutual partitions of $\{A, B\}$; note that A_1 and B_1 are the same as previously defined. Since S', T' are independent (Theorem 6.7), and also $S' \| C_2 \cup b$ and $T' \| C_1 \cup a$, the sets $S^* = S' \cup C_2 \cup b$ and $T^* = T' \cup C_1 \cup a$ are independent. Let $S'_1, \ldots, S'_l$ and $T'_1, \ldots, T'_l$ be the mutual partitions of $\{S^*, T^*\}$; they have the same number of parts because $C_2 \| T^*$ and $C_1 \| S^*$. Also $a \leftrightarrow (S^* - C_2)$ implies $S'_l = C_2$, and examination of $N(a)$ shows $T'_1 = \langle a \rangle$. Similarly $T'_l = C_1$ and $S'_1 = \langle b \rangle$. A sketch of the structure we have developed, indicating edges guaranteed but not those forbidden or undecided, appears in Fig. 3.

Using these vertex subsets, we can describe most of the edges and non-edges of Type 1 and Type 3 **G**-graphs in a block adjacency matrix. The information we have

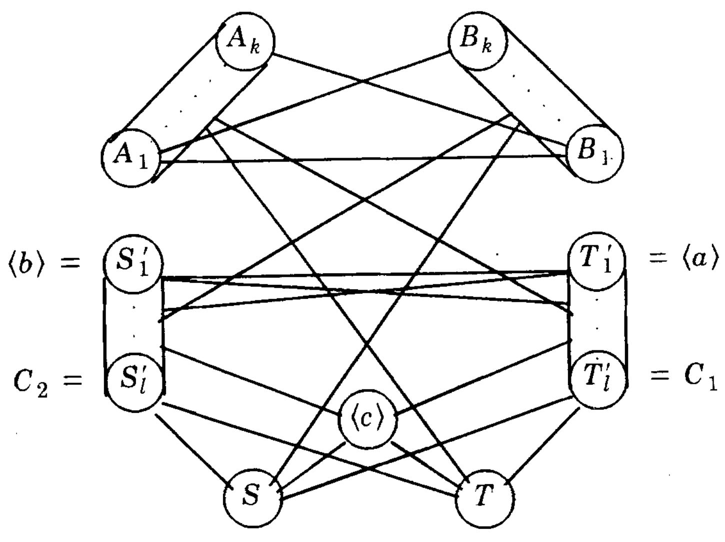

Fig. 3. Sketch of Type 1 and Type 3 **G**-graphs

determined is recorded in Figures 4 and 5. We complete this section by characterizing the remaining conditions for these graphs to be $2K_2$-free. The third condition is vacuous for Type 1 **G**-graphs.

Lemma 15. *A graph with block adjacencies as in Figure 4 or Figure 5 is $2K_2$-free if and only if the following all hold.*

1. *The subgraphs induced by $S \cup T$, $S \cup S'$, and $T \cup T'$ are $2K_2$-free.*
2. *$T \subset N(u) \cup N(u')$ when uu' is an edge of S; similarly for edges in T.*
3. *Suppose $u \| w$ with $u \in S$ and $w \in T$. If $u \leftrightarrow A_j$ is false for some $j > 1$, then $w \leftrightarrow (B_{k+2-j} \cup \cdots \cup B_k)$. Similarly if $w \leftrightarrow B_j$ is false.*

Proof. The subgraph induced by $V(G) - S - T$ is $2K_2$-free; we need only consider edges corresponding to question marks in Figs. 4 and 5. The case of four vertices in S, S', T, T' is handled by (1). (2) is forced by $b \leftrightarrow T$ for edges of S and by $a \leftrightarrow S$ for edges of T; no other $2K_2$ can use an edge of S or T. For (3), consider the nonadjacencies guaranteed in the mutual partitions of $\{A, B\}$; $w \leftrightarrow A_j$ & $u \leftrightarrow (B_{k+2-j} \cup \cdots \cup B_k)$ force $w \leftrightarrow (B_{k+2-j} \cup \cdots \cup B_k)$. (3) also eliminates the possibility of a $2K_2$ containing only one vertex of $S \cup T$. □

As yet we know little about S and T; in particular, we do not know whether these are independent sets. In the next two sections, we will characterize all Type 1 and Type 3 **G**-graphs in which S and T are independent sets. Unfortunately, there do exist **G**-graphs of both Types in which S and T are not independent sets; we will construct arbitrarily large examples. Fortunately, such graphs have a smaller ratio of v/r, and none yet discovered violates Conjecture 1 or 2.

Before embarking on the separate study of Type 1 and Type 3 graphs, we prove a lemma that is applicable to appropriate vertex subsets of arbitrary **G**-graphs. It will be applied to **G**-graphs of each Type, but these applications require considerable structural knowledge and therefore come late in the subsequent sections. Hence we have placed this lemma here instead of earlier; it can be contrasted with Lemma 14.

Lemma 16. *Suppose that I and K are disjoint vertex subsets of a **G**-graph and there is a vertex $a \notin I \cup K$ such that $a \leftrightarrow I$ and $a \| K$. If I is independent, no vertex of K is adjacent to all of I, and all vertices of K have equal degree in $I \cup K$, then I and K have partitions into $I_1, \ldots, I_m$ and $K_1, \ldots, K_m$ such that each K_i is independent, $I_i \| K_i$,* and *$K_i \leftrightarrow I_j \cup K_j$ if $i \neq j$. If vertices of I also have equal degree in $I \cup K$, then $|I_i|$ and $|K_i|$ are constant over i.*

Proof. We show first that vertices of I with common non-neighbors in K have identical neighbors in K. For $u, u' \in I$, select $w \in \bar{N}(uu') \cap K$. If $N(u) \cap K \neq N(u') \cap K$, then we may select $x \in K \cap N(u' | u)$ (by symmetry). Now $a \leftrightarrow u$ forces $w \| x$, and then $N(w) \cap I \subset N(x) \cap I$ (Lemma 1). Since x, w have equal-sized neighborhoods in $I \cup K$, we may select $y \in K \cap N(w | x)$. Now $y \leftrightarrow w$ & $a \leftrightarrow I$ forces $y \leftrightarrow \bar{N}(w) \cap I$. By Lemma 1, this implies $x \leftrightarrow I$ or $y \leftrightarrow I$, which contradicts the hypothesis.

Now partition I into maximal sets $I_1, \ldots, I_m$ with identical neighborhoods in K. By the preceding paragraph, the sets $K_i = K \cap \bar{N}(I_i)$ are disjoint. They also exhaust

K, since no vertex of K is adjacent to all of I. Finally, $a \leftrightarrow I_i$ forces K_i independent, and $K_i \leftrightarrow I_j$ & $I_i \leftrightarrow K_j$ forces $K_i \leftrightarrow K_j$. □

7. Bounds and Partial Characterization for Type 1 G-graphs

For Type 1 **G**-graphs, $A = A_1$ and $B = B_1$. This means we have determined all edges in these graphs except for the edges involving S and T. For easy reference, this information appears in Fig. 4, with question marks where we do not know all the edges. By Lemma 15, the problem of constructing Type 1 **G**-graphs reduces to that of inserting edges involving $S \cup T$ so as to satisfy Lemma 15 and maintain regularity by appropriate choices for the sizes of the other sets. The simplest choice is S, T independent; we characterize the resulting graphs in Theorem 8. More complicated choices are considered subsequently. Meanwhile, in Theorem 7 we derive some constraints on the sizes of the blocks. The first result verifies the weaker part of Conjecture 1 for Type 1 **G**-graphs; r is even.

	S	$C_2 \leftarrow S^* \rightarrow \langle b \rangle$ $S'_l \cdots S'_1$	A	$\langle c \rangle$	B	$\langle a \rangle \leftarrow T^* \rightarrow C_1$ $T'_1 \cdots T'_l$	T
S	?	1 ? 0	0	1	1	1	?
$\uparrow C_2 = S'_l$	1					0 0 0	
S^*	?	0	0	1	1	1 0 0	1
$\downarrow \langle b \rangle = S'_1$	0					1 1 0	
A	0	0	0	0	1	1	1
$\langle c \rangle$	1	1	0	0	0	1	1
B	1	1	1	0	0	0	0
$\uparrow \langle a \rangle = T'_1$		0 1 1					0
T^*	1	0 0 1	1	1	0	0	?
$\downarrow C_1 = T'_l$		0 0 0					1
T	?	1	1	1	0	0 ? 1	?

Fig. 4. Block adjacency matrix for Type 1 **G**-graphs

Theorem 7. *If abc is a c-critical Type* 1 *triangle, then r is even. In addition,*
1. $r/2 = |A| = |B| = |T^*| + |T| = |S^*| + |S|$.
2. $|S| = |T^*| - q = |T'| + 1$ and $|T| = |S^*| - q = |S'| + 1$.
3. $|S| + |T| = r/2 - q$, $|S^*| + |T^*| = r/2 + q$, $|S'| + |T'| = r/2 - 2 - q$.
4. *S, S' T, T', are all nonempty.*
5. *For $u \in S^*$, $|N(u) \cap (S \cup T^*)| = |T^*| - q$. Similarly for $w \in T^*$.*
6. *For $u \in S'_j$, $|N(u) \cap S| = \sum_{i=1}^{j=1} |T'_{l-i}|$. Similarly for $w \in T'_j$.*
7. *$S \| S'$ iff $l = 2$ iff $T \| T'$.*
8. $e(S \cup T) = (\sum_{i+j \le l} |S'_j \| T'_j|) - q(r/2 - q)$.

