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Extremal Problems for Finite Sets
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Progress and New Directions in Dimension Theory for Finite Partially Ordered Sets

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0. Introduction

In this paper we discuss combinatorial problems for finite partially ordered sets, using the concept of dimension as a unifying theme. A special effort is made to provide a sketch of the background behind the problems and to indicate why efforts to resolve them are worthwhile. Some new results are included, but our emphasis is on providing a motivating framework for future research.

We consider a *partially ordered set,* or *poset* for short, as a pair (X, P) where X is a set (usually finite) and P is a reflexive, antisymmetric and transitive binary relation on X. We prefer to write $x \leq y$ in P rather than $(x, y) \in P$. Also, when x and y are incomparable in P, we will write $x \| y$ in P. The *dimension* of a poset (X, P), denoted $\dim(X, P)$, is the least positive integer t for which there exists a family $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ of linear extensions of P so that $P = L_1 \cap L_2 \cap \cdots \cap L_t$. We refer the reader to the survey articles [36], [59], and [62] and the monograph [63] for additional background material on partially ordered sets and dimension theory. Additional combinatorial problems for finite posets are given in [61].

The remainder of this article is divided into eight sections with each section featuring one or two central problems. In each section, we will also present some related conjectures and research problems.

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1. Planarity and Genus

One of the most exciting developments in dimension theory since the concept was first introduced by Dushnik and Miller [15] some 50 years ago is Schnyder's recent theorem [47] which characterizes planar graphs in terms of the dimensions of associated posets. For a graph $\mathbf{G} = (V, E)$, define the *vertex-edge* poset $\mathbf{P_G}$ as the poset (X, P) where $X = V \cup E$ and $x < e$ in P if and only if $x \in V$, $e \in E$ and x is an end point of e. This poset is also called the *incidence* poset of G.

Theorem 1.1. (Schnyder [47]). *A graph* $\mathbf{G}$ *is planar if and only if the dimension of its vertex-edge poset* $\mathbf{P_G}$ *is at most three.* ■

Schnyder's theorem is a particularly striking result, and its proof contains some novel concepts for planar triangulations which are of substantial independent interest. Schnyder also proposed to investigate the dimension of the poset consisting of the vertices, edges and faces of a planar map, partially ordered by inclusion. For a planar map M, we call this poset the *vertex-edge-face* poset of M. The *vertex-face* poset of a planar map is defined similarly. Recall that a poset (X, P) is *t-irreducible* for some $t \geq 2$ if $\dim(X, P) = t$ and $\dim(X - \{x\}, P(X - \{x\})) = t - 1$, for every $x \in X$.

Theorem 1.2. (Brightwell/Trotter [11], [12]). *Let M be a planar map.*

i. *The dimension of the vertex-edge-face poset of M is at most 4, even if M is allowed to have loops and multiple edges.*
ii. *If M is a 3-connected planar map with no loops or multiple edges, then the dimension of the vertex-edge-face poset of M is exactly 4. In fact, the vertex-face poset of M is 4-irreducible.* ■

There are several important observations to be made about the preceding theorem. First, by the well-known characterization theorem of Steinitz [52], the 3-connected planar maps with no loops or multiple edges are exactly those planar maps associated with convex polytopes in $\mathbb{R}^3$. Second, while Theorem 1.2 can be considered as a generalization of Theorem 1.1 and contains an alternate proof of the primary direction of Schnyder's theorem, it requires some new ideas for planar graphs which are of independent interest. Third, in the proof of Theorem 1.2, the second part is actually proved first, and the first part is then derived from it. In this derivation, the proof of the first part is reduced to the case where M is a 2-connected map with no loops or multiple edges.

Problem 1. Let G be a graph of genus $n > 0$ and let M be the map obtained by embedding G without edge crossings on a sphere with n handles. Find the maximum value $f(n)$ of the dimension of the vertex-edge-face poset of M. ■

We comment that the existence of the function $f(n)$ in Problem 1. is easily established by induction on n, once the result is known for $n = 0$. Of course, Theorem 1.2 asserts that $f(0) = 4$, but a weaker result, say $f(0) \leq 10^{10}$, would suffice to prove that the function $f(n)$ exists.

Problem 2. Find a simple argument, avoiding the elaborate machinery of Theorems 1.1 and 1.2, to show that the dimension of the vertex-edge-face poset of a planar map is at most 10^{10}. ■

Problem 3. Which planar maps have vertex-edge-face posets with dimension exactly 4? ■

Yannakakis [71] showed that the problem "Is $\dim(X, P) \leq t$?" is NP-complete for each fixed $t \geq 3$.

Problem 4. For fixed $t \geq 4$, is it NP-complete to determine whether the dimension of the vertex–edge poset of a graph is at most t? ■

Note that the answer to Problem 4 is no when $t = 3$, since by Schnyder's theorem, this is equivalent to testing the graph for planarity.

