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Balancing Pairs in Partially Ordered Sets

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Dedicated to Paul Erdös on his eightieth birthday.

ABSTRACT

J. Kahn and M. Saks proved that if P is a partially ordered set and is not a chain, then there exists a pair $x, y \in P$ so that the number of linear extensions of P with x less than y is at least 3/11 and at most 8/11 of the total number of linear extensions. S. S. Kislitsyn and, independently M. Fredman and N. Linial conjectured that this result is still true with 3/11 and 8/11 replaced by 1/3 and 2/3. In this manuscript, we produce some new inequalities for linear extensions of partially ordered sets, and we give new proofs of some known inequalities. We also conjecture a cross product inequality which we are able to verify for width two posets. As a consequence of our approach to studying balanced pairs, we are able to improve the Kahn/Saks result by a small positive constant.

1. Introduction

Given a finite partially ordered set (poset) P and a pair x, y of distinct elements of P, let $\mathrm{Prob}(x > y)$ denote the number of linear extensions of P in which $x > y$ divided by the total number of linear extensions. If $x < y$ in P, $\mathrm{Prob}(x > y) = 0$, while $\mathrm{Prob}(x > y) = 1$ if $x > y$ in P. On the other hand, if x and y are incomparable in P, then $0 < \mathrm{Prob}(x > y) < 1$. In 1969,

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[2] This is an extended abstract of a full length article which will appear elsewhere.

S. S. Kislitsyn [9] made the following conjecture, which remains of the most intriguing problems in the combinatorial theory of posets.

Conjecture 1.1. *If P is a poset which is not a chain, then there exists an incomparable pair $x, y \in P$ so that*

$$1/3 \leq \mathrm{Prob}(x > y) \leq 2/3. \qquad \blacksquare$$

Conjecture 1.1 was also made independently by both M. Fredman and N. Linial, and many papers on this subject attribute the conjecture to them. It is now known as the 1/3—2/3 conjecture. Linial [11] showed that the conjecture holds for width two posets, and P. Fishburn, W. G. Gehrlein and W. T. Trotter [2] showed that it holds for height two posets.

Following [7], we let $\delta(P)$ denote the largest positive number for which there exists a pair $x, y \in P$ with $\delta(P) \leq \mathrm{Prob}(x > y) \leq 1 - \delta(P)$. Using this terminology, we can state the principal result of [7] as follows.

Theorem 1.2. *If P is not a chain, then $\delta(P) \geq 3/11$.* $\blacksquare$

The original motivation for studying balancing pairs in posets was the connection with sorting. The problem was to answer whether it is always possible to determine an unknown linear extension with $O(\log t)$ rounds (questions) where t is the number of linear extensions. The answer would be yes if one could prove that exists an absolute constant δ_0 so that $\delta(P) \geq \delta_0$ for any P which is not a chain.

Although the Kahn/Saks bound given in Theorem 1.2 is the best known bound valid for all finite posets, other proofs bounding $\delta(P)$ away from zero have been given. In [8], L. Khachiyan uses geometric techniques to show $\delta(P) \geq 1/e^2$. Kahn and Linial [6] provide a short and elegant argument using the Brunn/Minkowski theorem to show that $\delta(P) \geq 1/2e$. In [4], J. Friedman also applies geometric techniques to obtain even better constants when the poset satisfies certain additional properties. Kahn and Saks conjectured that $\delta(P)$ approaches 1/2 as the width of P tends to infinity. In [10], J. Komlós provides support for this conjecture by showing that for every $\epsilon > 0$, there exists a function $f_\epsilon(n) = o(n)$ so that if $|P| = n$ and P has at least $f_\epsilon(n)$ minimal points, then $\delta(P) > 1/2 - \epsilon$.