Proof. Recall $|\langle c \rangle| = |C_1| = |C_2| = q$ and compare (block) neighborhoods as listed in Fig. 4. (1): $N(C_1)$ vs. $N(C_2)$ yields $|A| = |B|$, and $\bar{N}(c) = \langle c \rangle \cup A \cup B$ yields

$|A| + |B| = r$; hence $|A| = |B| = r/2$. $N(c)$ vs. $N(B)$ yields $|A| = |T^*| + |T|$, and $N(c)$ vs. $N(A)$ yields $|B| = |S^*| + |S|$. (2): $N(C_2)$ vs. $N(b)$ and $B(C_1)$ vs. $N(a)$. (3): Apply $r = d(c) = |S| + |S^*| + |T^*| + |T|$ and (2). (4): First S and T are nonempty by (2), then $|S| \geq 2q$ and $|T| \geq 2q$ by Theorem 6.5, and finally S' and T' are nonempty by (2). (5, 6): $N(u)$ vs. $N(b)$. (7): $N(S'_1)$ vs. $N(S'_2)$.

(8): Let $U = S \cup T$; $e(U) = \frac{1}{2}[r(|S| + |T|) - e(U, \bar{U})]$. By (1) and Figure 4, $e(U, S^* \cup T^*) = |S^*|(|T| + |T^*| - q) + |T^*|(|S| + |S^*| - q) - 2\sum_{i+j \leq l}|S'_i||T'_j|$, and $e(U, A \cup \langle c \rangle \cup B) = (r/2 + q)(|S| + |T|)$. Using (2), the computation simplifies to the formula claimed. □

Letting S, S', T, T' be as small as possible yields our first Type 1 **G**-graph. This graph J is a member of several classes, one of which we present immediately.

Example 4. Let J be a graph having block adjacency matrix as in Fig. 4, with the parameter l set to 2, all unknown entries set to 0, and block sizes as follows: set $|A| = |B| = 5$, $|\langle c \rangle| = |C_1| = |C_2| = 1$, and $|S| = |T| = |\langle a \rangle| = |\langle b \rangle| = 2$. In fact, $q = 1$ and $r = 10$ imply these block sizes (Theorem 7). By inspection (Lemma 15), J is a $\mathbf{G}_{10}$-graph with 21 vertices. The triangles using one vertex each of $\langle a \rangle$, $\langle b \rangle$, $\langle c \rangle$ are critical Type 1 triangles.

More generally, define J_l for each value of $l \geq 2$ in Figure 4. To complete the block adjacency matrix, let $\{S, T\}$ be independent sets, and let $S_1, \ldots, S_{l-1}$ and $T_1, \ldots, T_{l-1}$ be a mutual partition of $\{S, T\}$ with $S_i \leftrightarrow T_j$ if $i + j < l$ and $S_i \| T_j$ if $i + j \geq l$. The remaining adjacencies are $S_i \leftrightarrow S'_j$, $T_i \leftrightarrow T'_j$ if $i + j > l$ and $S_i \| S'_j$, $T_i \| T'_j$ if $i + j \leq l$, in accordance with Theorem 7.6. Maintaining regularity requires satisfying the constraints of Theorem 7. Set $|A| = |B| = 4l - 3$, $|\langle c \rangle| = |C_1| = |C_2| = 1$, and $|S_i| = |T_i| = |S'_i| = |T'_i| = 2$ for $1 \leq i < l$. The graph J_l is regular of degree $r = 8l - 6$ and has $16l - 11 = 2r + 1$ vertices. Note that $J_2 = J$ and that setting $l = 1$ yields a 5-cycle, which is a **G**-graph but not Type 1. □

The graphs of Example 4 characterize the Type 1 **G**-graphs for which S and T are independent sets.

Theorem 8. *If abc is a c-critical Type* 1 *triangle and S, T are independent, then G is the q-fold expansion of the graph J_l of Example* 4, *for some $l \geq 2$.*

Proof. When S is independent, Theorem 7.6 implies that $S'_l, \ldots, S'_1$ is the S-partition of S^*. Since $S'_l = C_2 \leftrightarrow S$ and $S'_1 = \langle b \rangle \| S$, the S^*-partition of S is some $S_{l-1}, \ldots, S_1$, with $S_i \leftrightarrow S'_j$ if $i + j > l$ and $S_i \| S'_j$ if $i + j \leq l$. Similarly, $T'_l, \ldots, T'_1$ is the T-partition of T^*, and we obtain the T^*-partition $T_{l-1}, \ldots, T_1$ of T. Given $x \in S_i$ and $y \in S_j$ with $i \leq j$, we have $N(x) - T \subseteq N(y) - T$, with equality if and only if $i = j$. With T independent, this forces $N(x) \cap T \supseteq N(y) \cap T$, with equality if and only if $i = j$. Therefore, $S_1, \ldots, S_{l-1}$ is the T-partition of S; similarly, $T_1, \ldots, T_{l-1}$ is the S-partition of T.

This establishes blocks and block adjacencies as in J_l, and we need only determine the block sizes. $N(T_i)$ vs. $N(T_{i+1})$ and $N(S'_i)$ vs. $N(S'_{i+1})$ yield $|S_1| = |T'_2| = |S_2| = \cdots = |T'_{l-1}| = |S_{l-1}| = s$. Also $N(S_i)$ vs. $N(S_{i+1})$ and $N(T'_i)$vs. $N(T'_{i+1})$ yield $|T_1| = |S'_2| = |T_2| = \cdots = |S'_{l-1}| = |T_{l-1}| = t$. Now $\langle a \rangle$ vs. T_1 (or $\langle b \rangle$ vs. S_1) yields $s = 2q = t$. Finally, $r = d(c) = 2q(4l - 4) + 2q = q(8l - 6)$. Since $|A| = |B| = r/2$, this completes the description of the graph as the q-fold expansion of J_l. □

Next we drop the requirement that both S and T be independent. Our discussion of Type 1 **G**-graphs thus far has been symmetric in A vs. B and S vs. T; the next example departs from this. These graphs will characterize the Type 1 **G**-graphs in which AC is independent.

Example 5. We construct L_m based on the matrix of Fig. 4. Set $l = 2$. To specify the remaining adjacencies, let T be the complete m-partite graph with partite sets $\{T_i\}$ of size 2, and let S be an independent set of size $2m^2$ partitioned into sets $\{S_i\}$ of size $2m$. Put $S_i \| T_i$, but $S_i \leftrightarrow T_j$ for $i \neq j$. Since $l = 2$, $2K_2$ can only occur within $S \cup T$. Since nonadjacent vertices of T have the same neighborhood and S is independent, none occur. It now suffices to specify set sizes for regularity. Let $|\langle c \rangle| = |C_1| = |C_2| = 1$, $|\langle b \rangle| = 2m$, $|\langle a \rangle| = 2m^2$, and $|A| = |B| = 1 + 2m + 2m^2$. Now L_m has $2r + 1$ vertices and is $(2 + 4m + 4m^2)$-regular. Note that L_1 is J of Example 4. □

The condition of the next theorem is equivalent to AC independent (and symmetric to BC independent).

Theorem 9. *If abc is a c-critical Type* 1 *triangle and* $S \| (S \cup S')$, *then G is the q-fold expansion of the graph* L_m *of Example* 5, *for some* $m \geq 1$.

Proof. By Theorem 7.7, $S \| S'$ implies $l = 2$. Hence vertices of T have identical neighborhoods outside $S \cup T$ and identical degrees inside. With S independent and $S \leftrightarrow a \| T$, Lemma 16 reduces the block adjacency matrix to that of L_m. Now consider the set sizes. When $S \| S'$, we have $d(ua|b) = 0$ (Fig. 4), and then $|T_i| = 2q$ (Theorem 6.5). Now $N(a)$ vs. $N(T_i)$ yields $|S_i| = |C| + |T - T_i| = 2mq$, also $N(b)$ vs. $N(C_2)$ yields $|\langle a \rangle| = |S| = 2m^2 q$, and $N(a)$ vs. $N(C_1)$ yields $|\langle b \rangle| = |T| = 2mq$. Finally, $r = |N(c)| = 2q + 4qm + 4qm^2$, so $|A| = |B| = r/2 = (1 + 2m + 2m^2)q$, and G is the q-fold expansion of L_m. □

There are also Type 1 **G**-graphs in which neither S nor T is independent. The following example introduces another operation for building large **G**-graphs from smaller ones.

Example 6. Let H be an s-regular **G**-graph with $2s + 4q$ vertices. We construct a collection $f(H)$ of Type 1 $\mathbf{G}_r$-graphs with $r = 4s + 10q$ and $v = 2r + q$. Let $l = 2$, and let the subgraph induced by $S \cup T$ be H. Allocate the vertices of H equally to S and T in a way that satisfies the condition of Lemma 15.2. Since $l = 2$ implies $S \| S'$ and $T \| T'$, Lemma 15 says any resulting graph is $2K_2$-free. Since $l = 2$, we need only specify block sizes to satisfy regularity. Let $|C_1| = |C_2| = |\langle c \rangle| = q$, $|A| = |B| = 2s + 5q$, and $|\langle a \rangle| = |\langle b \rangle| = s + 2q$. Counting vertex neighborhoods confirms that each such graph is $4s + 10q$-regular and has $8s + 21q$ vertices.

