ELSEVIER

Discrete Mathematics 136 (1994) 101–117

DISCRETE MATHEMATICS

On the fractional dimension of partially ordered sets

Stefan Felsner[a,b,1], William T. Trotter[a,c,*]

[a] *Bell Communications Research, 445 South Street, 2L-367, Morristown, NJ 07962, USA*
[b] *Technische Universität Berlin, Fachbereich Mathematik, Strasse des 17. Juni 135, 1000 Berlin 12, Germany*
[c] *Department of Mathematics, Arizona State University, Tempe AZ 85287, USA*

Received 23 September 1993; revised 21 December 1993

Abstract

We use a variety of combinatorial techniques to prove several theorems concerning fractional dimension of partially ordered sets. In particular, we settle a conjecture of Brightwell and Scheinerman by showing that the fractional dimension of a poset is never more than the maximum degree plus one. Furthermore, when the maximum degree k is at least two, we show that equality holds if and only if one of the components of the poset is isomorphic to $\boldsymbol{S}_{k+1}$, the 'standard example' of a $k+1$-dimensional poset. When $w \geqslant 3$, the fractional dimension of a poset $\boldsymbol{P}$ of width w is less than w unless $\boldsymbol{P}$ contains $\boldsymbol{S}_w$. If $\boldsymbol{P}$ is a poset containing an antichain A and at most n other points, where $n \geqslant 3$, we show that the fractional dimension of $\boldsymbol{P}$ is less than n unless $\boldsymbol{P}$ contains $\boldsymbol{S}_n$. If $\boldsymbol{P}$ contains an antichain A such that all antichains disjoint from A have size at most $w \geqslant 4$, then the fractional dimension of $\boldsymbol{P}$ is at most $2w$, and this bound is best possible.

Keywords: Partially poset; Dimension; Fractional dimension; Degree

1. Introduction

In this paper, we consider a *partially ordered set* $\boldsymbol{P}$ as a pair (X, P), and refer to X as the *ground* set and P as the *partial order*. We find it convenient to use the short form *poset* for a partially ordered set, and we use the term *subposet* to refer to a poset induced by a subset of the ground set.

Let $\boldsymbol{P}=(X,P)$ be a poset and let $\mathscr{F}=\{M_1,\dots,M_t\}$ be a multiset of linear extensions of P. Brightwell and Scheinerman [3] call $\mathscr{F}$ a *k-fold realizer* of P if for each incomparable pair (x, y), there are at least k linear extensions in $\mathscr{F}$ which reverse the pair (x, y), i.e., $|\{i\colon 1 \leqslant i \leqslant t,\ x > y \text{ in } M_i\}| \geqslant k$. The *fractional dimension* of $\boldsymbol{P}$, denoted by $\operatorname{fdim}(\boldsymbol{P})$, is then defined in [3] as the least real number $q \geqslant 1$ for which there exists

[1] The research was supported by DFG under grant FE-340/2-1.
* Corresponding author.

SSDI 0012-365X(94)00109-V

a k-fold realizer $\mathscr{F}=\{M_1,\dots,M_t\}$ of P so that $k/t\geqslant 1/q$ (it is easily verified that the least upper bound of such real numbers q is indeed attained). Using this terminology, the *dimension* of $\boldsymbol{P}$, denoted by $\dim(\boldsymbol{P})$, is just the least t for which there exists a 1-fold realizer of P. It follows immediately that $\mathrm{fdim}(\boldsymbol{P})\leqslant\dim(\boldsymbol{P})$, for every poset $\boldsymbol{P}$.

In this paper, we will find it convenient to use the following probabilistic interpretation of this concept. Let $\boldsymbol{P}=(X,P)$ be a poset and let $\mathscr{F}=\{M_1,\dots,M_t\}$ be a multiset of linear extensions of P. We consider the linear extensions of $\mathscr{F}$ as outcomes in a uniform sample space. For an incomparable pair (x,y), the *probability that x is over y in $\mathscr{F}$* is given by

$$\mathrm{Prob}_{\mathscr{F}}[x>y]=\frac{1}{t}\,|\{i\colon 1\leqslant i\leqslant t,\ x>y \text{ in } M_i\}|.$$

The fractional dimension of $\boldsymbol{P}$ is then the least rational number $q\geqslant 1$ so that there exists a multiset $\mathscr{F}=\{M_1,\dots,M_t\}$ of linear extensions of P with $\mathrm{Prob}_{\mathscr{F}}[x>y]\geqslant 1/q$, for every incomparable pair (x,y).

For each $n\geqslant 3$, the height two poset containing n minimal elements $\{x_1,x_2,\dots,x_n\}$ and n maximal elements $\{y_1,y_2,\dots,y_n\}$ with $x_i<y_j$ if and only if $i\neq j$, for all $i,j=1,2,\dots,n$ will be denoted by $\boldsymbol{S}_n$. The poset $\boldsymbol{S}_n$ is known as the *standard example* of a n-dimensional poset. As noted in [3], $\mathrm{fdim}(\boldsymbol{S}_n)=\dim(\boldsymbol{S}_n)=n$.