None of the arguments in [6], [7] and [8] yields an efficient algorithm for the original sorting problem since they do not provide an efficient method for determing how to locate the balancing pair. In [5], Kahn and J. Kim have taken a totally different approach to the sorting problem. Using a concept of entropy for posets, they show the existence of a polynomial time algorithm for sorting in $O(\log t)$ rounds. Their algorithm shows how to efficiently locate pairs to use in queries so that, regardless of the responses, the determination of the unknown linear extension is made in $O(\log t)$ rounds. However, at individual rounds, the pairs need not be balanced in the sense that for a given pair (x, y) used in the algorithm, $\text{Prob}(x > y)$ may be arbitrarily close to zero.

In this paper, we will concentrate entirely on the issue of balancing pairs—setting aside for the time being the algorithmic questions. First, we intend to develop some new combinatorial lemmas for linear extensions of posets. One immediate benefit will be to provide simple proofs of lemmas first developed in [7]. Second, we will obtain a modest improvement in Theorem 1.2. Third, we will make a conjecture which, if true, would imply an even stronger result, one which is best possible if we extend our study of balancing pairs to countably infinite posets of width two.

2. Notation and terminology

As much as is possible, we will use the notation and terminology of [7], and we will assume the reader is familiar with the concepts and proof techniques of this paper. In particular, we consider the sample space of all linear extensions of a poset P with all linear extensions being equally likely. For a random linear extension L and a point $x \in P$, $h_L(x)$ denotes the *height* of x in L, i.e., if L orders the points in P as $x_1 < x_2 < \ldots < x_n$ and $x = x_i$, then $h_L(x) = i$. We denote by $h(x)$ the expected value of $h_L(x)$. When $(x, y) \in P \times P$ is a fixed ordered pair of incomparable points, then for each positive integer i, we let a_i denote the probability that $h_L(y) - h_L(x) = i$, and we let b_i denote the probability that $h_L(x) - h_L(y) = i$. We also set $b = b_1$ and let $B = \sum_i b_i = \text{Prob}(x > y)$. Then we set $\epsilon = b/B$.

The following lemmas are proven in [7].

Lemma 2.1. $a_1 = b_1 = b$. ■

Lemma 2.2. $a_2 + b_2 \leq a_1 + b_1$. ■

Lemma 2.3. *For each* $i \geq 1$, $a_{i+1} \leq a_i + a_{i+2}$ *and* $b_{i+1} \leq b_i + b_{i+2}$. ■

Lemma 2.1 is trivial, but already Lemma 2.2 requires a clever little argument. Lemma 2.3 is more substantial.

The basic approach of [7] may now be summarized as follows. Since P is not a chain, it follows that we may choose an ordered pair (x, y) with $0 \leq h(y) - h(x) < 1$. Then Kahn and Saks show that $3/11 < B = \text{Prob}(x > y) < 8/11$. The argument cannot be made just on the basis of these lemmas. Kahn and Saks also derive in [7] the following nonlinear inequality, which is critical to their argument.

Theorem 2.4. *The sequences* $\{a_i : i \geq 1\}$ *and* $\{b_i : i \geq 1\}$ *are log-concave, i.e., for each* $i \geq 1$, $a_{i+1}^2 \geq a_i a_{i+2}$ *and* $b_{i+1}^2 \geq b_i b_{i+2}$. ∎

3. Obstacles and pitfalls

As Kahn and Saks point out in [7], the value of the constant in Theorem 1.2 could be improved if we could show that there exists a positive absolute constant γ so that if P is not a chain, then if is always possible to find an ordered pair (x, y) with $0 \leq h(y) - h(x) \leq 1 - \gamma$. However, nobody has yet been able to settle whether such a γ exists. If it does, then as shown by Saks in [12], it must satisfy $\gamma \leq .133$. Even this value would not be enough to prove $\delta(P) \geq 1/3$. However, it is of interest to determine the maximum value of $|h(y) - h(x)|$ which allows one to conclude that $1/3 \leq \text{Prob}(x > y) \leq 2/3$. It is relatively easy to modify the Kahn/Saks proof technique to obtain the next result, which is clearly best possible.