For the degenerate case $H = 4K_1$, we have $s = 0$ and $q = 1$, and the resulting graph $f(H)$ is J of Example 4. Suppose H is a Type 1 $\mathbf{G}_s$-graph with critical triangle $a_H b_H c_H$; we have specified $|\langle c_H \rangle| = 4q$. Examination of Figure 4 shows that the requirement of Lemma 15.2 is satisfied by placing $A_H \cup S_H^* \cup S_H$ in S, $B_H \cup T_H^* \cup T_H$ in T, and splitting $\langle c_H \rangle$ equally between S and T. As another example, consider the graphs G_k of Example 1, which are Type 3 **G**-graphs except for G_1, the 5-cycle. If H is a $4q$-fold expansion of G_k, place the images of $1, \ldots, 2k$ in S, of $2k + 1, \ldots, 4k$ in T, and split the images of 0 equally between S and T. □

Note that f "preserves" both Conjecture 1 and Conjecture 2. For Conjecture 1, if $s/(4q)$ is an even integer, then so is $(4s + 10q)/q$. If the graph H used by f satisfies Conjecture 2, then it is a $4q$-fold expansion of a $\mathbf{G}_{s/(4q)}$-graph on $2s/(4q) + 1$ vertices. Let H' be the (s/q)-regular 4-fold expansion of this graph. Since $s + 2q = q(s/q + 2)$ and $2s + 5q = q(2s/q + 5)$, any graph in $f(H)$ is the q-fold expansion of the corresponding graph in $f(H')$, which is $(4s/q + 10)$-regular and has $8s/q + 21$ vertices. Finally, note that any application of f yields a graph with $v/r \geq \frac{21}{10}$, with equality only when $s = 0$ and $q = 1$, which is the degenerate case yielding J.

The requirement of Lemma 15.2 is quite restrictive. We do not know whether $f(H)$ is nonempty when H is an arbitrary **G**-graph of Type 2 or 3. If $H = G_k^{4q}$, then $f(H)$ contains only one graph. More generally, suppose H is a $4q$-fold expansion of a $\mathbf{G}_t$-graph H' on $2t + 1$ vertices, with $t = s/4q$ (i.e., suppose H satisfies Conjecture 2). Call the independent set expanded from each vertex of H' a "clump". We claim that the set of clumps that are "split" by having vertices in both S and T in forming $f(H)$ form an independent set in H' with identical neighborhoods (the only such sets in G_k are single vertices). If clumps corresponding to two adjacent vertices are split, then there are edges between them in S and in T. Hence these two clumps together have edges to members of all $2t + 1$ clumps, violating t-regularity. With the split clumps forming an independent set U, any edge from a clump in U to another clump X yields an edge in S or in T between U and X. This member of X must be adjacent to members of all clumps in U on the other side. Hence $X \leftrightarrow U$.

Without assuming Conjecture 2, we must leave $f(H)$ as described in Example 6. Nevertheless, like the previous constructions, the operation f characterizes a class of **G**-graphs.

Theorem 10. *If abc is a c-critical Type* 1 *triangle, $S \cup T$ induces a regular subgraph H, and $S \| S'$, then H is a* **G***-graph and $G \in f(H)$, where $f(H)$ is defined as in Example* 6.

Proof. By Theorem 7.7, $S \| S'$ implies $l = 2$. Since G is $2K_2$-free, the subgraph H must also be $2K_2$-free, and the adjacencies must be as described in Example 6. For $G \in f(H)$, we need only show that the set sizes must be as in Example 6. Suppose H is s-regular and has n vertices. Since $l = 2$, we have $|\langle a \rangle| = |S| = n_1$ and $|\langle b \rangle| = |T| = n_2$. As usual, $|C_1| = |C_2| = |\langle c \rangle| = q$ and $|A| = |B| = r/2$. Hence $d(u) - d(w) = |\langle a \rangle| - |\langle b \rangle|$ if $u \in S$ and $w \in T$, which implies $n_1 = n_2 = n/2$. Futhermore, $d(u) = r = s + 3q + r/2 + n/2$ and $d(c) = r = 2n + 2q$. Solving for n and r yields $n = 2s + 4q$ and $r = 4s + 10q$. Hence H is a **G**-graph and $G \in f(H)$. □

We do not know a common generalization of J_l and L_m, nor can we strengthen Theorem 9 to characterize all Type 1 **G**-graphs with at least one of S, T independent. However, we have characterized all Type 1 **G**-graphs with large vertex/degree ratio. This yields a partial proof of Conjectures 1 and 2 for Type 1 **G**-graphs, since J_l and L_m have $q = 1$.

Theorem 11. *If G is a Type 1* **G***-graph and is not an expansion of any J_l or L_m,* then *$v/r < 29/14$.*

Proof. By Theorem 8, we may assume ww' is an edge in T. Now $S \leftrightarrow a$ & $w \leftrightarrow w'$ *force* $S \subset N(w) \cup N(w')$. With Fig. 4 ($|A| = r/2$ by Theorem 7.1), this yields $d(w) + d(w') \geq |S| + 2|S^*| + r + 4q$. By Theorem 9, avoiding L_m requires an edge uu' from S to $S \cup S'$. If $u, u' \in S$, then as above we get $d(u) + d(u') \geq |T| + 2|T^*| + r + 4q$. Summing these and using Theorem 7 yields $4r \geq 7r/2 + 9q$, meaning $v \leq (2 + 1/18)r$. If $u \in S$ and $u' \in S'$, then $l > 2$, and $T \leftrightarrow b$ & $u \leftrightarrow u'$ force $T \subset N(u) \cup N(u')$. This time $d(u) + d(u') > 2q + r + 2|T^*| + |T|$. Now the sum is $4r > 7r/2 + 7q$, meaning $v < (2 + 1/14)r$. □

8. Bounds and Partial Characterization for Type 3 G-Graphs

For Type 3 **G**-graphs with c-critical abc, we conduct a similar analysis. As obtained in Section 6, we have mutual partitions $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ of $\{A, B\}$ and $S_1', \ldots, S_l'$ and $T_1', \ldots, T_l'$ of $\{S^*, T^*\}$. To have link edges for $\{A, C\}$ and $\{B, C\}$, we must have $k \geq 2$ (Lemma 7.4). The resulting block adjacency matrix replacing Fig. 4 appears in Fig. 5.

	S	$C_2 \leftarrow S^* \rightarrow \langle b \rangle$ $S_l' \cdots S_1'$	$A_1, \ldots, A_k$	$\langle c \rangle$	$B_k, \ldots, B_1$	$\langle a \rangle \leftarrow T^* \rightarrow C_1$ $T_1' \cdots T_l'$	T
S	?	1 ? 0	0 ? ?	1	1	1	?
$C_2 = S_l'$	1		0 1 1			0 0 0	
⋮	?	0	0 ? ?	1	1	1 0 0	1
$\langle b \rangle = S_1'$	0		0 0 0			1 1 0	
A_1	0	0 0 0			1 1 1		
⋮	?	1 ? 0	0	0	0 1 1	1	1
A_k	?	1 ? 0			0 0 1		
$\langle c \rangle$	1	1	0	0	0	1	1
B_k			1 0 0			0 ? 1	?
⋮	1	1	1 1 0	0	0	0 ? 1	?
B_1			1 1 1			0 0 0	0
$\langle a \rangle = T_1'$		0 1 1			0 0 0		0
⋮	1	0 0 1	1	1	? ? 0	0	?
$C_1 = T_l'$		0 0 0			1 1 0		1
T	?	1	1	1	? ? 0	0 ? 1	?

Fig. 5. Block adjacency matrix for Type 3 **G**-graphs

The additional flexibility resulting from $k > 1$ makes Type 3 **G**-graphs considerably harder to characterize. In addition to the extra question marks in the matrix, it is no longer true that S and T must be nonempty. However, it is easy to characterize the Type 3 **G**-graphs with $S = \varnothing$ (which happens only if $T = \varnothing$ also).

Theorem 12. *If abc is a c-critical Type 3 triangle with $S = \varnothing$ and the mutual partitions of $\{A, B\}$ have k parts, then G is the q-fold expansion of the 2k-regular $4k + 1$-vertex* **G***-graph H_k of Example* 1.

Proof. With $S = \varnothing$, the requirements of regularity and the adjacencies recorded in Fig. 5 imply that $A_k, \ldots, A_1$ is the S^*-partition of A and $S'_l, \ldots, S'_1$ is the A-partition of S^*. Since $A_1 \| S^*$ and $\langle b \rangle = S_1 \| A$, Lemma 14 yields $k = l$, with $A_i \| S'_j$ when $i + j \leq k + 1$ and $A_i \leftrightarrow S'_j$ when $i + j > k + 1$. By Theorem 6.5, $S = \varnothing$ if and only if $T = \varnothing$. Hence we similarly have $B_k, \ldots, B_1$ and $T'_l, \ldots, T'_1$ as mutual partitions of B and T^*, with $k = l$, $B_i \| T'_j$ when $i + j \leq k + 1$, and $B_i \leftrightarrow T'_j$ when $i + j > k + 1$.

Under the cyclic ordering $S'_k, \ldots, S'_1, A_1, \ldots, A_k, \langle c \rangle, B_k, \ldots, B_1, T'_1, \ldots, T'_k$ (see Fig. 3), G now has the same block adjacency matrix as G_k of Example 1. Regularity forces the blocks to be the same size; $|\langle c \rangle| = q$ implies $G = G_k^q$. □

When S and T are nonempty, more complicated graphs are possible. The proofs and results here are analogous to but more complicated than those in Section 7. It is possible to combine some of this with the results of Section 7, but we feel that the exposition is much clearer when the simpler setting of Type 1 **G**-graphs is considered first. For Type 3 **G**-graphs, we have not shown that r is even, and the results about block sizes have an additional variable $p = |A_1| = |B_1|$. In Example 8 we will see Type 3 **G**-graphs with $|A| \neq |B|$, meaning that $|A| = r/2$ cannot be proved. Note also the absence of the conclusion that S, T must be nonempty.