There is no analogue of Schnyder's theorem for genus larger than zero; the complete bipartite graph $K_{m,m}$ has large genus when m is sufficiently large, but the dimension of its vertex–edge poset is at most 4. Similarly, there is no analogue of the second part of Theorem 1.2 for convex polytopes in $\mathbb{R}^d$ when $d \geq 4$. This is due to the fact that when $d \geq 4$, there exist convex polytopes in $\mathbb{R}^d$ for which there exists an arbitrarily large subset S of the vertex set V so that each pair of vertices from S determines an edge of the convex polytope. For this reason, the vertex–edge poset has arbitrarily large dimension (see Theorem 3.2). However, there may still be interesting bounds in terms of other combinatorial parameters of the poset.

The following result is an easy exercise.

Theorem 1.3. *There exists an absolute constant $c > 0$ so that if G is a graph with chromatic number $n \geq 1$, then the dimension of the vertex–edge poset of G is at most $c \log \log n$.* ■

On the other hand, Křiz and Nešetřil [39] have constructed a family of graphs $\{G_n : n \geq 1\}$ so that the dimension of the vertex–edge poset of G_n is at most 10 for each $n \geq 1$, yet the chromatic number of G_n tends to infinity as n increases.

2. Geometric Containment Orders

Let $\mathcal{F}$ be any family of sets. We call a poset (X, P) an *$\mathcal{F}$-containment order* if it is possible to choose for each $x \in X$ a set S_x in $\mathcal{F}$ so that $x \leq y$ in P if and only if $S_x \subseteq S_y$.

Example. [15]. If $\mathcal{F}$ is the set of closed intervals of $\mathbb{R}$, then a finite poset is an $\mathcal{F}$-containment order if and only if it has dimension at most two. ■

There are several quite natural instances of geometric containment orders which have been studied in the literature:

1. Angular regions in the plane [22], [23], [24]
2. Polygonal regions in the plane [28], [45], [49]
3. Spheres in $\mathbb{R}^d$ [9], [13], [21], [47]

The special case of spheres (disks) in the plane $\mathbb{R}^2$ has attracted the attention of many researchers and is the source of a most perplexing problem. Perhaps these posets should be called *disk containment orders,* but most authors have called them *circle containment orders.* Observe that by the example given previously, any two-dimensional poset is a circle containment order; in fact, we may require their centers to be collinear. It follows from the "degrees of freedom" theory developed by Alon and Scheinerman [1] that not all four-dimensional posets are circle containment orders.

For an integer $n \geq 3$, let $\mathcal{F}_n$ consist of all polygonal regions in the plane $\mathbb{R}^2$ whose boundary is a regular polygon with n sides with one side having a fixed orientation (say horizontal). We prefer to refer to a poset which is an $\mathcal{F}_n$-containment order as an *n-gon containment order.*

Example. A finite poset is a 3-gon containment order if and only if it has dimension at most three. ■

Many researchers have independently discovered the following elementary result.

Theorem 2.1. *Let (X, P) be a finite poset of dimension at most 3. Then (X, P) is an n-gon containment order, for each $n \geq 3$.* ■

Consider a fixed poset (X, P) of dimension at most 3, and let the integer n in Theorem 2.1 tend to infinity. Then n-gons are very close to being circles, suggesting the following problem.

Problem 5. Is every finite poset of dimension 3 a circle containment order? ■

The qualifying word "finite" cannot be deleted from this problem. Scheinerman and Weirman [46] showed that the countably infinite poset $\mathbb{Z}^3$ is not a circle containment order, and Hurlbert [32] has found an even simpler proof that $\mathbb{N}^3$ is not a circle containment order.

The importance of this question continues to grow. Kobbe and Andriev proved that every planar graph $\mathbf{G} = (V, E)$ can be represented by a family $\{D_x : x \in V\}$ of disks in the plane so that the disks have disjoint interiors and xy is an edge in G exactly when D_x and D_y are tangent. Scheinerman [47] has used this theorem to establish the following analogue of Schnyder's result.

Theorem 2.2 (Scheinerman [47]). *A finite graph G is planar if and only if its vertex-edge poset is a circle containment order.* ■

In recent years, Kobbe and Andriev's theorem has been given new proofs by Thurston [53], Pulleyblank [40], and Brightwell and Scheinerman [10]. In each case, the new proofs actually give somewhat stronger results. Many interesting questions remain.

Problem 6. Find the minimum family $\mathcal{C}$ of posets so that a poset is a circle containment order if and only if it does not contain a poset from $\mathcal{C}$ as a subposet. ■

Problem 7. Is it NP-complete to determine whether a poset is a circle containment order? ■

The following tantalizing problem appears in Brightwell and Winkler [13].