This paper is organized as follows. In Section 2, we introduce some terminology necessary for our theorems and review facts about lexicographic sums. In Section 3, we formulate the results relating the fractional dimension of a poset to its degree. In Section 4, we present a key technical lemma involving the algorithmic transformation of linear orders into linear extensions and present the proof for the main theorem of Section 3. In Section 5, we present the forbidden subposet characterization of the degree inequality. In Sections 6 and 7, we derive two other forbidden subposet characterizations of inequalities for fractional dimension. In both cases equality is only possible when P contains a ‘full dimensional’ standard example as a subposet.

In Section 8, we present an inequality relating the fractional dimension of a poset $\boldsymbol{P}=(X,P)$ to the width of the subposet induced by $X-A$, where A is an antichain. The bounds obtained in this case are slightly stronger than the corresponding bounds for dimension. We also prove that our bounds are best possible. In Section 9, we characterize Hiraguchi's inequality for fractional dimension. Finally, in Section 10, we propose some new problems for fractional dimension.

2. Notation and terminology

Let $\boldsymbol{P}=(X,P)$ be a poset. We denote the set of incomparable pairs of $\boldsymbol{P}$ by $\mathrm{inc}(\boldsymbol{P})$. Now let $\mathscr{F}=\{M_1,M_2,\dots,M_t\}$ be a nonempty multiset of linear orders on X. We call $\mathscr{F}$ a *multirealizer* of P if $P=\bigcap\mathscr{F}$, i.e., each M_i is a linear extension of P and $\mathrm{prob}_{\mathscr{F}}[x>y]>0$, for every $(x,y)\in\mathrm{inc}(\boldsymbol{P})$.

When $\mathscr{F}$ is a multirealizer of P, we define the *value* of $\mathscr{F}$, denoted $\text{value}_P(\mathscr{F})$, as the least rational number $q \geqslant 1$, so that $\text{prob}_{\mathscr{F}}[x>y] \geqslant 1/q$, for every $(x, y) \in \text{inc}(\boldsymbol{P})$. The fractional dimension of $\boldsymbol{P}$ is just the least rational $q \geqslant 1$ so that $\boldsymbol{P}$ has a multirealizer $\mathscr{F}$ of value $1/q$.

Recall that an incomparable pair (x, y) in a poset $\boldsymbol{P}=(X, P)$ is *critical* if any point less than x is less than y and any point greater than y is greater than x. We denote the set of all critical pairs of $\boldsymbol{P}$ by $\text{crit}(\boldsymbol{P})$. The following elementary proposition, first noted by Rabinovitch and Rival [13], is stated formally to emphasize the fundamentally important role played by critical pairs in dimension theory.

Proposition 2.1. *Let $\boldsymbol{P}=(X, P)$ be a poset and let $\mathscr{F}$ be a multiset of linear extensions of P. Then $\mathscr{F}$ is a multirealizer of P if and only if $\text{prob}_{\mathscr{F}}[x>y]>0$, for every $(x, y) \in \text{crit}(\boldsymbol{P})$.*

We say that a linear order L *reverses* the critical pair (x, y) if $x>y$ in L. Similarly, we say L *reverses* the set $S \subset \text{crit}(\boldsymbol{P})$ if L reverses each pair in S.

The following remark is noted in [3].

Proposition 2.2. *Let $\boldsymbol{P}=(X, P)$ be poset and let $q \geqslant 1$ be a rational number. Then $\text{fdim}(\boldsymbol{P}) \leqslant q$ if and only if there exists a multiset $\mathscr{F}$ of linear extensions of P so that $\text{Prob}_{\mathscr{F}}[x>y] \geqslant 1/q$, whenever $(x, y) \in \text{crit}(\boldsymbol{P})$.*

Let $\boldsymbol{P}=(X, P)$ be a poset, and let $\mathscr{D}=\{\boldsymbol{Y}_x=(Y_x, Q_x)\colon x \in X\}$ be a family of posets indexed by the point set X of $\boldsymbol{P}$. The *lexicographic sum* of $\mathscr{D}$ over $\boldsymbol{P}$ is the poset $\boldsymbol{S}$ whose point set consists of all pairs of the form (x, y_x), where $x \in X$ and $y \in Y_x$. The partial order on $\boldsymbol{S}$ is defined by setting $(x_1, y_{x_1}) \leqslant (x_2, y_{x_2})$ in $\boldsymbol{S}$ if and only if

(1) $x_1<x_2$ in P, or

(2) $x_1=x_2$ and $y_{x_1} \leqslant y_{x_2}$ in Q_{x_1}.

Both dimension and fractional dimension are well behaved with respect to lexicographic sums. The statement for dimension is due to Hiraguchi [10], and the statement for fractional dimension is given in [3].