Theorem 3.1. *Let* (x, y) *be distinct points in a poset* P, *and suppose that* $0 \leq h(y) - h(x) \leq 2/3$. *Then* $1/3 \leq \text{Prob}(x > y) \leq 2/3$. ∎

There is another more serious obstacle. If we extend the concept to countably infinite posets, then the 1/3—2/3 conjecture is false. As was discovered independently by G. Brightwell and Trotter, there is a countably infinite width two semiorder P with $\delta(P) = (5 - \sqrt{5})/10 \approx .27639$. This example is constructed as follows. The poset P has as its point set $X = \{x_i : i \in \mathbb{Z}\}$ with: $x_i < x_j$ in P if and only if $j > i + 1$ in $\mathbb{Z}$. If we define the finite poset P_n to be the subset of P consisting of all points whose subscripts in absolute value are at most n, then it is an easy exercise to show that

$$\lim_{n \to \infty} \text{Prob}(x_0 > x_1) = (5 - \sqrt{5})/10.$$

This example is striking on two counts. First, as noted in Section 1, Linial showed in [11] that $\delta(P) \geq 1/3$, for any *finite* width two poset. Second, this infinite poset is a semiorder, and Brightwell showed in [1] that $\delta(P) \geq 1/3$, for any *finite* semiorder—regardless of its width.

Also, there is a six point poset (see [13], for example) containing an incomparable pair x, y with $h(y) - h(x) = 1$ and $\text{Prob}(x > y) = 3/11$. This example shows that the Kahn/Saks inequality in Theorem 1.2 is, in some sense, best possible.

4. Some new inequalities

We begin this section with a generalization of Lemma 2.2.

Lemma 4.1. *If x and y are incomparable, then for each $i \geq 2$, $b_2 + a_i \leq b_1 + a_1 + a_2 + \cdots + a_{i-1}$.*

Proof. For each integer k, let E_k denote the set of linear extensions of P in which y appears exactly k positions above x. Now let $i \geq 2$. We describe an injection ϕ from $E_{-2} \cup E_i$ to $E_{-1} \cup E_1 \cup E_2 + \cdots \cup E_{i-1}$ First consider a linear extension L in E_i. We distinguish two cases. In Case 1, there is at least one point between x and y in L which is larger than x in P. In Case 2, there is no such point. In Case 1, let $\{u_1, u_2, \ldots, u_r\}$ be the nonempty set of points which are between x and y in L and are greater than x in P. We assume the points have been labelled so that $u_1 < u_2 < \cdots < u_r$ in L. Set $x = u_0$ and $y = u_{r+1}$. For each $i = 0, 1, \ldots, r$, let S_i denote the linear order induced on the set of points which lie between u_i and u_{i+1} in L. Then let U denote the linear order on the points below x in L and let H denote the linear order on the points above y in L. With this notation, the linear order L can be decribed as

$$L = U < x < S_0 < u_1 < S_1 < u_2 < S_2 < \cdots < u_r < S_r < y < H.$$

We associate with L the linear extension $\phi(L)$ in $E_1 \cup E_2 + \cdots \cup E_{i-1}$ defined as follows.

$$\phi(L) = U < S_0 < x < S_1 < u_1 < S_2 < u_2 < S_3 < \cdots < u_{r-1} < S_r < y < u_r < H.$$

If Case 2 holds, we define U and H as before and set M to be the linear order determined by the points between x and y in L. Now we take $\phi(L)$ in E_{-1} as

$$\phi(L) = U < M < y < x < H.$$

If L is in E_{-2} we may write $L = U < y < u < x < H$. If $u < x$ in P then $\phi(L) = U < u < y < x < H$ in E_{-1}, otherwise $\phi(L) = U < x < y < u < H$ in E_1.