Problem 8. Does there exist a finite poset which is not a sphere containment order in $\mathbb{R}^d$, for any $d \geq 1$? ■

3. Families of Subsets

For an integer $n \geq 1$, let $[n] = \{1, 2, \ldots, n\}$. Then let $\binom{[n]}{k} = \{S \subseteq [n] : |S| = k\}$, for each $k = 0, 1, 2, \ldots, n$. Also, for distinct positive integers $k_1, k_2, \ldots, k_t \leq n$, let $P(k_1, k_2, \ldots, k_t; n)$ denote the poset (X, P) where $X = \cup_{j=1}^{t} \binom{[n]}{k_j}$ and $S \leq T$ in P if and only if $S \subseteq T$.

Example. For $n \geq 3$, the poset $P(1, n-1; n)$ is an n-irreducible poset on $2n$ points. This is called the "standard" example of an n-dimensional poset, and is also denoted $\mathbf{S_n}$. ■

The following result gives the precise dimension of $P(1, k; n)$ when k is relatively large in comparison to n, in particular when $2\sqrt{n} \leq k < n$.

Theorem 3.1. (Dushnik [14]). *Let $n \geq 4$ and let j be any integer with $2 \leq j \leq \lfloor n^{1/2} \rfloor$. If k is a positive integer satisfying $\left\lfloor \frac{n+j^2-j}{j} \right\rfloor \leq k - 1 \leq \left\lfloor \frac{n+(j-1)^2-j+1}{j-1} \right\rfloor$, then* $\dim(P(1, k; n)) = n - j + 1$. ■

For fixed k and n tending to infinity, the following estimates are known.

Theorem 3.2. (Spencer [50]).

(i) $\dim(P(1, 2; n)) = (1 + o(1)) \lg \lg n$.

(ii) For each $k \geq 2$, there exists a constant $c_k > 0$ so that $\dim(P(1, k; n)) \leq c_k \lg \lg n$. ■

Little else is known about $\dim(P(1, k; n))$, although some special cases are treated in [64].

Problem 9. Estimate $\dim(P(1, k; n))$, the dimension of the poset of 1 and k element subsets of an n element set ordered by inclusion when $2 \leq k < 2\sqrt{n}$. ■

Hurlbert has investigated some more general families and has obtained the following inequalities for families symmetric about the middle of the subset lattice.

Theorem 3.3. (Hurlbert [31]).

(i) $n - 2 \leq \dim(P(2, n-2; n) \leq n - 1$.

(ii) If $n > 3k$, then $n - k \leq \dim(P(k, n-k; n)) \leq n - 1$. ■

Problem 10. For large n, is $\dim(P(2, n-2; n)$ equal to $n-2$ or $n-1$? ■

Problem 11. For fixed k and n suitably large, is $\dim(P(k, n-k; n)) = \dim(P(1, n-k; n))$? ■

The estimation of the dimension of the middle two levels of the subset lattice is of particular interest. The only known inequalities are the following trivial bounds $\lg \lg n \leq \dim(P(1, 2; n+2)) \leq \dim(P(n, n+1; 2n+1) \leq 2n$.

Problem 12. Estimate $\dim(P(n, n+1; 2n+1))$, the dimension of the poset consisting of the middle two levels of the subset lattice of all subsets of a $2n+1$ element set. ■

4. Removal Theorems

One of the best known inequalities in the dimension theory is the following important result, known as Hiraguchi's inequality.

Theorem 4.1. (Hiraguchi [30]). *If (X, P) is a poset, then* $\dim(X, P) \leq |X|/2$, *when* $|X| \geq 4$. ■

The original proof of this inequality, as well as a somewhat more polished version due to Bogart [6], proceeded by induction on $|X|$. By ad-hoc techniques, the result is first established for small posets, those with $4 \leq |X| \leq 7$. The general result follows then by showing that if (X, P) is any poset with $|X| \geq 8$, then either (i) (X, P) contains a two element subset $S = \{x, y\}$ so that $\dim(X, P) \leq 1 + \dim(X - S, P(X - S))$, or (ii) X contains a four element subset $T = \{x, y, z, w\}$ so that $\dim(X, P) \leq 2 + \dim(X - T, P(X - T))$.

In 1974, Kimble [38] and Trotter [56] independently found a new proof of Hiraguchi's inequality by establishing the following result.

Theorem 4.2. (Kimble [38], Trotter [56]). *Let (X, P) be a poset and let A be an antichain in (X, P). Then* $\dim(X, P) \leq \max\{2, |X - A|\}$. ■

The inequality in 4.2 combined with the elementary bound, $\dim(X, P) \leq \text{width}(X, P)$, yields 4.1. However, the following natural question remains open (see also Kelly [35]).