Proposition 2.3. *Let $\boldsymbol{S}$ be the lexicographic sum of $\mathscr{D}=\{\boldsymbol{Y}_x\colon x \in X\}$ over $\boldsymbol{P}=(X, P)$. Then*

(1) $\dim(\boldsymbol{S})=\max\{\dim(\boldsymbol{P}), \max_{x \in X} \dim(\boldsymbol{Y}_x)\}$.

(2) $\text{fdim}(\boldsymbol{S})=\max\{\text{fdim}(\boldsymbol{P}), \max_{x \in X} \text{fdim}(\boldsymbol{Y}_x)\}$.

A poset $\boldsymbol{S}$ is *decomposable* with respect to lexicographic sums if it is isomorphic to a lexicographic sum of a family $\mathscr{D}=\{\boldsymbol{Y}_x\colon x \in X\}$ over a poset $\boldsymbol{P}=(X, P)$, where $|X| \geqslant 2$, and $|Y_x| \geqslant 2$ for at least one $x \in X$. Otherwise, we say $\boldsymbol{S}$ is *indecomposable*.

Given a point x in a poset $\boldsymbol{P}=(X, P)$, let $D_P(x)$ denote the set of all points which are less than x in P, and $U_P(x)$ the set of points which are greater than x in P. Whenever the meaning is clear from the context, we will drop the subscripts and just write $D(x)$ and $U(x)$. Two points x and y are said to be *interchangeable* in a poset $\boldsymbol{P}=(X, P)$ if

$D(x)=D(y)$ and $U(x)=U(y)$. Thus x and y are interchangeable if and only if both (x,y) and (y,x) are critical pairs. Note that a poset with an interchangeable pair is decomposable with respect to lexicographic sums *unless* it is a 2-element antichain.

3. Fractional dimension and degree

For a point x in a poset $\boldsymbol{P}=(X,P)$, define the *degree of* x, denoted $\deg(x)$, as the number of points comparable (but not equal) to x in $\boldsymbol{P}$. Then let $\Delta(\boldsymbol{P})$ denote the *maximum degree* of $\boldsymbol{P}$, i.e., $\Delta(\boldsymbol{P})=\max\{\deg(x)\colon x\in X\}$.

The following theorem is proved by Brightwell and Scheinerman in [3].

Theorem 3.1. *If* $\boldsymbol{P}=(X,P)$ *is a poset, then* $\operatorname{fdim}(\boldsymbol{P})\leqslant 2+\Delta(\boldsymbol{P})$.

In Section 4, we prove the following improved bound which was conjectured in [3].

Theorem 3.2. *If* $\boldsymbol{P}=(X,P)$ *is a poset and is not an antichain, then* $\operatorname{fdim}(\boldsymbol{P})\leqslant 1+\Delta(\boldsymbol{P})$.

As noted in [3], $\operatorname{fdim}(\boldsymbol{S}_n)=n=1+\Delta(\boldsymbol{S}_n)$, for all $n\geqslant 3$, so Theorem 3.2 is best possible. In Section 4, we show that when the maximum degree of a connected poset $\boldsymbol{P}$ is at least 2, the inequality in Theorem 3.2 is strict, unless $\boldsymbol{P}$ is the standard example of dimension $1+\Delta(\boldsymbol{P})$. This result has much the same flavor as Brooks' theorem [4], which asserts that the chromatic number of a connected graph is at most one more than the maximum degree, with equality holding if and only if the graph is a complete graph or an odd cycle. It also provides another instance in which the role played by standard examples in dimension theory is analogous to the role played by complete graphs in the theory of graph coloring. This parallel is explored in greater detail in the monograph [17].

We will in fact prove a result that is slightly stronger than Theorem 3.2. Given a point x in a fixed poset $\boldsymbol{P}=(X,P)$, let $D[x]=D(x)\cup\{x\}$. Then let $\deg_D(x)=|D(x)|$, and define $\Delta_D(\boldsymbol{P})=\max\{\deg_D(x)\colon x\in X\}$. The set $U[x]$ and the quantities $\deg_U(x)$, $\Delta_U(\boldsymbol{P})$ are defined dually.

With this notation, we can now state the result we actually prove.

Theorem 3.3. *If* $\boldsymbol{P}=(X,P)$ *is a poset, then* $\operatorname{fdim}(\boldsymbol{P})\leqslant 1+\Delta_D(\boldsymbol{P})$.

Note that Theorem 3.3 has Theorem 3.2 as an immediate corollary. Since fractional dimension is dual, i.e., a poset and its dual have the same fractional dimension, the following result will also be an immediate corollary to Theorem 3.3.

Corollary 3.4. *If* $\boldsymbol{P}=(X,P)$ *is a poset, then* $\operatorname{fdim}(P)\leqslant 1+\min\{\Delta_D(P),\Delta_U(P)\}$.