It is straightforward to verify that the function ϕ is an injection. The desired inequality is obtained by taking cardinalities and dividing by the total number of linear extensions of P. ■

We next use the proof of the preceding lemma to provide a simple proof of Lemma 2.3.

Proof of Lemma 2.3. Let L be a linear extension of P in which y is exactly $i+1$ positions above x. We associate with L another linear extension $\phi(L)$ defined as follows. We consider three cases. Case 1 holds if x is incomparable with the point u immediately above it in L. In this case we exchange u and x to obtain $\phi(L)$ in E_i. Case 2 holds if these two points are comparable and there is an element v immediately below and incomparable with x. In this case we exchange v and x to obtain $\phi(L)$ in E_{i+2}. In Case 3, observe that $x < u$ if u is the point immediately over x in L. Also, either x is the least element of L, or the element v immediately below x in L is comparable with x. In this case we form $\phi(L)$ as in Case 1 in the proof of Lemma 4.2. Since S_0 is empty in this case, we have $\phi(L)$ in E_i.

Given $\phi(L)$ in E_i we only have to inspect the comparability status of x and the element immediately below x in $\phi(L)$ to decide if the rule of Case 1 or Case 3 has been applied to obtain $\phi(L)$. Hence ϕ is an injection. ■

5. Our approach to balancing pairs

Without loss of generality, we may restrict our attention to posets in which no point is comparable to all others. For such posets, note that if $x < y$ in P, then $h(y) > 1 + h(x)$. We then choose three points x, y and z with $h(x) \leq h(y) \leq h(z) \leq 2 + h(x)$, and consider some cases depending on the subposet of P formed by $\{x, y, z\}$. Taking advantage of duality, we need only consider the following four situations (we write $u \| v$ when u and v are incomparable).

Case A: $x < z$ and $y < z$ in P.

Case B: $y < z$, $x \| y$ and $x \| z$ in P.

Case C: $\{x, y, z\}$ is a 3–element antichain.

Case D: $x < z$, $x \| y$ and $y \| z$ in P.

We will show that there exists an absolute constant γ_0 so that if Cases A, B or C hold, then one of the three pairs (x, y), (y, z) and (x, z) is an incomparable pair witnessing $\delta(P) \geq 3/11 + \gamma_0$. Case D will require a more detailed analysis.

For integers i, j, let $p(i, j)$ denote the probability that $h(y) - h(x) = i$ and $h(z) - h(y) = j$. In arguments to follow, we will continue to use the previous definitions for b, B, a_i, b_i, ϵ for the pair (x, y). When y and z are incomparable, the quantities which are analogous to b and B will be denoted r and R, respectively.

6. Cases A, B and C, the easy cases

In this section, we show that Cases A, B and C lead to a stronger conclusion than given in Theorem 1.2. In fact, in Cases A and B, we show that P satisfies the 1/3—2/3 conjecture. Throughout this section, we assume that x, y and z have been chosen so that $h(x) \leq h(y) \leq h(z) \leq 2 + h(x)$. We begin with Case A.

Theorem 6.1. *If Case* A *holds, then*

$$1/3 \leq \text{Prob}(x < y) \leq 2/3.$$

Proof. $\text{Prob}(x > y) = B \geq b$, and $h(z) - h(x) \leq 2$ require that $2 \geq B + 2b + 3(1 - B - b)$ so that $B \geq 1/3$. ■

Theorem 6.2. *If Case* B *holds, then*

$$1/3 \leq \text{Prob}(x < y) \leq 2/3.$$

Proof. Obviously, $h(z) - h(y) = 1 + \sum_{i,j}(j-1)p(i,j) \geq 1 + p(-1,2) + 2\sum_{i \geq 3} p(-1,i) + \sum_{i \geq 2} p(1,i)$. Using $\sum_{i \geq 1} p(1,i) = \sum_{i \geq 2} p(-1,j) = b_1$ and $p(-1,2) = p(1,1) = p(-2,1) \leq b_2$ we obtain $h(z) - h(y) \geq 1 + b_2 + 2(b_1 - b_2) + (b_1 - b_2) = 1 + 3b_1 - 2b_2$.