Problem 13. Let (X, P) be a poset with $|X| \geq 3$. Is it always the case that X contains a two element subset $S = \{x, y\}$ so that $\dim(X, P) \leq 1 + \dim(X - S, P(X - S))$. ■

Problem 13 appears to be surprisingly difficult. A number of properties have been given so that if a two element set satisfies any one of them, then its removal decreases the dimension by at most one. Examples of such removal theorems may be found in [6], [8], [31] [65] and [66].

Let (X,P) be a poset and let $(x,y) \in X \times X$. We call (x,y) a *critical pair* (also, a *nonforced pair*) if (1) $x \| y$ in P and (2) for all $z, w \in X$, $z < x$ in P implies $z < y$ in P and $y < w$ in P implies $x < w$ in P. In [65] Trotter conjectured that the removal of any critical pair decreases the dimension by at most one.

Example. (Reuter [43]). In the poset on Fig. 1, (x,y) is a critical pair. However, (X,P) has dimension 4 and removing the 2 element set $\{x,y\}$ leaves a two-dimensional poset. ■

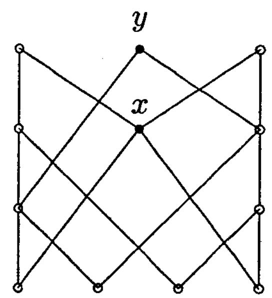

Figure 1.

Example. (Kierstead and Trotter [37]). For $n \geq 5$, the poset on Fig. 2 is n dimensional, and (x,y) is a critical pair. However, removing x and y leaves an $n-2$ dimensional poset. ■

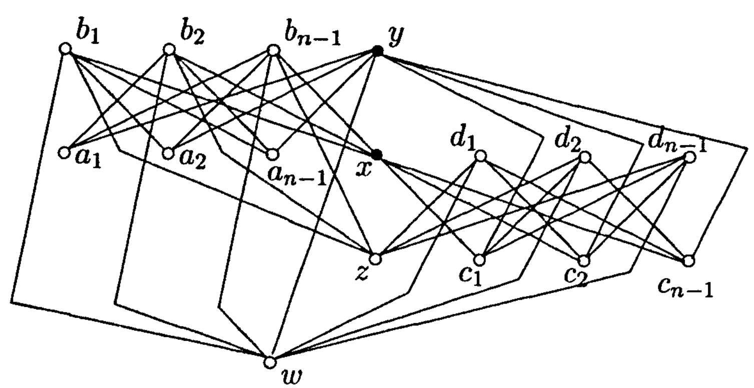

Figure 2.

Despite this negative evidence, many researchers believe that the answer to the following question (see [35] and [36]) is yes.

Problem 14. Let (X,P) be a poset with $|X| \geq 3$. Is it always the case that for every $x \in X$, there exists some $y \in X - \{x\}$ so that removing the 2 element subset $\{x,y\}$ decreases the dimension by at most one? ■

In [58], Trotter gave a forbidden subposet characterization of the inequality in Theorem 4.2. The following characterization problem will no doubt be even harder.

Problem 15. For each $n \geq 4$, find the minimum list $\mathcal{C}_n$ of posets so that if (X,P) is a poset of width n, then $\dim(X,P) < n$ unless it contains a poset from $\mathcal{C}_n$ as a subposet. ■

We require $n \geq 4$ in Problem 15 because the problem is trivial for $n = 2$ and easily solved for 3. When $n = 3$, we need only examine the list of all 3-irreducible posets (see [33] or [69]) and choose those which have width 3. If Problem 15 is really difficult, perhaps the reason is an affirmative answer to the following question.

Problem 16. For each $n \geq 3$, is it NP-complete to determine whether a width n poset has dimension n? ■

Clearly, the answer to the question in Problem 16 is no when $n = 1$, 2 or 3, so it only makes sense to consider $n \geq 4$.

Peter Fishburn [20] has posed the following challenging problem.

Problem 17. For each $n \geq 2$, find the minimum number of incomparable pairs in a poset of dimension n. ■

Although Fishburn noted $f(3) = 7$, he conjectured $f(n) = n^2$ for $n \geq 4$ and that the standard example $\mathbf{S_n}$ is the unique extremal poset. Qin [41] has verified this conjecture when $n = 4$, but his argument is very difficult. For $n = 5$, Qin is only able to show $f(5) \geq 24$, and this requires a very complicated argument.

The following dual problem is also of interest.

Problem 18. What is the minimum number of comparable pairs in a poset of dimension n? ■

I suspect that the answer to Problem 18 may be as small as $cn^2/\log^2 n$. It is certainly no more than $cn^2/\log n$, by the results of Section 6.