4. Linear orders and linear extensions

In [8], Füredi and Kahn showed that dimension theory problems could be formulated in terms of linear orders on the ground set instead of linear extensions. Let $\boldsymbol{P}=(X,P)$ be a poset and let L be any linear order on the ground set X. When $S\subset X$ and $x\in X-S$, we write $x>S$ in L when $x>s$ in L, for every $s\in S$. We let $\mathscr{C}(L)=\{(x,y)\colon (x,y)\in \mathrm{crit}(\boldsymbol{P})$ and $x>D[y]$ in $L\}$. Füuredi and Kahn noted that the following statement holds.

Proposition 4.1. *Let $\boldsymbol{P}=(X,P)$ be a poset and let L be any linear order on X. Then there exists a linear extension M of P with $\mathscr{C}(L)\subseteq\mathscr{C}(M)$.*

Given a poset $\boldsymbol{P}=(X,P)$ and a linear order L on X, we let $\mathscr{C}^*(L)=\mathscr{C}(L)\cup\{(x,y)\colon (x,y)$ is a critical pair, $x>D(y)$ in L, and $|D(y)|=\Delta_D(\boldsymbol{P})\}$.

Lemma 4.2. *Let $\boldsymbol{P}=(X,P)$ be a poset with no interchangeable pairs, and let L be any linear order on X. Then there exists a linear extension M of P with $\mathscr{C}^*(L)\subseteq\mathscr{C}(M)$.*

Proof. In order to simplify arguments to follow, we present an algorithmic proof. We describe a deterministic algorithm, LinEx, which takes an arbitrary linear order L on X as input and outputs a linear extension $M=\mathrm{LinEx}(L)$ satisfying the conclusion of the lemma.

Set $L_0=L$; for $j\geqslant 0$, we obtain L_{j+1} from L_j by moving an element of X to an alternate position according to rules which we describe below. The procedure halts when we have a linear extension $M=L_j$ satisfying the conclusion of the lemma. We divide the process into two phases. The first phase just transforms L into a linear extension N with $\mathscr{C}(L)\subseteq\mathscr{C}(N)$ and $\mathscr{C}^*(L)\subseteq\mathscr{C}^*(N)$, so in this phase, we will already be proving a modest extension of Proposition 4.1.

Phase 1: *Obtaining a linear extension.* This phase ends when L_j is a linear extension of P. If it is not, among all pairs (u,v) with $u<v$ in P and $v<u$ in L_j, choose one for which the number of points between them in L_j is minimum. If there is more than one such pair, we choose one for which v is as low as possible in L_j.

Claim 1. *If w is between u and v in L_j, then w is incomparable with both u and v.*

Proof. Suppose to the contrary that w is between u and v in L_j, but that w is comparable to u. Now $w<u$ in L_j. Also, there are fewer points between w and u than between u and v. If $u<w$ in P, then we contradict our choice of the pair (u,v), as we would prefer the pair (u,w). On the other hand, if $w<u$ in P, then $w<v$ in P, and we would prefer (v,w) to (u,v). The argument when a point between u and v is comparable to v is dual.

Now form L_{j+1} from L_j by moving v to a point immediately over u. From Claim 1, we note that L_{j+1} preserves exactly one more comparable pair of $\boldsymbol{P}$ than does L_j. This insures that Phase 1 will terminate.

Claim 2. $\mathscr{C}(L_j)\subseteq\mathscr{C}(L_{j+1})$.

Proof. Suppose to the contrary that there exists a critical pair $(x,y)\in\mathscr{C}(L_j)-\mathscr{C}(L_{j+1})$, and suppose v was placed over u in the construction of L_{j+1}. Then $v\in D[y], v<x$ in L_j, but $x<v$ in L_{j+1}. However, $u<v$ in P now implies that $u<y$ in P. Now $(x,y)\in\mathscr{C}(L_j)$ requires $x>u$ in L_j. This already gives $x>v$ in L_{j+1}, a contradiction. □

The argument for Claim 2 also establishes the next claim, which we state for emphasis.

Claim 3. $\mathscr{C}^*(L_j)\subseteq\mathscr{C}^*(L_{j+1})$.

We may now assume that Phase 1 terminates with a linear extension N of P.

Phase 2: *Doing more.* Set $N_0=N$. This phase produces a sequence $\{N_j: j\geqslant 0\}$ of linear extensions of P and terminates when $M=\text{LinEx}(L)=N_j$ satisfies $\mathscr{C}^*(M)=\mathscr{C}(M)$. At each intermediate stage, we will preserve the key properties that $\mathscr{C}(N_j)\subseteq\mathscr{C}(N_{j+1})$ and $\mathscr{C}^*(N_j)\subseteq\mathscr{C}^*(N_{j+1})$.

Suppose that there is some pair $(x,y)\in\mathscr{C}^*(N_j)-\mathscr{C}(N_j)$. Choose one such pair and note that $x<y$ in N_j, but $z<x$ in N_j for each $z\in X$ with $z<y$ in P.