This leads to a correlation between the height of x and y and their probability of being reversed and close to each other:

$$h(y) - h(x) \leq 2 - (h(z) - h(y)) \leq 1 - 3b_1 + 2b_2.$$

Now suppose that there are sequences $\{a_i\}_{i \geq 1}$ and $\{b_i\}_{i \geq 1}$ satisfying the conditions of Lemmas 2.1, 2.2, 2.3 and Theorem 2.4, together with

$\sum_{i\geq 1}(a_i + b_i) = 1$ and $\sum_{i\geq 1} i(a_i - b_i) \leq 1 - 3b_1 + 2b_2$ and $\sum_{i\geq 1} b_i \leq B$. Then the "packed" sequences $a_i = b(1+\epsilon)^i$ and $b_i = b(1-\epsilon)^i$ also satisfy all these conditions. Therefore we can analyze the situation with the techniques of [7]. It turns out that the worst case occurs in Case 2 with $k = 3$. For this value of k, it may be verified that $B \geq 0.335$, which is a little more than what is claimed in the statement of the theorem. ■

Before presenting the result for Case C, we need the following technical lemma which is also proved with the Kahn/Saks techniques. The basic idea behind this lemma is that a pair of incomparable points which are relatively close in expected height must be a balancing pair if they are unlikely to appear consecutively in a random linear extension.

Lemma 6.3 *For every pair γ_1, C of positive contants, there exists a positive constant γ_2 so that if x and z are incomparable points in a poset P and $0 \leq h(z) - h(x) \leq C$, then either $1/e - \gamma_1 \leq \mathrm{Prob}(x > z) \leq 1 + \gamma_1 - 1/e$ or $b = \mathrm{Prob}(h_L(z) - h_L(x) = 1) \geq \gamma_2$.* ■

Theorem 6.4. *There exists an absolute constant $\gamma_3 > 0$ so that if Case C holds, then $\delta(P) > \gamma_3 + 3/11$.* ■

Sketch of the proof. Apply Lemma 6.3 with $\gamma_1 = 0.1$ and $C = 2$, and let γ_2 be the constant provided by the theorem. We may then assume that the probability that z covers x is at least γ_2. It follows that either $B - b \geq \gamma_2/2$ or $R - r \geq \gamma_2/2$. A straightforward calculation then shows that at least one of $\mathrm{Prob}(x > y)$ and $\mathrm{Prob}(y > z)$ must be larger than $3/11$ by a postive constant expressible as a function of γ_2. ■

We close this section with the following easy technical extension to Theorems 6.1, 6.2 and 6.4. This result allows us to increase the difference in expected heights between x and z to a quantity strictly larger than 2.

Theorem 6.5. *For all (sufficiently small) $\gamma_4, \gamma_5 > 0$, there exists a $\gamma_6 > 0$ so that if x and y satisfy $h(x) \leq h(y) \leq 1 + \gamma_4 + h(x)$ and $B - b \geq \gamma_5$, then $\delta(P) > 3/11 + \gamma_6$.* ■

7. An application of linear programming

In this section, we begin to analyze Case D. Recall that $B = \mathrm{Prob}(x > y)$ and $R = \mathrm{Prob}(y > z)$. We state the following result using the notation introduced in Section 5.

Theorem 7.1. *$B + R \geq 6/11$ so that $\delta(P) \geq 3/11$.*

Sketch of proof. We outline the steps necessary to prove this inequality. The detailed computations are straightforward. First, produce a lower bound on $h(z) - h(x)$ by assuming that

(1.) $h(z) - h(x) = 1$ whenever $h_L(x) - h_L(y) \geq 2$.