5. Interval Orders and Shift Graphs

For $n \geq 4$, let G_n denote the graph whose vertex set is $\binom{[n]}{3}$ and whose edge set consists of those pairs $\{\{i_1, i_2, i_3\}\ \{i_2, i_3, i_4\}\}$ with $1 \leq i, < i_2 < i_3 < i_4 \leq n$. Erdős and Hajnal [16] showed that the chromatic number of G_n is $(1 + o(1))\lg\lg n$. In fact the chromatic number of G_n is exactly the least t so that the lattice of all subsets of $\{1, 2, \ldots, t\}$ contains at least n antichains. So $\chi(G_n) = \lg\lg n + (\frac{1}{2} + o(1))\lg\lg\lg n$. A poset (X, P) is called an **interval order** if it is isomorphic to a poset (Y, Q) where Y is a family of closed intervals of the real line $\mathbf{R}$ and $[a, b] < [c, d]$ in Q if and only if $b < c$ in $\mathbf{R}$.

Rabinovitch [42] proved that the dimension of an interval order can be bounded in terms of its height, i.e., the maximum number of points in a chain (see also [7]).

Theorem 5.1. (P. Hajnal, Füredi, Rödl, Trotter). *The maximum value $f(n)$ of the dimension of an interval order of height n is given by:*

$$f(n) = \lg\lg n + \left(\frac{1}{2} + o(1)\right)\lg\lg\lg n. \qquad \blacksquare$$

To show an appropriate lower bound for the function $f(n)$ is easy. The canonical interval order $\mathbf{I}_n$ consisting of all intervals with integer endpoints from $\{1, 2, \ldots, n\}$ has dimension at least as large as the chromatic number of the double shift graph G_n defined above. To show an upper bound of the same form is the real challenge. This is accomplished in two steps. First, we show that if $m = \binom{t}{\lfloor \frac{t}{2} \rfloor}$ and $n \leq 2^m$, then $\dim(\mathbf{I}_n)$ is at most $t + 3$. It is just a technical step to extend the proof to general interval orders.

Many interesting problems involving dimension of interval orders remain.

Problem 19. Does there exist a function $f : \mathbf{N} \to \mathbf{N}$ so that if $n \geq 4$ and (X, P) is any interval order of dimension at least $f(n)$, then (X, P) contains a subposet isomorphic to the canonical interval order of all intervals with integer end points from $[n]$? $\blacksquare$

Results obtained by Felsner and Morvan [19] concerning staircases in interval orders of large dimension appear to support an affirmative answer to Problem 19.

Problem 20. What is the maximum value of the dimension of an interval order of width n? $\blacksquare$

Problem 21. For fixed $t \geq 3$, is it NP-complete to determine whether the dimension of an interval order is at most t? $\blacksquare$

Problem 22. For integers n, t, how many t-irreducible interval orders have exactly n points? $\blacksquare$

6. Random Posets

In [17], Erdős, Kierstead and Trotter investigated the dimension of random posets using the following model. The point set X is the union of two antichains $A \cup A'$. Each of A and A' are n element antichains. The only pairs $(x, y) \in X \times X$ for which it is possible to have $x < y$ in P belong to $A \times A'$. For each $(a, a') \in A \times A'$, take $\text{Prob}[a < a' \text{ in } P] = p$ where $p = p(n)$ satisfies $0 < p < 1$. Furthermore, events corresponding to distinct pairs from $A \times A'$ are independent. Call the resulting sample space $\Omega(n, p)$ Among the principal results of [ ] are:

Theorem 6.1. (Erdős, Kierstead, Trotter [17]).

i. *For every $\varepsilon > 0$, there exists $\delta > 0$ so that if $n^{-1+\varepsilon} < p \leq 1/\log n$, then $\dim(X, P) > \delta pn \log pn$, for almost all $(X, P) \in \Omega(n, p)$.*
ii. *For every $\varepsilon > 0$, there exists $\delta > 0$ so that if $1/\log n \leq p < 1 - n^{-1+\varepsilon}$, then $\dim(X, P) > \max\left\{\delta n,\ n - \frac{n \log 1/p}{\delta \log n}\right\}$, for almost all $(X, P) \in \Omega(n, p)$.*
iii. *There exist positive constants c_1, c_2 so that if $p = 1/2$, then $n - \frac{c_1 n}{\log n} < \dim(X, P) < n - \frac{c_2 n}{\log n}$, for almost all $(X, P) \in \Omega(n, 1/2)$.* $\blacksquare$

One of the most significant consequences of Theorem 6.1 is the information we gain about the relative tightness of upper bounds on dimension expressed in terms of maximum degree. For a poset (X, P) and a point $x \in X$, define the **degree** of x to be the number of elements comparable to x. Then let $\Delta(X, P)$ denote the **maximum degree** among all the points in X. Rödl and Trotter first showed the existence of a function $f : \mathbf{N} \to \mathbf{N}$ so that $f(k)$ is the maximum dimension of a poset (X, P) for which $\Delta(X, P) \leq k$.