Claim 4. *No point between* x *and* y *in* N_j *is comparable to* y.

Proof. Suppose to the contrary that u is between x and y in N_j, but that u is comparable to y. Since N_j is a linear extension of P, we know that $u<y$ in P. However, this contradicts the assumption that $(x,y)\in\mathscr{C}^*(L)$. □

Form the linear order N_{j+1} from N_j by moving y immediately below x. In view of Claim 4, we know that N_{j+1} is a linear extension of P.

Claim 5. $\mathscr{C}(N_j)\subseteq\mathscr{C}(N_{j+1})$.

Proof. Suppose to the contrary that $(u,v)\in\mathscr{C}(N_j)-\mathscr{C}(N_{j+1})$. Then it follows that $u=y$. Since $|D(y)|$ is maximum, y is maximal element in $\boldsymbol{P}$. However, since (y,v) is a critical pair, we know $D(y)\subseteq D(v)$, and thus $D(y)=D(v)$. So v is also a maximal point, and $U(y)=U(v)=\emptyset$. We conclude that y and v are interchangeable. □

Exactly the same argument establishes the following claim.

Claim 6. $\mathscr{C}^*(N_j)\subseteq\mathscr{C}^*(N_{j+1})$.

Phase 2 terminates because $\mathscr{C}(N_j)$ is a proper subset of $\mathscr{C}(N_{j+1})$, as the second set contains (x, y), but the first does not. This completes the proof of the lemma. $\square$

We now present the proof of Theorem 3.3.

Proof of Theorem 3.3. Clearly, we may assume that $\boldsymbol{P}$ is indecomposable with respect to lexicographic sums; in particular, $\boldsymbol{P}$ has no interchangeable pair.

Let $|X| = n$ and set $t = n!$. Then consider the set $\mathscr{F} = \{L_1, L_2, \ldots, L_t\}$ of all linear orders on the ground set X. Note that for a point x and a set S with $x \notin S$, the fraction of linear orders in which x is higher than all points in S is exactly $1/(|S|+1)$. It follows, that a critical pair (x, y) with $|D(y)| < k$ is in $\mathscr{C}(L)$ for $n!/(|D[y]|+1)$ different permutations L. Also, a critical pair (x, y) with $|D(y)| = k$ is in $\mathscr{C}^*(L)$ for $n!/(|D(y)|+1)$ different permutations L.

Apply the algorithm LinEx defined in the previous section, and let $\mathscr{S} = \{M_1, M_2, \ldots, M_t\}$, where $M_i = \text{LinEx}(L_i)$, for each $i = 1, 2, \ldots, t$. Lemma 4.2 implies $\text{Prob}_{\mathscr{S}}[x > y] \geqslant 1/(|D[y]|+1)$, when $|D(y)| < k$ and $\text{Prob}_{\mathscr{S}}[x > y] \geqslant 1/(|D(y)|+1)$, when $|D(y)| = k$. Thus, $\text{Prob}_{\mathscr{S}}[x > y] \geqslant 1/(k+1)$, for every critical pair (x, y) in $\boldsymbol{P}$. $\square$

We remark that local exchange arguments of the type used in this argument have also been applied in [5].

5. Characterizing the case of equality

As noted by Brightwell and Scheinerman, the standard example of an n-dimensional poset satisfies fdim $(\boldsymbol{P}) = 1 + \Delta(\boldsymbol{P})$, so Theorem 3.2 is best possible. In this section, we will show that the standard examples are the only connected posets for which the inequality in Theorem 3.2 is tight. Before presenting the details of the argument, we need a little more notation.

Let $\boldsymbol{P} = (X, P)$ be a poset, and let $\mathscr{F}$ be a multirealizer of $\boldsymbol{P}$ with value $(\mathscr{F}) = r$. We define $\text{ess}(\mathscr{F}) = \{(x, y) \in \text{crit}(\boldsymbol{P})\colon \text{prob}_{\mathscr{F}}[x > y] = 1/r\}$. The pairs in $\text{ess}(\mathscr{F})$ are called the *essential* pairs of $\mathscr{F}$.

We call a multirealizer $\mathscr{F}$ of a poset $\boldsymbol{P} = (X, P)$ an *optimal* family for $\boldsymbol{P}$ if value $(\mathscr{F}) = \text{fdim}(\boldsymbol{P})$. Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be multirealizers of $\boldsymbol{P}$. We denote by $\mathscr{F}_1 \cup \mathscr{F}_2$ the union of the two multisets, defined so that if a linear order appears n times in $\mathscr{F}_1$ and m times in $\mathscr{F}_2$, it appears $n+m$ times in $\mathscr{F}_1 \cup \mathscr{F}_2$. The following elementary proposition is stated formally for emphasis.