(2.) $h(z) - h(x) = 1$ whenever $h_L(y) - h_L(z) \geq 2$.

(3.) $h(z) - h(x) = 6$ whenever $h_L(y) - h_L(x) \geq 2$, $h_L(z) - h_L(y) \geq 2$ and $h_L(z) - h_L(x) \geq 6$.

Use this lower bound and the inequality $2 \geq h(z) - h(x)$ to obtain the following inequality:

$$5(B+R)+4p(1,1)+2p(1,2)+2p(2,1)+2p(2,2)+p(2,3)+p(3,2) \geq 4. \quad (7.1)$$

Note that

$$p(1,1) + p(1,2) = p(-1,2) + p(-1,3) \leq B \quad (7.2)$$

and

$$p(1,1) + p(2,1) = p(2,-1) + p(3,-1) \leq R. \quad (7.3)$$

Also note that

$$\sum_j p(2,j) \leq \sum_j p(1,j) + p(3,j) \leq 1 - B - \sum_j p(2,j) \quad (7.4)$$

so that

$$\sum_j p(2,j) \leq 1/2 - B/2. \quad (7.5)$$

Using a symmetric inequality, we obtain

$$\sum_j p(2,j) + p(j,2) \leq 1 - (B+R)/2. \quad (7.6)$$

Next, since $2p(1,1) = p(2,-1) + p(-1,2)$, observe that

$$2p(1,1) + p(1,2) + p(2,1) + 2p(2,2) \leq \sum_j p(2,j) + p(j,2) \leq 1 - (B+R)/2. \quad (7.7)$$

The desired result follows by applying inequalities (7.2), (7.3) and (7.7) to the left hand side of (7.1). ■

As will be clear from the results which follow, Theorem 7.1 does something more than just prove a special case of Theorem 1.2. Since we are combining a series of inequalities, it is clear that we get an improvement whenever any of the individual inequalities used in the argument are not tight.

8. The cross product conjecture

We made some effort to construct a poset which would show that the inequality obtained in Theorem 7.1 is best possible, but were not successful. We then turned our attention to the infinite width two semiorder which violates the 1/3–2/3 conjecture. In this poset, the following identity holds.

$$p(1,1)p(2,2) = p(1,2)p(2,1) \tag{8.1}$$

Consideration of several examples leads us to make the following "cross product" conjecture.

Conjecture 8.1. *Let x, y and z be distinct points in a poset P and let $p(i,j)$ denote the probability that x is i positions below y and y is j positions below z. Then*

$$p(1,1)p(2,2) \leq p(1,2)p(2,1). \qquad \blacksquare$$

We can provide some additional motivation for this conjecture with the following result.

Theorem 8.2. *If x, y and z are three distinct points of a poset P, $h(x) \leq h(y) \leq h(z) \leq 2 + h(x)$, $x < z$ in P, and the cross product Conjecture 8.1 is true, then $\delta(P) \geq (5-\sqrt{5})/10$.*

Sketch of proof. Clearly, we may assume that x is incomparable to y and y is incomparable to z. To simplify the computational efforts, we let $X = B+R$, $x_1 = p(1,1)$, $x_2 = p(1,2)+p(2,1)$, $x_3 = p(2,2)$, $x_4 = p(1,3)+p(3,1)$, $x_5 = p(2,3) + p(3,2)$ and $x_6 = p(1,4) + p(4,1)$. Then inequality (7.1) becomes

$$5X + 4x_1 + 2x_2 + 2x_3 + x_5 \geq 4. \tag{8.2}$$

We can add $p(1,3)$ and $p(3,1)$ to the left sides of (7.2) and (7.3) respectively. Then add the resulting inequalities to obtain

$$X \geq 2x_1 + x_2 + x_4. \tag{8.3}$$

We then observe that

$$p(2,3) \leq p(1,3) + p(1,4) \tag{8.4}$$

and

$$p(3,2) \leq p(3,1) + p(4,1) \tag{8.5}$$

so that

$$x_5 \leq x_4 + x_6. \tag{8.6}$$

Since the sum of all probability is one, we have

$$X + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 1. \tag{8.7}$$

The cross product inequality implies

$$x_1 x_3 \leq (x_2^2)/4. \tag{8.8}$$

With some calculations, one can show that subject to the inequalities (8.2), (8.3), (8.6), (8.7), and (8.8), the minimum value of X is $(5-\sqrt{5})/5$. ■

We believe that a more general form of the cross product conjecture is valid.