Theorem 6.2. (Füredi/Kahn [25]). *If (X, P) is a poset, $|X| = n \geq 2$ and $\Delta(X, P) = k \geq 1$, then*

i. $\dim(X, P) \leq 10k \log n$, *and*

ii. $\dim(X, P) \leq 50k(\log k)^2$ ■

It is interesting to note that the inequalities in Theorem 6.2 are universal and apply for all posets satisfying the hypothesis as opposed to holding for almost all posets in some probability space. However, probabilistic techniques are used in proving the two inequalities. In particular, the so-called Lovász Local Lemma [18] is essential to the proof of the second part.

It is easy to see that Theorem 6.1 implies that the first inequality in Theorem 6.2 is best possible up to the value of the constant $c_1 = 10$. Although, we do not entirely settle the question of the accuracy of the second part, we at least get a super-linear lower bound.

Problem 23. For each $k \geq 2$, let $f(k)$ denote the maximum dimension of a poset with $\Delta(X, P) = k$, It is known that there are positive constants c_1, c_2 so that $c_1\, k(\log k) < f(k) < c_2\, k(\log k)^2$. Provide better estimates for $f(k)$. ■

Strangely enough there is a technical problem whose resolution could shed some light on Problem 23. We pose the problem using the notation of Section 2.

Problem 24. There exist positive constants c_1, c_2 so that $c_1(\log n)^2 < \dim(P(1, \log n; n)) < c_2(\log n)^3$, for all $n \geq 10$. Provide a better estimate for $\dim(P(1, \log n; n))$. ■

We comment that if one can show an upper bound of the form $\dim(P(1; \log n)\,;\, n) < c_3(\log n)^{2+\delta}$ where $0 \leq \delta < 1$, then there is a positive constant c_4 so that the dimension of a poset (X, P) with $\Delta(X, P) = k \geq 2$ does not exceed $c_4 k(\log k)^{1+\delta}$.

A listing of problems in the dimension theory for random posets is given in [17]. Here are some of the most important ones.

Problem 25. For fixed $t = 3, 4, 5, \ldots$, what is the threshold probability p_t for which we first expect that a random poset will have dimension at least t? ■

Problem 26. What is the expected value of $\dim(X, P)$ when p is very small, say $p < n^{-1+\epsilon}$? Or when p is very close to one, say $p > 1 - n^{-1+\epsilon}$. ■

Problem 27. What is the expected value of $\dim(X, P)/n$ when $p = 1/\log n$. ■

Here is an interesting extremal problem for which these random techniques yield partial results. For integers n, k with $3 \leq k \leq n/2$, let $f(n, k)$ be the maximum dimension of a poset on n points which does not contain a k dimensional standard example $\mathbf{S}_k$. Although $f(3, 7) = 3$, we know $f(k, 2k+1) = k - 1$ for all $k \geq 4$. However, this is extraordinarily difficult to show (see [38], for example).

Problem 28. Find a short proof that for $k \geq 4$, any poset on $2k + 1$ points which does not contain $\mathbf{S}_k$ has dimension at most $k - 1$. ■

Problem 29. Find the maximum value $f(n, k)$ of a poset on n points which does not contain a k-dimensional standard example $\mathbf{S}_k$ as a subposet. ■

When $1 \ll k \ll n$, the random model gives a lower bound of the form $f(n, k) \geq n^{1-\frac{1}{k}}$. This is obtained simply by using probability $p = n^{-\frac{1}{k}}$.

Dimension problems for two other models of random posets have been investigated. Erdős, Kierstead and Trotter [17] also studied the set of all labelled posets on n elements with each one being equally likely. They produced upper and lower bounds on the expected value of the dimension of a random poset in this sample space.

Problem 30. Let $\Omega(n)$ consist of all labelled posets (X, P) with $X = [n]$. Find the positive constant c so that $\lim_{n\to\infty} \Big(n - E(\dim(X, P))\Big)/\log n = c$. ■

In [2] Albert and Frieze study random posets defined as follows. First choose a random graph G on the vertex set $[n]$. Then direct all edges from the smaller to larger vertices and set $i < j$ in P if there is a directed path from i to j in G. They show that the expected value of the dimension of a random poset is almost surely at least $\frac{1-\epsilon}{2}\sqrt{\frac{2\log n}{\log 2}}$ and at most $\sqrt{\frac{2\log n}{\log 2}}$. They conjecture that the dimension is sharply concentrated at the upper bound.

Although it does not necessarily make a well-defined problem, I strongly suspect that it is worthwhile investigating the dimension of height one posets whose minimal elements form a ground set S and whose maximal elements correspond to blocks in a design on S. The partial order is defined in the natural way: an element $x \in S$ is less than a block B if $x \in B$. This problem is particularly interesting in the case of a (v, k, λ)-design.

7. Crowns and Cycles

Baker, Fishburn and Roberts [4] first used the term *crown* to identify one of the 3-irreducible posets shown on Fig. 3.