Proposition 5.1. *If $\mathscr{F}_1$ and $\mathscr{F}_2$ are optimal multirealizers of a poset $\boldsymbol{P}$, then $\mathscr{F}_1 \cup \mathscr{F}_2$ is also an optimal multirealizer. Furthermore,*

$$\text{ess}(\mathscr{F}_1 \cup \mathscr{F}_2) = \text{ess}(\mathscr{F}_1) \cap \text{ess}(\mathscr{F}_2).$$

We may then say that a critical pair $(x,y)\in\text{crit}(\boldsymbol{P})$ is an *essential pair of* $\boldsymbol{P}$ if $(x,y)\in\text{ess}(\mathscr{F})$, for every optimal multirealizer $\mathscr{F}$. We let $\text{ess}(\boldsymbol{P})$ denote the set of all essential pairs of $\boldsymbol{P}$. Note that $\text{ess}(\boldsymbol{P})$ is always nonempty when $\boldsymbol{P}$ is not a chain. When L is a linear order on X, we let $\text{ess}(L)=\mathscr{C}(L)\cap\text{ess}(\boldsymbol{P})$ and $\text{ess}^*(L)=\mathscr{C}^*(L)\cap\text{ess}(\boldsymbol{P})$. The next statement is an immediate consequence of the preceding defintions.

Proposition 5.2. *Let* $\boldsymbol{P}=(X,P)$ *be a poset and let* $\mathscr{F}$ *be an optimal multirealizer for* $\boldsymbol{P}$. *If* $M_1, M_2\in\mathscr{F}$ *and* $\text{ess}(M_1)\subseteq\text{ess}(M_2)$, *then* $\text{ess}(M_1)=\text{ess}(M_2)$.

We are now ready to present our characterization of Theorem 3.3.

Theorem 5.3. *If* $\boldsymbol{P}=(X,P)$ *is a poset with* $\Delta_D(\boldsymbol{P})\geqslant 2$, *then* $\text{fdim}(\boldsymbol{P})<1+\Delta_D(\boldsymbol{P})$, *unless* $\boldsymbol{P}$ *contains a standard example of a poset of dimension* $1+\Delta_D(\boldsymbol{P})$.

Proof. Let $\boldsymbol{P}=(X,P)$ be a poset, $k=\Delta_D(\boldsymbol{P})$, $\text{fdim}(\boldsymbol{P})=1+k$ and $|X|=n$. Without loss of generality, we may assume that $\boldsymbol{P}$ is indecomposable with respect to lexicographic sums; in particular, we may assume $\boldsymbol{P}$ has no interchangeable pair. Again, let $\mathscr{S}$ denote the family of linear extensions of P constructed in the proof of Theorem 3.3. given in the preceding section. Then $\mathscr{S}$ is optimal. We now derive some elementary statements about the essential pairs reversed by a linear extension in $\mathscr{S}$.

Claim 1. *Let* L *be any linear order, and let* $M=\textsc{LinEx}(L)$. *Then* $\text{ess}^*(L)=\text{ess}(M)$.

Proof. Let $(x,y)\in\text{crit}(\boldsymbol{P})$ and let L' be a random linear order on X. The probability that (x,y) belongs to $\text{ess}^*(L')$ is at least $1/(k+1)$. Should this inequality not be tight, then it follows that the pair (x,y) does not belong to $\text{ess}(\boldsymbol{P})$. If there is any linear order L on X for which there exists a pair $(x,y)\in\text{ess}(\textsc{LinEx}(L))-\text{ess}^*(L)$, then we would conclude that $\text{Prob}_{\mathscr{S}}[x>y]>1/(k+1)$, and thus $(x\,y)\notin\text{ess}(\boldsymbol{P})$. $\square$

Claim 2. *If* $(x,y)\in\text{ess}(\boldsymbol{P})$, *then* $|D(y)|=\Delta_D(\boldsymbol{P})$.

Proof. Suppose to the contrary that $r=|D(y)|<\Delta_D(\boldsymbol{P})=k$. If $r<k-1$, then the probability that $(x,y)\in\mathscr{C}(L)$ for a random linear order on X is $1/(r+2)$, which is greater than $1/(k+1)$. Hence $(x,y)\notin\text{ess}(\boldsymbol{P})$, a contradiction.

Now suppose $r=k-1$ and let $D(y)=\{y_1,y_2,\dots,y_{k-1}\}$. Also let L be any linear order on X so that the lowest $k+1$ elements of L are $\{x\}\cup D[y]$ with $y_1<y_2<\cdots<y_{k-1}<y<x$ in L. Let L' be obtained from L by switching the positions of x and y. Then $(x,y)\notin\text{ess}(L')$. We conclude from Claim 1 and Proposition 5.2 that there is a pair (a,b) in $\text{ess}^*(L')-\text{ess}^*(L)$. It is easy to see that this requires $(a,b)=(y,z)$, for some z which is higher than x in L. This is in turn requires $\deg_D(z)=k$ and hence $D(z)=\{x\}\cup D(y)$.