Conjecture 8.3. *Let x, y and z be distinct points in a poset P and let $p(i,j)$ denote the probability that x is i positions below y and y is j positions below z. If i and j are positive, then*

$$p(i,j)p(i+1,j+1) \leq p(i,j+1)p(j+1,i). \qquad ■$$

Although we have not been able to verify the cross product inequality in general, we have been able to verify Conjecture 8.3 for width two posets.

9. Improving the Kahn/Saks bound

In this section, we outline the proof of the following theorem; the critical part of the argument involves a more detailed analysis of Case D.

Theorem 9.1. *There exists an absolute constant $\gamma_0 > 0$ so that if P is a poset and is not a chain, then $\delta(P) > 3/11 + \gamma_0$.* ■

Sketch of proof. Let $|P| = n$; without loss of generality, we may assume $n \geq 6$. Choose a set $U = \{u_1, u_2, \ldots, u_6\}$ of six points from P so that

$$h(u_1) \leq h(u_2) \leq \cdots \leq h(u_6) \leq 5 + h(u_1).$$

Using the results of Section 6, we may assume that

(1.) $u_i \| u_{i+1}$ in P, for $i = 1, 2, 3, 4, 5$.

(2.) $u_i < u_j$ in P, whenever $j \geq i + 2$.

Also, we may assume that $h(u_{i+1})$ is almost exactly $1 + h(u_i)$. For the remainder of this outline, we assume $h(u_{i+1})$ is exactly $1 + h(u_i)$. The steps necessary to handle the approximations are routine. Now we return to the linear programming argument in Section 7. It follows that we may apply the arguments in this section to each triple $(x, y, z) \in \{(u_1, u_2, u_3,), (u_2, u_3, u_4), (u_3, u_4, u_5), (u_4, u_5, u_6)\}$. It is easy to see that the inequality $B + R \geq 6/11$ is tight only when

(3.) $p(1, 1) = p(-1, 2) = p(2, -1) = 2/11$, and

(4.) $p(1, 2) = p(2, 1) = p(-1, 3) = p(3, -1) = p(2, 2) = 1/11$.

Let M be a linear extension of U. Consider the probability q_M that a random linear extension of P has the points of U appearing consecutively in the same order as M. It is straightforward to use properties (3.) and (4.) to show that if the inequality $\delta(P) \geq 3/11$ is tight, then $q_M = 1/11$. However, this is impossible since there are 13 linear extensions of U. The contradiction is enough to show that $\delta(P)$ exceeds $3/11$ by some absolute constant. ■

10. Concluding remarks

We consider the research techniques behind the results outlined in this paper as perhaps of greater importance than the results themselves. First, we have done considerable experimentation with computers and several commercially available optimization packages. A linear programming package first discovered the proof in section 3. This result and the fact that the solution is unique is key to making any improvement in the Kahn/Saks bound. Similarly, a non-linear solver first discovered that the cross product inequality is sufficient to show $\delta(P) \geq (5 - \sqrt{5})/10$ in Case D. We also found the computer of great value in analyzing the proof of Theorem 6.2. Numerical evidence suggests that in the case of a 3–element antichain, we can actually conclude that $\delta(P) \geq 0.3$. If this is true, then the proof of the cross product conjecture would imply that $\delta(P) \geq (5 - \sqrt{5})/10$ for any poset P which is not a chain.

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