Theorem 13. *If abc is a c-critical Type 3 triangle, then*
1. $|A_1| = |B_1| = p$ *and* $|A| + |B| = r$. *Also* $|A| = |T^*| + |T|$ *and* $|B| = |S^*| + |S|$.
2. $|S| + |A| - p = |T^*| - q = |T'| + 1$ *and* $|T| + |B| - p = |S^*| - q = |S'| + 1$.
3. $|S| + |T| = p - q$ *and* $|S^*| + |T^*| = (r - p) + q$.
4. *For* $u \in S^*$, $|N(u) \cap (S \cup T^* \cup A - A_1)| = |T^*| - q$. *Similarly for* $w \in T^*$.
5. *For* $u \in S'_j$, $|N(u) \cap (S \cup A - A_1)| = \sum_{i=1}^{j-1} |T'_{l-i}|$. *Similarly for* $w \in T'_j$.
6. $S' \| (S \cup A)$ *iff* $l = 2$ *iff* $T' \| (T \cup B)$.
7. *If S or T is nonempty, then* $S \leftrightarrow (A - A_1)$ *and* $T \leftrightarrow (B - B_1)$ *cannot both hold.*

Proof. (1–6): Same neighborhood comparisons as for Theorem 7. (7): By Theorem 6.5, we may assume both S and T are nonempty and choose a vertex in each of S, T, A_k, B_k. If the claim is false, then every vertex has at least two neighbors among these four (see Fig. 5), yielding the contradiction $4r \geq 2v$. □

In light of Theorem 12, we may assume that S and T are nonempty. Our first such Type 3 **G**-graphs can be viewed as another generalization of the ubiquitous graph J of Example 4.

Example 7. In Fig. 5, let $l = 2$; we define a graph M_k for any $k \geq 2$. Let S and T be independent sets with mutual partitions $S_1, \ldots, S_k$ and $T_1, \ldots, T_k$ satisfying $T_k \| S_k$, so that $S_i \leftrightarrow T_j$ when $i + j \leq k$ and $S_i \| T_j$ when $i + j > k$. For the remaining question marks in Fig. 5, put $A_i \leftrightarrow S_j$ and $B_i \leftrightarrow T_j$ if $i + j > k + 1$, and put $A_i \| S_j$ and $B_i \| T_j$ if $i + j \leq k + 1$. To define the set sizes, set $|C_1| = |C_2| = |\langle c \rangle| = 1$, $|\langle a \rangle| = |\langle b \rangle| = 6k - 4$, $|A_1| = |B_1| = 6k - 1$, and $|S_k| = |T_k| = 2$, and let the remaining $4(k - 1)$ unspecified sizes for S_i, A_i, T_i, B_i be 3.

Mutual partitions avoid $2K_2$ in the subgraph induced by two independent sets. To verify Lemma 15.3, suppose $u \in S_i$, $w \in T_j$, $x \in A_s$, $y \in B_t$ with $u \| w$ and $x \| y$. Then $i + j \geq k + 1$ and $s + t > k + 1$. This implies $i + s > k + 1$ or $j + t > k + 1$, which means $u \leftrightarrow x$ or $w \leftrightarrow y$, and $2K_2$ is avoided. Finally, summing the set sizes for neighbors of each block confirms that M_k is an $18k - 8$-regular **G**-graph with $36k - 15$ vertices. Setting $k = 1$ collapses M_k to J (abc is no longer Type 3).

There is another family closely related to M_k, which we call M'_k. Again set $l = 2$, but this time let the S, T-partitions be $S_1, \ldots, S_{k-2}$ and $T_1, \ldots, T_{k-2}$ with $T_{k-2} \| S_{k-2}$. Put $A_i \leftrightarrow S_j$ and $B_i \leftrightarrow T_j$ if $i + j > k$, otherwise $A_i \| S_j$ and $B_i \| T_j$. In particular, note that $A_2 \| S$. This is made to work by setting $|B_k| = |C_2|$ and $|A_k| = |C_1|$, all of which equal 1 along with $|\langle c \rangle|$. Also set $|\langle a \rangle| = |\langle b \rangle| = |A_1| = |B_1| = 6k - 11$, and let the remaining $4(k - 2)$ unspecified sizes for S_i, A_i, T_i, B_i be 3. The resulting graph M'_k is an $18k - 32$-regular **G**-graph with $36k - 63$ vertices. When $k = 2$, S and T vanish and M'_k degenerates to the graph G_2 of Example 1, with vertex/degree ratio 9/4. For $k > 2$, if we use the block ordering $C_2, S_{k-2}, \ldots, S_1, \langle b \rangle, A_1, \ldots, A_k, \langle c \rangle, B_k, \ldots, B_1, \langle a \rangle, T_1, \ldots, T_{k-2}, C_1$, then the block adjacency matrix of M'_k is the same as that of G_k, except for additional "block" adjacencies $C_2 \leftrightarrow S \leftrightarrow C_1 \leftrightarrow T \leftrightarrow C_2$. □

Not surprisingly, these characterize the graphs satisfying appropriate conditions; the proof is similar to that of Theorem 8. When comparing neighborhoods, we henceforth adopt the stereotypic "U vs. W" in place of "$N(U)$ vs. $N(W)$".

Theorem 14. *If abc is a c-critical Type 3 triangle, S, T are independent sets, and $S' \| (S \cup A)$ (or $T' \| (T \cup B)$), then G is the q-fold expansion of M_k or M'_k, for some $k \geq 2$.*

Proof. By Theorem 13.6, $S' \| (S \cup A)$ and $l = 2$ and $T' \| (T \cup B)$. Let $S_1, \ldots, S_h$ and $T_1, \ldots, T_h$ be the mutual partitions of S, T; note that $S_h \| T$ and $T_h \| S$ (Theorem 6.5). We need only determine the S, A and T, B adjacencies. This implies that $S_h, \ldots, S_1$ is the A-partition of S and $T_h, \ldots, T_1$ is the B-partition of T. The A, B adjacencies force $A_k, \ldots, A_2$ to be the S-partition of $A - A_1$ and $B_k, \ldots, B_2$ to be the T-partition of $B - B_1$. We have $A_k \leftrightarrow S$ or $S_1 \| A$, and $B_k \leftrightarrow T$ or $T_1 \| A$. Note that A_k vs. B_k implies $A_k \leftrightarrow S$ if and only if $B_k \leftrightarrow T$. Also, $A_2 \| S$ if $|B_k| = |C_2| = q$, in which case the S-partition of A is $A_k, \ldots, A_3, (A_2 \cup A_1)$. Since this reduces h, we have $|B_k| = q$ if and only if $|A_k| = q$. Hence the completion of the block adjacency matrix depends on whether $A_k \leftrightarrow S$ and on whether $|B_k| = q$. If ε of these two things happens, then $h = k - \varepsilon$.

If $h = k$, then we have the block adjacency matrix of M_k. Now T_{k-i} vs. T_{k-i-1} and A_i vs. A_{i+1} successively yield $|S_1| = |B_2| = |S_2| = \cdots = |B_k| = |S_k| + q$, and $\langle a \rangle$ vs. T_1 yields $|S_k| = 2q$. We similarly obtain the corresponding sizes for $\{T_i, A_i\}$. Also, C_2 vs. $\langle b \rangle$ yields $|\langle a \rangle| = (6k - 4)q$, and similarly $|\langle b \rangle| = (6k - 4)q$. Now A_k vs. $\langle c \rangle$ yields $|B_1| = |\langle b \rangle| + |S_1| = (6k - 1)q$, and similarly $|A_1| = (6k - 1)q$. Hence G is the q-fold expansion of M_k.

If $h = k - 2$, then $A_k \leftrightarrow S$ and $|B_k| = q$, and we have the block adjacency matrix of M'_k. Now A_i vs. A_{i+1} and T_{k-2-i} vs. T_{k-3-i} successively yield $|B_2| = |S_1| = |B_3| = |S_2| = \cdots = |B_{k-1}| = |S_{k-2}| = |B_k| + 2q = 3q$, and similarly for sizes in T, A because we also have $|A_k| = q$. Next C_2 vs. $\langle b \rangle$ yields $|\langle a \rangle| = (6k - 11)q$, and similarly $|\langle b \rangle| = (6k - 11)q$. Finally A_k vs. $\langle c \rangle$ yields $|B_1| = |\langle b \rangle|$, and similarly $|A_1| = |\langle a \rangle|$. Hence G is the q-fold expansion of M'_k.

If $h = k - 1$, we have two cases to consider. First suppose $A_k \leftrightarrow S$ and $|B_k| > q$.

Now T_{k-i-1} vs. T_{k-i} yields $|S_i| = |B_{i+1}|$ for $i = 1, \ldots, k-2$, and B_1 vs. T_1 yields $|S_{k-1}| = |B_k| + 2q$. Hence $|S| = |B| - p + 2q$, which implies $|S^*| = p - 2q$ (Theorem 13.1). However, A_k vs. $\langle c \rangle$ implies $|\langle b \rangle| = p$ and hence $|S^*| = q + p$.