Let L'' be obtained from L by moving y_r immediately above x. Now $\text{ess}(L'')\subsetneq\text{ess}(L)$, which is a contradiction. $\square$

Proof of Theorem 5.3 (*conclusion*). Now fix an essential pair (y_{k+1}, z_{k+1}) of $\boldsymbol{P}$ and let $D(z_{k+1})=\{y_1, y_2, \ldots, y_k\}$. Then let R be any linear order on X in which the $k+1$ lowest elements are $\{y_1, y_2, \ldots, y_{k+1}\}$ with $y_1<y_2<\cdots<y_{k+1}$ in R. Note that $(y_{k+1}, z_{k+1}) \in \mathrm{ess}^*(R)$.

For each $i=1,2,\ldots,k$, let R_i be the linear order on X obtained by moving y_i immediately over y_{k+1} in R. Now choose an integer $i \in \{1,2,\ldots,k\}$. Then $(y_{k+1}, z_{k+1}) \notin \mathrm{ess}^*(R_i)$. It follows that there is a pair in $\mathrm{ess}^*(R_i)-\mathrm{ess}^*(R)$. Claim 2 implies that this pair has the form (y_i, z_i) with $|D(z_i)|=k$ and hence $D(z_i)=\{y_1, y_2, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{k+1}\}$. We conclude that the subposet of $\boldsymbol{P}$ induced by $\{y_1, y_2, \ldots, y_{k+1}\} \cup \{z_1, z_2, \ldots, z_{k+1}\}$ is a standard example of dimension $k+1$. □

We note that Theorem 15 in [3] is a special case of the preceding theorem. We also note that the characterization for Theorem 3.2 follows immediately.

Corollary 5.4. *If* $\boldsymbol{P}=(X,P)$ *is a poset with* $3 \leqslant \mathrm{fdim}(\boldsymbol{P})=1+\Delta_D(\boldsymbol{P})$, *then one of the connected components of* $\boldsymbol{P}$ *is isomorphic to the standard example of dimension* $1+\Delta_D(\boldsymbol{P})$.

6. Fractional dimension and width

The following inequality is due to Dilworth [6].

Theorem 6.1. *Let* $\boldsymbol{P}=(X,P)$ *be a poset. Then* $\dim(\boldsymbol{P}) \leqslant \mathrm{width}(\boldsymbol{P})$.

The forbidden subposet characterization problem for the inequality in Theorem 6.1 remains an unsolved (and we suspect very challenging) problem, although in [15], some infinite families of examples are constructed which must be present in the solution. Also, it is not known whether it is NP-complete to determine whether the dimension of a poset is less than its width. Using the techniques developed in the preceding sections, we can answer these questions completely for fractional dimension.

Let $\boldsymbol{P}=(X,P)$ be a poset and let S and T be disjoint subsets of X. When M is a linear extension of P, we say S is *over* T in M, and write S/T in M, when $x>y$ in M, for every $(x,y) \in \mathrm{inc}(\boldsymbol{P})$, with $x \in S$ and $y \in T$. The following elementary result is due to Hiraguchi [10].

Proposition 6.2. *Let* $\boldsymbol{P}=(X,P)$ *be a poset and let* $C \subset X$ *be a chain. Then there exist linear extensions* M_1, M_2 *of* P *with* $C/(X-C)$ *in* M_1 *and* $(X-C)/C$ *in* M_2.

Recall that Dilworth's theorem [6] asserts that a poset $\boldsymbol{P}$ of width w can be partitioned into w chains. As a consequence, the inequality in Theorem 6.1 is an immediate corollary to Proposition 6.2.

In what follows, we will concentrate on chain *decompositions* (also called *chain covers*), i.e., we will write the ground set as a union of not necessarily disjoint subsets, each of which is a chain.

Theorem 6.3. *Let* $\boldsymbol{P}$ *be a poset. Then* $\mathrm{fdim}(\boldsymbol{P}) \leqslant \mathrm{width}(\boldsymbol{P})$. *Furthermore,* $\mathrm{fdim}(\boldsymbol{P}) < \mathrm{width}(\boldsymbol{P})$, *unless*

(1) width $(\boldsymbol{P})=2$; *or*

(2) width $(\boldsymbol{P})=w$ *for some* $w \geqslant 3$, *and* $\boldsymbol{P}$ *contains the standard example of a t-dimensional poset as a subposet.*

Proof. The first statement is trivial. Suppose the second is false and choose a counterexample $\boldsymbol{P}=(X,P)$ with X as small as possible. Let $w=\mathrm{width}(\boldsymbol{P})=\mathrm{fdim}(\boldsymbol{P}) \geqslant 3$. The minimality of $|X|$ requires that fdim $(\boldsymbol{Q})<w$, for any proper subposet $\boldsymbol{Q}$ of $\boldsymbol{P}$.