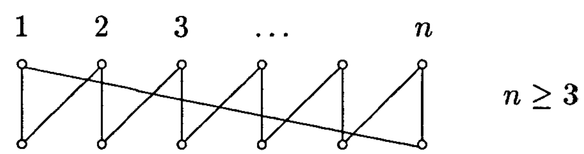

Figure 3.

For integers n, k with $n \geq 3$ and $k \geq 0$, Trotter [54] defined the *generalized crown* $\mathbf{S}_n^k$ as a height one poset with $n+k$ minimal elements $a_1, a_2, \ldots, a_{n+k}$ and $n+k$ maximal elements $b_1, b_2, \ldots, b_{n+k}$. The only order relations are those of the form $a_i < b_j$ where j is congruent to one of the following integers (modulo $n+k$) $i+k+1, i+k+2, i+k+3, \ldots, i+k+n-1$. The original 3-irreducible crowns are those in subfamily $\{\mathbf{S}_3^k : k \geq 0\}$, and the standard examples are those in $\{\mathbf{S}_n^0 : n \geq 0\}$. Trotter [54] gave the general formula $\dim(\mathbf{S}_n^k) = \lceil 2(n+k)/(k+2) \rceil$ and showed that for each $t \geq 3$, there are infinitely many (generalized) crowns which are t-irreducible.

Define a poset (X, P) to be *transitive* if for every $x, y \in X$ with height (x) = height (y), there exists an isomorphism φ of (X, P) with $\varphi(x) = y$. Of course, a generalized crown is transitive, so that for every $t \geq 3$, there are infinitely many t-irreducible transitive posets. Other examples of irreducible transitive posets have been constructed by Kelly [34], but Kelly's construction produces just one poset for each $t \geq 3$.

Problem 31. Let $t \geq 4$. Are there infinitely many t-irreducible transitive posets which are not generalized crowns? ■

A poset (X, P) is said to be *cycle-free* if its comparability graph is a rigid circuit graph, i.e., it does not contain induced cycles of four or more vertices. Alternately, a poset is cycle-free if it does not contain any 3-irreducible crown as a subposet and it does not contain the 4-element poset shown on Fig. 4 as a subposet.

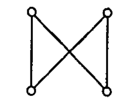

Figure 4.

Theorem. (Spinrad and Ma [51]). *If (X, P) is a cycle-free poset, then* $\dim(X, P) \leq 4$. ■

Qin [41] has found a clever argument to show the existence of 4-dimensional cycle-free posets, but Qin's example is not 4-irreducible. In fact, Qin's example contains more than 10^{40} points. It must certainly be the case that a smaller example exists.

Problem 32. Find all 4-irreducible cycle-free posets and find one with less than 100 points. ■

If there are infinitely many 4-irreducible cycle-free posets, then the following problem makes sense.

Problem 33. Is it NP-complete to determine whether a cycle-free poset has dimension at most 3? ■

The rigid circuit graphs are exactly those graphs which are the intersection graphs of families of subtrees of a (graph-theoretic) tree. However, not all such graphs are comparability graphs, and it is useful to consider cycle-free posets in terms of posets which are themselves called trees.

A poset is called a *forest* if it does not contain the 3-element poset shown on Fig. 5 as a subposet.

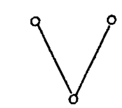

Figure 5.

A poset is a *tree* if it is a forest and it has a greatest element. Every component of a forest is a tree. Of course, a linear order is a special case of a tree. As noted by Wolk [70], the dimension of a tree is at most 2. It is an easy exercise to see that posets which are forests or trees can be represented by subgraphs of graph theoretic trees in the following manner. Consider a graph theoretic tree T and a family $\mathcal{T}$ of subtrees of T satisfying:

Intersection/Inclusion Property. If $T_1, T_2 \in \mathcal{T}$, $T_1 \neq T_2$ and $T_1 \cap T_2 \neq \emptyset$, then either $T_1 \subseteq T_2$ or $T_2 \subseteq T_1$.

Clearly, the poset $(\mathcal{T}, \subseteq)$ is a forest. Just on its own merits, it is of interest to revisit the subject of geometric containment orders and consider tree-like families $\mathcal{F}$ which satisfy the Intersection/Inclusion property given above.

To connect this subject with cycle-free posets, we consider the following construction. Let T be a graph theoretic tree and let $\mathcal{A}, \mathcal{B}$ be families of subtrees of T each satisfying the Intersection/Inclusion property. Define a poset $(\mathcal{A} \cup \mathcal{B}, P)$ by:

(1) $A_1 \leq A_2$ in P if $A_2 \subseteq A_1$, for all $A_1, A_2 \in \mathcal{A}$,

(2) $B_1 \leq B_2$ in P if $B_1 \subseteq B_2$, for all $B_1, B_2 \in \mathcal{B}$, and

(3) $A \leq B$ in P if $A \cap B \neq \emptyset$, for all $A \in \mathcal{A}$, $B \in \mathcal{B}$.