Finally, suppose $S_1 \| A$ (and $T_1 \| B$) but $|B_k| = |A_k| = q$. Now $\langle a \rangle$ vs. T_1 yields $|S_{k-1}| = |C_1| + |C_2| = 2q$. Also, A_i vs. A_{i+1} and T_{k-1-i} vs. T_{k-2-i} successively yield $|B_2| = |S_1| = |B_3| = |S_2| = \cdots = |B_{k-1}| = |S_{k-2}| = |B_k| = q$. Hence $|S| = kq$ and $|B| = p + (k-1)q$. Furthermore, A_k vs. $\langle c \rangle$ implies $|S_1| + |\langle b \rangle| = p$, or $|\langle b \rangle| = p - q$. This implies $|S^*| = p$, which contradicts Theorem 13.1. □

Next we allow T to have edges but keep S independent. The examples that result are our first c-critical **G**-graphs with $|A| \neq |B|$.

Example 8. In the structure of the adjacency matrix in Fig. 5, let $l = 2$ and $k = 2$. For the remaining question marks, put $S \| A_2$ but $T \leftrightarrow B_2$. Let S and T each consist of m blocks of vertices with identical neighborhoods, such that S is independent and $T_i \| (S_i \cup T_i)$, but $T_j \leftrightarrow (S_i \cup T_i)$ if $j \neq i$. To complete specification of the resulting graph P_m, put $|\langle c \rangle| = |B_2| = |C_1| = |C_2| = 1$, $|T_i| = 2$, $|S_i| = |\langle b \rangle| = |A_2| = 2m + 1$, and $|A_1| = |B_1| = |\langle a \rangle| = 2m^2 + 2m + 1$. By Lemma 15.3, P_m is $2K_2$-free, and counting the neighborhoods in each class shows that it is $4m^2 + 8m + 4$-regular with $8m^2 + 16m + 9$ vertices. Setting $m = 0$ collapses this to the graph G_2 of Example 1. □

The proof of the corresponding characterization is similar to that of Theorem 9.

Theorem 15. *If abc is a c-critical Type* 3 *triangle with* $(S \cup S') \| (S \cup A))$, *then G is the q-fold expansion of* P_m, *for some* $m \geq 1$.

Proof. In the structure of Fig. 5, we have $l = 2$ and $T' \| (T \cup B)$ (Theorem 13.6). By Lemma 15.3, $S \| A$ implies $T \leftrightarrow (B - B_1)$. Now $B - B_1$ has constant neighborhood outside A, forcing $k = 2$. Now consider the subgraph induced by $S \cup T$. Since $N(w) \cap (B \cup T^*) = B_2 \cup C_1$ for all $w \in T$, the degree of w in $S \cup T$ is constant. By Lemma 16, we can partitioned S and T into equivalence classes $S_1, \ldots, S_m$ and $T_1, \ldots, T_m$ such that $\bar{N}(S_i) \cap T = T_i = \bar{N}(T_i) \cap T$.

Hence G has the block adjacency matrix of P_m, and it remains to determine the set sizes. With $S \| (S \cup S' \cup A)$, Theorem 6.5 says $|T_i| = 2q$. Now A_1 vs. A_2 yields $|B_2| = |C_2| = q$ and B_1 vs. B_2 yields $|A_2| = |C_1| + |T| = (2m+1)q$. Also $\langle a \rangle$ vs. C_1 yields $|\langle b \rangle| = |B_2| + |T| = (2m+1)q$ and C_1 vs. T_i yields $|S_i| = |\langle b \rangle|$. $\langle b \rangle$ vs. C_2 yields $|\langle a \rangle| = |S| + |A_2| = (2m^2 + 3m + 1)q$. Finally, Theorem 13.1 yields $|A_1| = |B_1| = (2m^2 + 3m + 1)q$ and $r = (4m^2 + 8m + 4)q$. This expresses G as the q-fold expansion of P_m. □

We do not have a common generalization of M_k and P_m nor a way to eliminate the extra independence hypotheses in these theorems. Nevertheless, we can prove there are no Type 3 **G**-graphs with $v/r > 33/16$ besides G_k. The next theorem completes our partial proof of Conjectures 1 and 2 for Type 3 **G**-graphs. Although it is easy to show $v/r \leq 37/18$ when S, T each has an edge (count the neighborhoods of the four end-points and use Lemma 15.2 and Theorem 13), handling the cases where S or T is independent requires a more subtle argument that also covers the non-independent case.

Theorem 16. *If abc is a c-critical Type 3 triangle for which S or T is nonempty, then* $v \leq 33/16r$, *with equality only for* P_1.

Proof. We take a weighted sum of eleven vertex neighborhoods. By Theorem 13.7 and symmetry, we may assume there exist $u \in S$ and $x \in (A - A_1)$ with $u \| x$. Theorem 6.5 guarantees a $w \in T$ with $u \| w$, and then $w \leftrightarrow B_k$ (Lemma 15.3). Use u, w, and one vertex from each of C_2, $\langle b \rangle$, A_1, A_2, $\langle c \rangle$, B_k, B_1, $\langle a \rangle$, C_1 and weight their neighborhoods as indicated in Table 1. By Lemma 15.3, each vertex of T is adjacent to u or B_k. Now every vertex is counted at least 16 times in the 33 neighborhoods, so $33r \geq 16v$.

Table 1. Neighborhood counting for Type 3 G-graphs

Nbhd weight	3	1	3	6	3	1	1	6	6	1	2
Vert locat	$u \in S$	C_2	$\langle b \rangle$	A_1	A_2	$\langle c \rangle$	B_k	B_1	$\langle a \rangle$	C_1	$w \in T$
S	?	1	0	0	0	1	1	6	6	1	?
C_2	3	0	0	0	3	1	1	6	0	0	2
S'	?	0	0	0	?	1	1	6	6	0	2
$\langle b \rangle$	0	0	0	0	0	1	1	6	6	0	2
A_1	0	0	0	0	0	0	1	6	6	1	2
$A - A_1$	?	1	0	0	0	0	0	6	6	1	2
$\langle c \rangle$	3	1	3	0	0	0	0	0	6	1	2
B_k	3	1	3	6	0	0	0	0	0	1	2
$B - B_1 - B_k$	3	1	3	6	3	0	0	0	0	1	?
B_1	3	1	3	6	3	0	0	0	0	0	0
$\langle a \rangle$	3	0	3	6	3	1	0	0	0	0	0
T'	3	0	3	6	3	1	?	?	0	0	?
C_1	3	0	0	6	3	1	1	0	0	0	2
T	?	1	3	6	3	1	?	0	0	1	?

The bound 33/16 is achieved by P_1. If we require equality, each vertex must be counted exactly 16 times. Hence $B - B_1 - B_k = \varnothing$ and $k = 2$. To avoid exceeding 16 in other neighborhood counts, we have $T \leftrightarrow B_k$, and all remaining question marks must be 0. This yields $S' \| (S \cup A)$, so $l = 2$ (Theorem 13.6). Finally, the weights must be proportional to the sizes of the corresponding sets, and we have the expansion of P_1. □

9. Bounds and Partial Characterization for Type 2 G-Graphs

For the remainder of the paper, we consider **G**-graphs with a c-critical Type 2 triangle abc; Theorem 4 implies that $\{A, B\}$ is linked. We may assume that $\{A, C\}$ is the unique non-linked pair, and that the mutual partitions of $\{A, B\}$ have k parts $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$, with $A_i \leftrightarrow B_j$ if $i + j \leq k + 1$ and $A_i \| B_j$ if $i + j > k + 1$. The Type 2 **G**-graphs H_m of Example 3 have $\{A, B\}$ totally linked and hence $k = 1$ for each c-critical triangle. We have only one example of a Type 2 **G**-graph with

$k = 2$ (and its expansions). It has other critical triangles, some of which are Type 2 with $k = 1$, and others of which are Type 1 and show the graph isomorphic to our old friend J of Example 4! We have no examples of Type 2 **G**-graphs with $k > 2$. As we shall see, the Type 2 **G**-graphs with large vertex/degree ratio must have $k = 1$; we will describe all Type 2 **G**-graphs with $v/r \geq \frac{21}{10}$.

We drop our previous usage of S, S', T, T' and introduce a new partition. Let $S' = AC \cap \bar{N}(B_1)$, $S = AC - S'$, $T' = BC \cap \bar{N}(A_1)$, $T = BC - T'$, $R' = ABC \cap \bar{N}(B_1)$, and $R = ABC - R'$. Also let $A' = A - A_1$ and $B' = B - B_1$. These definitions hold for the remainder of the paper.

Lemma 17 contains counterparts of earlier results for Type 1 and Type 3 graphs. For example, Lemma 17.1 says that $C_1 = \varnothing$ and $C_2 = C$ in the xy-partition of C; hence we no longer discuss the xy-partition. We also no longer have $AC \leftrightarrow B$ and $BC \leftrightarrow A$, but we can say something about which edges are present or missing.

Lemma 17. *If abc is a c-critical Type 2 triangle in G, then*
1. $A_1 \| C$ *and* $B \leftrightarrow C \leftrightarrow A'$ *(in particular,* $A \| C$ *if and only if* $k = 1$).
2. $(R' \cup S') \leftrightarrow (C \cup A)$ *and* $T' \leftrightarrow B$
3. $S' \leftrightarrow B'$ *and* $T' \leftrightarrow A'$.
4. $(R \cup S \cup T') \| A_1$ *and* $(R' \cup S' \cup T) \| B_1$.
5. $(R \cup S) \leftrightarrow B$ *and* $T \leftrightarrow A$.
6. S' *and* $T' \cup C$ *are independent sets of size at least q.*
7. $(R' \cup S') \leftrightarrow T'$.