It is an easy exercise to show that $(\mathcal{A} \cup \mathcal{B}, P)$ is cycle-free. Furthermore, every cycle-free poset has such a representation.

Problem 33. Let $\mathcal{A}$ and $\mathcal{B}$ satisfy the Intersection/Inclusion property. Define a partial order on P by the three rules given for cycle-free posets. What posets arise and what can be said about their dimension if $\mathcal{A}$ and $\mathcal{B}$ are families of :

(1) Intervals of the real line?

(2) Boxes in $\mathcal{R}^d$?

(3) Disks in $\mathcal{R}^2$?

(4) Spheres in $\mathcal{R}^d$?

(5) Arcs on a circle? ∎

The concept of dimension can be generalized in many ways. While we have limited our discussion to the original Dushnik/Miller concept of dimension, we cannot help mentioning one new variant. Behrendt [5] proposed to define the *tree-dimension* of a poset (X, P) as the least number t for which there exist partial orders $P_1, P_2, \ldots, P_t$ on X so that $P = P_1 \cap P_2 \cap \cdots \cap P_t$ and (X, P_i) is a tree, for each $i = 1, 2, \ldots, t$.

Problem 35. Let (X, P) be a poset with tree-dimension t. If we remove a point from (X, P), by how much can the tree-dimension decrease? increase? ∎

Finally, before leaving the subject of trees, we should remark that Trotter [68] proposed to call any poset a *tree* if its cover graph is a graph theoretic tree. This is a more inclusive definition of a tree and allows them to have dimension 1, 2, or 3. In fact, dimension 3 is characterized by the two posets of Fig. 6, both of which are 3-irreducible.

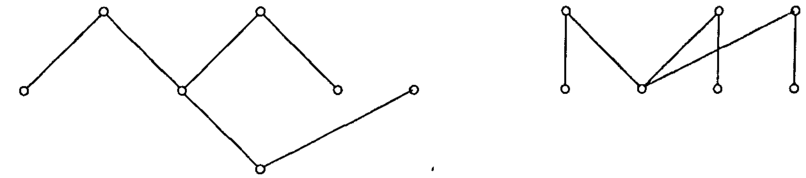

Figure 6.

8. Lattices and Products

B. Sands posed the following problem.

Problem 36. For each $n \geq 2$ what is the smallest $f(n)$ so that there is a lattice L on $f(n)$ points so that L is an n-dimensional poset? ∎

Sands remarked that the subset lattice shows $f(n) \leq 2^n$, and in the absence of a better example, asked whether $f(n)$ might equal 2^n. Ganter, Nevermann, Reuter and Stahl [27] showed that the dimension of the lattice of all partitions of an n-element set is at least $\frac{3}{8}\binom{n}{2}$ and at most $\binom{n}{2}$. This shows $f(n) < c^{\log^2 n}$. However, Füredi and Kahn [26] have shown that the poset of points and lines of a finite projective plane or order m has dimension at least $m/2 \log m$. This shows $f(n) < cn^2 \log^2 n$, but it is not even known whether $f(n)/n$ tends to infinity. Attila Sali [44] has a new approach to this problem and has proposed an extremal set theory conjecture whose resolution would imply that $f(n)/n \to \infty$.

If (X, P) and (Y, Q) are posets, the *cartesian product* $(X, P) \times (Y, Q)$ is the poset (Z, R) where $Z = X \times Y$ and $R = \{((x_1, y_1), (x_2, y_2)) : (x_1, x_2) \in P$ and $(y_1, y_2) \in Q\}$. Trivially, $\dim((X, P) \times (Y, Q)) \leq \dim(X, P) + \dim(Y, Q)$, and Baker [3] showed that equality holds if (X, P) and (Y, Q) have distinct greatest and least elements, i.e., both have a "zero" and a "one."

Problem 37. For integers $m, n \geq 3$, what is the least $f(m, n)$ for which there exist posets (X, P) and (Y, Q) so that $\dim(X, P) = m$, $\dim(Y, Q) = n$ and $\dim((X, P) \times (Y, Q)) = f(m, n)$? ∎

Problem 38. For $n \geq 3$, what is the least $g(n)$ for which there exists an n-dimensional poset (X, P) for which $\dim((X, P) \times (X, P)) = g(n)$? ∎

In [59] Trotter showed $f(n, n) \leq g(n) \leq 2n-2$ by proving that $\dim(\mathrm{S}_n \times \mathrm{S}_n) = 2n-2$ for $n \geq 3$. It is tempting to conjecture that $f(m, n) = m+n-2$ and $g(n) = 2n-2$, but I believe that $f(m, n)$ is much closer to $\max\{m, n\}$

and $g(n)$ is much closer to n. We just don't know enough about posets to come up with the right examples.

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