Proof. (1): If A has no vertex independent of C, then $z \| A$ for some $z \in C$, since $\{A, C\}$ is non-linked. This imples $A \leftrightarrow B$ (Lemma 6), so any A, B-edge is a link edge. Since $\{B, C\}$ is also linked by some yz, any edge between A and C forms a triangle with y, contradicting Lemma 7.4. Hence we may assume A has a vertex independent of C, and then $B \leftrightarrow C$ (Lemma 6). Now any A_1, C-edge violates Lemma 7.4. Finally, $A' \leftrightarrow a$ & $B_k \leftrightarrow C$ force $A' \leftrightarrow C$.

(2): Three applications of Lemma 7.1. (3): By $S' \leftrightarrow A_k$ & $b \leftrightarrow B'$; similarly for $T' \leftrightarrow A'$, (4): By Lemma 7.4, since every vertex of $R \cup S \cup T'$ has a neighbor in B_1; similarly for $R' \cup S' \cup T$ and A_1. (5): By $(R \cup S) \leftrightarrow c$ & $A_1 \leftrightarrow B$; similarly for $T \leftrightarrow A$. (6): By Remark 1, since now $\bar{N}(by) = S'$ for any $y \in B_1$ and $\bar{N}(ax) = T' \cup C$ for any $x \in A_1$. (7): By $(R' \cup S') \leftrightarrow a$ & $B_1 \leftrightarrow T'$. □

Lemma 17 makes no comment on adjacencies for AB. Here we can obtain $\langle c \rangle = AB \cup c$ as in Theorem 6.2 if $k = 1$. Even when $k > 1$, we know of no counterexample to this conclusion.

Lemma 18. *If abc is a c-critical Type 2 triangle, then* $AB - \langle c \rangle$ *is the disjoint union of sets* AB_1 *and* AB_2 *such that* $A_1 \leftrightarrow AB_1 \| B_1$ *and* $A_1 \| AB_2 \leftrightarrow B_1$. *If* $k = 1$, *these sets are empty, i.e.* $\langle c \rangle = AB \cup c$ *and* $|C| = 2q + |ABC|$.

Proof. Choose $x \in A_1$, $y \in B_1$, $u \in AB$. If $u \leftrightarrow x$, then $u \| B_1$ (Lemma 7.4). Now $u \leftrightarrow A_1$, since $\langle c \rangle = \bar{N}(x'y)$ for all $x' \in A_1$ (Theorem 4.2). Similarly $A_1 \| u \leftrightarrow B_1$ if $u \leftrightarrow y$. If $u \| x, y$, then $u \in \bar{N}(xy) = \langle c \rangle$. If $k = 1$ and $u \in AB_1$, then $u \leftrightarrow A$. Since also $u \leftrightarrow x$ forces $N(c) \subseteq N(x) \cup N(u)$, and since $x \leftrightarrow B$, we conclude that uax is a dominating triangle and $N(a|ux) \subseteq (AB \cup c) - u$, which contradicts the criticality of abc. The symmetric argument yields a contradiction when $u \in AB_2$. Theorem 4.2 and Lemma 5.1 yield $|C|$. □

These adjacency statements enable us to characterize large Type 2 **G**-graphs with $k > 1$.

Theorem 17. *If abc is a c-critical Type* 2 *triangle in G,* $v/r \geq \frac{21}{10}$, *and* $k > 1$, *then G is the q-fold expansion of the graph J of Example* 4.

Proof. Choose a, b, c and one vertex each from A_1, A_2, B_1, B_2, C, S', where sets are denoted as in Lemma 17. Counting the neighborhoods of these vertices with weights as indicated in Table 2 yields a total count as indicated there, where we have included $AB \leftrightarrow C$ (Lemma 8) and the results of Lemmas 17 and 18. By $T \leftrightarrow b \,\&\, S' \leftrightarrow A \cup C$, we have $w \leftrightarrow S'$ or $w \leftrightarrow A \cup C$ for any $w \in T$. Hence every vertex is counted at least 10 times, and $v \leq \frac{21}{10}r$. If equality holds, then every vertex must be counted exactly 10 times, so $ABC = AB - \langle c \rangle = T' = \varnothing$, $T \leftrightarrow S'$, and all other question marks become 0. Because the vertices were chosen arbitrarily from the specified sets, setting a question mark to 0 forces complete independence. We can now restrict our attention to the block adjacency matrix on $\langle a \rangle$, $\langle b \rangle$, $\langle c \rangle$, A_1, A_2, B_1, B_2, C, S_1. To achieve $v/r \geq \frac{21}{10}$ and regularity, the sizes of these sets must be in proportion to the weights in Table 2, because the vertices of any set whose size is less than $v/21$ times its weight will have more than $10v/21$ neighbors. With $|\langle c \rangle| = q$, this yields a $10q$-regular graph with $21q$ vertices. To transform this description into the q-fold expansion of J, relabel the sets listed above as T, A, C_1, $\langle b \rangle$, C_2, B, $\langle a \rangle$, S, $\langle c \rangle$, respectively, in the notation for J in Example 4. □

Table 2. Neighborhood counting for Type 2 **G**-graphs with $k > 1$

Nbhd weight	2	5	1	2	1	5	2	2	1	
Vert locat	a	b	c	A_1	A_2	B_1	B_2	C	S'	Total
a	0	5	1	2	1	0	0	0	1	10
b	2	0	1	0	0	5	2	0	0	10
$\langle c \rangle$	2	5	0	0	0	0	0	2	1	10
A_1	2	0	0	0	0	5	2	0	1	10
A'	2	0	0	0	0	5	?	2	1	10
B_1	0	5	0	2	1	0	0	2	0	10
B'	0	5	0	2	?	0	0	2	1	10
C	0	0	1	0	1	5	2	0	1	10
S'	2	0	1	2	1	0	2	2	0	10
S	2	0	1	0	?	5	2	?	?	10
T'	0	5	1	0	1	5	2	0	1	15
T	0	5	1	2	1	0	?	?	?	9
AB_1	2	5	0	2	?	?	?	2	?	11
AB_2	2	5	0	?	?	5	?	2	?	14
R'	2	5	1	2	1	0	?	2	?	13
R	2	5	1	0	?	5	2	?	?	15

In light of the somewhat unexpected appearance of J in Theorem 17, let us consider other alternate interpretations of J. In turns out that J has many critical triangles. In describing the corresponding structure, it is convenient to introduce $S'' = S - \langle b \rangle$ and $T'' = T - \langle a \rangle$.

Example 9. Given the description of J as in Theorem 17, choose $u \in S'$ and $x \in A_1$. We have $u \leftrightarrow C \cup B'$, $x \leftrightarrow B$, and $N(b) \subset N(u) \cup N(x)$. Hence aux is a dominating triangle. If $x \leftrightarrow BC$, then it is a critical triangle, with $N(u|ax) = C$. We also have $N(a|ux) = S_2 \cap \bar{N}(u)$ and $N(x|au) = B_1 \cup N(bc|u)$. Then $BC \leftrightarrow x \,\&\, u \leftrightarrow C$ force $N(bc|u) \leftrightarrow C$. By Theorem 4, $\{N(a|ux), N(x|au)\}$ is also linked, so this is another Type 2 triangle. However, $k = 1$ for this u-critical triangle aux. This relabeling corresponds to the fourth row in Table 3. Each row of Table 3 designates triangles obtained by taking a vertex of each of the three sets in the first column, the third set being the critical one. The entries in the interior of the table are the set names under the alternate description.

Table 3. Alternate interpretations of J

Set size	2	5	1	2	1	5	2	2	1	
Set name	$\langle a \rangle$	$\langle b \rangle$	$\langle c \rangle$	A_1	A_2	B_1	B_2	C	S'	
Triangle				New name of set						Notes
$\langle a \rangle\langle b \rangle\langle c \rangle$	$\langle a \rangle$	$\langle b \rangle$	$\langle c \rangle$	A_1	A_2	B_1	B_2	C	S'	†1
CB_1A_2	C	B_1	A_2	B_2	$\langle c \rangle$	$\langle b \rangle$	A_1	$\langle a \rangle$	S'	†1
B_2A_1S'	BC	A	C	$\langle b \rangle$	C	B	$\langle a \rangle$	AC	$\langle c \rangle$	†2
$\langle a \rangle A_1S'$	$\langle a \rangle$	A	S'	$\langle b \rangle$	S''	B	T''	C	$\langle c \rangle$	†3
CB_2S'	C	B	S''	T''	S'	A	$\langle b \rangle$	$\langle a \rangle$	$\langle c \rangle$	†3
$A_2\langle a \rangle S'$	$\langle b \rangle$	B	T'	T''	$\langle a \rangle$	A	C	S'	$\langle c \rangle$	†4
$\langle c \rangle CS'$	S'	A	$\langle a \rangle$	C	T'	B	T''	$\langle b \rangle$	$\langle c \rangle$	†4

†1 Type 2, $k = 2$, description in Theorem 17
†2 Type 1, graph J of Example 4!
†3 Type 2, $k = 1$, Lemma 20, Theorem 18
†4 Type 2, $k = 1$, Lemma 19

In the remainder of this paper, we study Type 2 **G**-graphs with $A \| C$; these are precisely those with $k = 1$ (Lemma 17.1) and include all those with $v/r > \frac{21}{10}$ (Theorem 17). The sketch in Fig. 6 applies for the remainder of the paper; known non-adjacencies are not indicated, and for clarity the forced edges $ABC \leftrightarrow \{\langle a \rangle, \langle b \rangle, \langle c \rangle\}$ and $R' \leftrightarrow T'$ are also omitted.

For easy reference, we collect the current information for Type 2 triangles with $k = 1$ in the block adjacency matrix of Fig. 7. Question marks denote unknown submatrices. If these are not constant, then these sets may break into smaller equivalence classes, but already every equivalence class is confined to one of these sets.

For the Type 2 graphs with $k = 1$, we consider two cases: $T' \neq \varnothing$ and $T' = \varnothing$. In each case, we find that such a graph has at most $\frac{21}{10}r$ vertices, with equality only for expansions of J. The main techniques are comparison of rows in Fig. 7 and

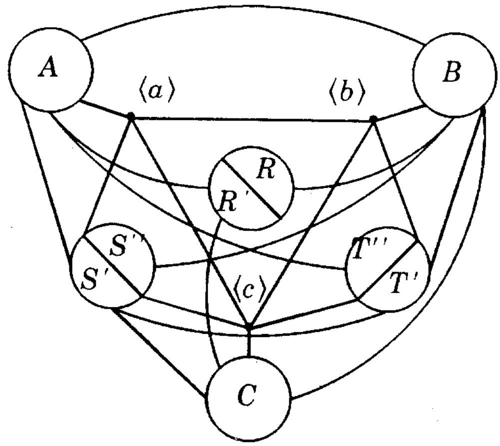

Fig. 6. Canonical sets for Type 2 c-critical abc with $A \| C$

							$AC-\langle b\rangle$		$BC-\langle a\rangle$		ABC	
	$\langle a\rangle$	$\langle b\rangle$	$\langle c\rangle$	A	B	C	S''	S'	T''	T'	R	R'
$\langle a\rangle$	0	1	1	1	0	0	1	1	0	0	1	1
$\langle b\rangle$	1	0	1	0	1	0	0	0	1	1	1	1
$\langle c\rangle$	1	1	0	0	0	1	1	1	1	1	1	1
A	1	0	0	0	1	0	0	1	1	0	0	1
B	0	1	0	1	0	1	1	0	0	1	1	0
C	0	0	1	0	1	0	?	1	?	0	?	1
S''	1	0	1	0	1	?	?	?	?	?	?	?
S'	1	0	1	1	0	1	?	0	?	1	?	?
T''	0	1	1	1	0	?	?	?	?	?	?	?
T'	0	1	1	0	1	0	?	1	?	0	?	1
R	1	1	1	0	1	?	?	?	?	?	?	?
R'	1	1	1	1	0	1	?	?	?	1	?	?

Fig. 7. Block adjacency matrix for Type 2 **G**-graphs with $A \| C$

Remark 1. Since $|C| = 2q + |ABC|$, any pair of vertices in a triangle has at least $2q + |ABC|$ common non-neighbors (by c-criticality), and the number of common non-neighbors is exactly q more than the number of common neighbors (Remark 1). When T'' and $\langle a\rangle$ are used together, we may use the alternate expression $T \cup a$; similarly for $S'' \cup \langle b\rangle = S \cup b$. We say that $\{B, C\}$ *generates a triangle* if some edge between B and C belongs to a triangle.

Lemma 19. *If abc is a c-critical Type 2 triangle with $A \| C$ and $T' \neq \emptyset$, then*
1. $|R| + |T'| = |S'| - q \geq q + |ABC|$, *also* $|T \cup a| \geq 2q + |ABC|$ *and* $|S \cup a| \geq q + |R'|$.
2. *If $\{B, C\}$ generates a triangle, then $v < \frac{21}{10}r$.*
3. *If $\{B, C\}$ generates no triangle, then $R \cup S'' \| C \leftrightarrow T'' \leftrightarrow S''$ and $S' \leftrightarrow R$.*

4. *If $v/r \geq 29/14$ and $S'' \leftrightarrow T''$, then $|R'| \leq q$ and S'' is an independent set.*
5. *If $v/r \geq \frac{21}{10}$, then G is the q-fold expansion of J.*

Proof. (1): If $y \in B$ and $w \in T'$, then byw is a triangle. From Figure 7, we have $\bar{N}(by) = S'$ and $N(by) = R \cup T'$, also $\bar{N}(wy) \subseteq T \cup a$ and $N(wy) \subseteq R \cup S \cup b$.

(2): Suppose $y \in B$ and $z \in C$ form a triangle with w; note that $w \in N(yz) \subseteq R \cup S''$. We have $|N(yz) \cap S''| \geq q + |R'|$. By (1) we may select some $u \in S'$; now $u \leftrightarrow z$ forces $N(b) \subseteq N(u) \cup N(z)$, which implies $\bar{N}(uz) \subseteq S'' \cup \langle b \rangle$. Since uzc is a triangle, we have $|\bar{N}(uz) \cap (S \cup b)| \geq 2q + |R| + |R'|$. Since $N(yz)$ and $\bar{N}(uz)$ are disjoint, together we have $|S \cup b| \geq 3q + |R| + 2|R'|$. Now $r = d(c) \geq 10q + 4|R| + 6|R'|$. This implies $v \leq \frac{21}{10}r$, with equality only if $ABC = \varnothing$ and $r = 10q$. In particular, $R = \varnothing$ and $w \in S''$. Equality also requires $|T'| = q$, $|C| = |S'| = |T \cup a| = 2q$, and $|S \cup b| = 3q$, which yield $|B| = 6q$ (Lemma 5.1). Since $d(yz) \geq q$, we conclude that z has $q, 6q, q, 2q$, neighbors in $\langle c \rangle, B, S'', S'$, respectively, which implies $z \parallel T''$. Now $w \leftrightarrow z \,\&\, b \leftrightarrow BC$ force $w \leftrightarrow BC$, yielding the contradiction $N(b) \subseteq N(w) - z$.

(3): If $\{B, C\}$ generates no triangle, then $C \parallel (R \cup S'')$. Also $\bar{N}(yz) = \langle a \rangle$ for any $y \in B$ and $z \in C$ (Theorem 3), and hence $T'' \leftrightarrow C$. Now $S'' \leftrightarrow a \,\&\, C \leftrightarrow T''$ force $S'' \leftrightarrow T''$, and $R \leftrightarrow b \,\&\, C \leftrightarrow S'$ force $R \leftrightarrow S'$.

(4): From (1), $r = d(c) \geq 8q + 4|R| + 6|R'|$. If $v/r \geq 29/14$, this implies $|R'| \leq q$. If uu' is an edge in S'', then $(N(b) \cup N(A)) \subseteq N(u) \cup N(u')$, which yields $S' \cup T' \cup ABC \subset N(u) \cup N(u')$. If we count $N(u)$, $N(u')$, and twice $N(B)$ (using $S'' \leftrightarrow T''$), we obtain $4r \geq 2v + 2 - |S'| + |T'| + |R| - |R'| = 2v + 2 - q - |R'|$, contradicting $|R'| \leq q$.

(5): If $v/r \geq \frac{21}{10}$, then the hypotheses of (3) and (4) hold, so S'' is independent. If $u \in S''$, then $u \parallel S'$ implies $N(u) \subseteq N(b)$. Hence Lemma 2 implies there exists $w \in S'$ with $w \leftrightarrow S''$ (this also holds vacuously if S'' is empty). We again count vertex neighborhoods, using this vertex $w \in S'$ and one vertex each from $\langle a \rangle, \langle b \rangle, \langle c \rangle, A, B, C, T'', T'$, weighted as indicated in the columns of Table 4. The count for vertices in each set appears in the rows. The 21 vertex neighborhoods count each vertex at least 10 times. Hence $v/r \leq \frac{21}{10}$. If equality holds, then each vertex must be counted exactly 10 times, which implies that $S'' = R = R' = \varnothing$ and each question mark in

Table 4. Neighborhood counting for some Type 2 G-graphs with $k = 1$

Neighbd weight	1	2	1	5	5	2	2	2	1	
Vertex location →	$\langle a \rangle$	$\langle b \rangle$	$\langle c \rangle$	A	B	C	$w \in S'$	T''	T'	Total
$\langle a \rangle$	0	2	1	5	0	0	2	0	0	10
$\langle b \rangle$	1	0	1	0	5	0	0	2	1	10
$\langle c \rangle$	1	2	0	0	0	2	2	2	1	10
A	1	0	0	0	5	0	2	2	0	10
B	0	2	0	5	0	2	0	0	1	10
C	0	0	1	0	5	0	2	2	0	10
S'	1	0	1	5	0	2	0	?	1	10
T''	0	2	1	5	0	2	?	0	?	10
T'	0	2	1	0	5	0	2	?	0	10
S''	1	0	1	0	5	0	2	2	?	11
R	1	2	1	0	5	0	2	?	?	11
R'	1	2	1	5	0	2	?	?	1	12

the other rows must become 0. Since $S'' = \varnothing$, we can now let w denote an arbitrary vertex of S'. The block adjacency matrix is now that of J, as described in the last two lines of Table 3. We obtain the q-fold expansion of J by letting the size of each set be q times its weight. Furthermore, this is the only way to achieve $v/r \geq \frac{21}{10}$ and regularity, because the vertices of any set whose size is less than $v/21$ times its weight will have more than $10v/21$ neighbors. □

The remaining case is $k = 1$ and $T' = \varnothing$. The adjacency information known at this point appears in Fig. 7.

Lemma 20. *If abc is a c-critical Type 2 triangle with $A \| C$ and $T' = \varnothing$, then*
1. $R = \varnothing$ *and* $\{\langle b\rangle, B\}$ *generates no triangle.*
2. S' *is an equivalence class of size q.*
3. *If* $S' \leftrightarrow S''$, *then* $S' \leftrightarrow (T'' \cup R' \cup S'')$.
4. *If* $S' \leftrightarrow S'' \| C$, *then* $|\langle a\rangle| = q$, $|A| = |\langle b\rangle| = |C| + |T''|$, *and* $|B| = 2|C| + |T''| + |S''|$.

Proof. (1): If $T' = \varnothing$, then $N(by) = R$ for all $y \in B$. If $R \neq \varnothing$, then $|R| = |\bar{N}(by)| - q \geq |C| - q = q + |ABC| > |R|$.

(2): Follows from (1), $\bar{N}(by) = S'$, and Theorem 3.

(3): By (2), we can pick $u \in S'$, $x \in A$ arbitrarily. Then axu is u-critical, since $T' = \varnothing$ implies $\bar{N}(ax) = C$ and $u \leftrightarrow C$. Let concatenations of A', X, U denote the sets in the vertex partition induced by axu as a dominating triangle, just as concatenations of A, B, C are used for the partition induced by abc. We have $U = C$, $A' = \langle b\rangle \cup (S'' \cap \bar{N}(u))$, $X = B \cup (T'' \cap \bar{N}(u))$, $A'X = S' \cup (R' \cap \bar{N}(u)) - u$, $XU = \langle a\rangle \cup (T'' \cap N(u)) - a$, $A'U = \langle c\rangle \cup (S'' \cap N(u))$, and $A'XU = (R' \cap N(u))$. Since $b \| C$, $\{A', U\}$ is not linked. However, $b \leftrightarrow (B \cup T'')$ and $B \leftrightarrow (\langle b\rangle \cup S'')$, so $\{A', X\}$ is linked. Also $C \leftrightarrow u \;\&\; b \leftrightarrow (T'' \cap \bar{N}(u))$ force $C \leftrightarrow (T'' \cap \bar{N}(u))$, so $\{X, U\}$ is linked. Therefore, axu is a Type 2 u-critical triangle with $\{A', U\}$ non-linked (i.e., a has the same role as before). By $u \leftrightarrow S''$, we have $A' \| U$ and the value of "k" for axu is 1. (Note: this is also implied by Theorem 17 if we assume $v/r \geq \frac{21}{10}$ and $G \neq J$.) This means that, in addition to $A'X = \langle u\rangle$ (Lemma 18), both A' and X are equivalence classes (as are A and B in Fig. 7). Hence there are none of the second type of vertex in the description of A', X, $A'X$ and we have $u \leftrightarrow (T'' \cup R' \cup S'')$.

(4): If $S'' \| C$, then $\{B, C\}$ generates no triangle. As in Lemma 19.3, Theorem 3 implies $\bar{N}(yz) = \langle a\rangle$ with size q, and hence $T'' \leftrightarrow C$. We obtain $|A|$ from $\langle a\rangle$ vs. $\langle c\rangle$, $|\langle b\rangle|$ from B vs. S', and $|B|$ from (3) and b vs. S'. □

This structural information enables us to characterize a class of Type 2 **G**-graphs. Although these seem like many assumptions, we shall see that they all hold when $v/r \geq \frac{21}{10}$ and $G \neq J$.

Theorem 18. *Suppose abc is a c-critical Type 2 triangle with $A \| C$, $T' = \varnothing$, $S' \leftrightarrow S'' \| C$, $T'' = \varnothing$, and S'' independent. Then G is the q-fold expansion of the graph H_m of example 3, for some $m \geq 2$.*

Proof. All the conclusions of Lemma 20 hold. Note that the relabeling of J in Example 9 that has $T' = \varnothing$ is forbidden by $S'' \| C$. In Fig. 8 we collect the current

Size Class	q $\langle a\rangle$	t $\langle b\rangle$	q $\langle c\rangle$	t A	$2t+\lvert S''\rvert$ B	t C	$\lvert S''\rvert$ S''	q S'	$t-2q$ R'
$\langle a\rangle$	0	1	1	1	0	0	1	1	1
$\langle b\rangle$	1	0	1	0	1	0	0	0	1
$\langle c\rangle$	1	1	0	0	0	1	1	1	1
A	1	0	0	0	1	0	0	1	1
B	0	1	0	1	0	1	1	0	0
C	0	0	1	0	1	0	0	1	1
S''	1	0	1	0	1	0	0	1	?
S'	1	0	1	1	0	1	1	0	1
R'	1	1	1	1	0	1	?	1	?

Fig. 8. Block adjacency matrix for certain Type 2 **G**-graphs

status of our block adjacency matrix, together with the known set sizes, using $t = |C| = 2q + |R'|$.

The only unknown adjacencies are in $S'' \cup R'$. Since $S'' \leftrightarrow B \| R'$, we can apply Lemma 16 to S'' and R'. We obtain partitions of S'' and R' into equivalence classes $S_1, \ldots, S_h$ and $R_1, \ldots, R_h$ such that $\bar{N}(S_i) \cap R' = R_i = \bar{N}(R_i) \cap R'$. Furthermore, the equivalence classes in S'' and in R' have the same size.

If $h = 0$, then $S'' = R' = \varnothing$ and $t = 2q$, and G is $6q$-regular on $13q$ vertices. As a degenerate instance of the encoding described below, we can express G as H_2. Hence assume $h > 0$. Now uy for $y \in B$, $u \in S_i$ is an edge not on a triangle, so $q = |\bar{N}(uy)| = |R_i|$ (Theorem 3). Any vertex outside R' now has $3t + |S''|$ neighbors, and $w \in R_i$ has $4t + |S''| - |S_i|$ neighbors, so $|S_i| = t$. We also have $t - 2q = |R'| = hq$, so $t = (h+2)q$ and $|S''| = t(t-2)$. Now G is the q-fold expansion of a graph that is isomorphic to H_t by setting $Q_1 = \langle b\rangle$, $Q_2 = B$, $Q_3 - U_1 - U_2 = S''$, $Q_4 = S'$, $u_1 = a$, $u_2 = c$, $Q_5 - u_1 - u_2 = R'$, corresponding to the c-critical triangle presented in Example 3. □

Finally, we conclude that we have found all the large **G**-graphs.

Theorem 19. *If G is a* **G***-graph with $v/r \geq \frac{21}{10}$, then G is one of H_2, J, or G_k for $1 \leq k \leq 5$.*

Proof. By Lemma 21.5 and Theorems 11, 12, 16, 17, 18, it remains only to prove that if $G \neq J$ and G has a Type 2 c-critical triangle abc with $T' = \varnothing$, $A \| C$, and $v/r \geq \frac{21}{10}$, then G satisfies the hypotheses of Theorem 18. We noted in the proof of Lemma 20.3 that Theorem 17 implies $S' \leftrightarrow S''$ when we assume $v/r \geq \frac{21}{10}$ and $G \neq J$. If $S'' \| C$ fails, then $\{B, C\}$ generates a dominating triangle uyz with $u \in S''$, $y \in B$, $z \in C$ (Lemma 4). The sets $\bar{N}(yz) \subseteq T'' \cup \langle a\rangle$ and $\bar{N}(uy) \subseteq T'' \cup R'$ are disjoint and together have at least $4q + 2|R'|$ vertices (Theorem 4.3). By Lemma 5.1, $|A| \geq 5q + 2|R'|$. By Lemma 20.3, any vertex of S' has at least $12q + 5|R'|$ neighbors, which requires $v \leq \frac{25}{12}r$.

If uu' is an edge in S'', then $N(uu') \supseteq \langle a\rangle \cup \langle c\rangle \cup B \cup S'$ and $\bar{N}(uu') \subseteq \langle b\rangle \cup A \cup C \cup S''$. Substituting in the known set sizes from Lemma 20.4 and applying Remark 1 yields $6q + 3|R'| + 2|T''||S''| \geq 4q + |B| = 8q + 2|R'| + |T''| + |S''|$.

The resulting $|R'| + |T''| \geq 2q$ yields $r = d(b) \geq 10q + |R'| + |S''|$, which requires $v/r < \frac{21}{10}$. Hence S'' is independent.

Finally, suppose there exists $w \in T''$. If $w \leftrightarrow S''$ (including $S'' = \varnothing$), then $\bar{N}(y) \subseteq \bar{N}(w) - c$ for any $y \in B$. Hence there exists $u \in S''$ with $u \| w$. Now $w \in \bar{N}(uy) = \bar{N}(u) \cap (T'' \cup R')$ is an equivalence class of size q, since uy belongs to no triangle. Hence $|T''| \geq q$. From c vs. w we have $|A| = |(S'' \cup T'' \cup R') - N(w)|$. Since $|A| = 2q + |R'| + |T''|$ (Lemma 20.4), we have $|S''| \geq 2q$, with equality only if $w \| S'' \cup T'' \cup R'$. Collecting the contributions to $d(c)$ from Lemma 20.4, we have $r \geq 6q + |S''| + 2|T''| + 3|R'| \geq 10q$, i.e. $v/r \leq \frac{21}{10}$. Equality requires $R' = \varnothing$ and $w \| S''$, but then $a \leftrightarrow S''$ & $T'' \leftrightarrow C$ force $a \leftrightarrow C$, which is a contradiction. □

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