Now suppose that $\boldsymbol{P}$ has fewer than w maximal elements. Then there exist a maximal element y and a chain decomposition $X=C_1 \cup C_2 \cup \cdots \cup C_w$ of $\boldsymbol{P}$ so that y belongs to two or more of the chains in the decomposition. Let $\boldsymbol{Q}=(Y,Q)$ be the subposet obtained by removing y from $\boldsymbol{P}$, and let fdim $(\boldsymbol{Q})=r<w$. Let $\mathscr{G}$ be an optimal realizer of $\boldsymbol{Q}$ and suppose that $\mathscr{G}$ consists of t linear extensions. For each $M \in \mathscr{G}$, form a linear extension M' of X by adding y at the top of M. Let $\mathscr{G}'$ denote the resulting family of linear extensions of P.

Then for each $i=1,2,\ldots,w$, let M_i be a linear extension of P with $(X-C_i)/C_i$ in M_i. Let $\mathscr{F}=\{M_1,M_2,\ldots,M_w\}$ and recall that value $(\mathscr{F})= 1/w$. Then for each $j \geqslant 2$, let $j\mathscr{F}$ denote the multirealizer of P consisting of j copies of each linear extension from $\mathscr{F}$.

Let $\mathscr{H}_j=\mathscr{G}' \cup j\mathscr{F}$ and observe, that for j sufficiently large, value $(\mathscr{H}_j)>1/w$, which is a contradiction.

The same argument with an appropriate $\mathscr{F}$ shows that for each maximal element y of $\boldsymbol{P}$, there is some $x \in X$ so that $(x,y) \in \mathrm{ess}(\boldsymbol{P})$.

Let $\{y_1,y_2,\ldots,y_w\}$ denote the set of maximal elements of $\boldsymbol{P}$, and choose an integer i from $\{1,2,\ldots,w\}$. Then choose an element $x_i \in X$ for which $(x_i,y_i) \in \mathrm{ess}(\boldsymbol{P})$. Since $(x_i,y_i) \in \mathrm{inc}(\boldsymbol{P})$, we know that $x_i \notin C_i$. Choose an integer $j=j_i$ so that $x_i \in C_j$. Then $x_i \leqslant y_j$. We now show that $x_i \leqslant y_k$ in P, for every $k=1,2,\ldots,w$, with $k \neq i$.

Suppose to the contrary that there is some integer $k \in \{1,2,\ldots,w\}$ with $k \neq i$, for which $x_i \nleqslant y_k$. It follows easily that there is a linear extension M_k' of P with $(X-C_k)/C_k$ and $x_i>y_i$ in M_k'. However, this implies that $(x_i,y_i) \notin \mathrm{ess}(\boldsymbol{P})$. The contradiction completes the proof of our assertion that $x_i \leqslant y_k$ in P, for every $k \neq 1,2,\ldots,w$, with $k \neq i$.

Now it follows that the points $\{x_1,x_2,\ldots,x_w\}$ form an antichain and the order induced on $\{x_1,x_2,\ldots,x_w\} \cup \{y_1,y_2,\ldots,y_w\}$ is isomorphic to $\boldsymbol{S}_w$. $\square$

7. The complements of antichains I

The following theorem was proved independently by Kimble [11] and Trotter [14].

Theorem 7.1. *Let* $\boldsymbol{P}=(X,P)$ *be a poset, let* $A\subset X$ *be an antichain and let* $n=|X-A|$. *Then* $\dim(\boldsymbol{P})\leqslant\max\{2,n\}$.

In [16], a forbidden subposet characterization of this inequality is obtained. The case where $|X-A|\leqslant 2$ is trivial, and the case $|X-A|=3$ includes some pathology associated with the family of 3-irreducible posets. However, for $|X-A|=n\geqslant 4$, there are exactly $2n-1$ forbidden subposets in the list.

For fractional dimension, the result is even more elegant.

Theorem 7.2. *Let* $\boldsymbol{P}=(X,P)$ *be a poset, let* $A\subset X$ *be an antichain and let* $n=|X-A|$. *Then* $\operatorname{fdim}(\boldsymbol{P})\leqslant\max\{2,n\}$. *Furthermore,* $\operatorname{fdim}(\boldsymbol{P})\leqslant\max\{2,n-(n-2)/(n-1)\}$, *unless*

(1) $n=2$ *and* $\boldsymbol{P}$ *contains a 2-element antichain; or*

(2) $n\geqslant 3$ *and* $\boldsymbol{P}$ *contains the standard example of an n-dimensional poset as a subposet.*

Proof. Again, the first statement is trivial. We provide a sketch of the proof of the second, omitting details for some routine parts of the argument. When $n\geqslant 4$, it is proved in [11] that the dimension of $\boldsymbol{P}$ is at most $n-1$, unless the n points of $X-A$ constitute an antichain on one side of A (that is, completely above or completely below A). When $n=3$, we leave it to the reader to show that fdim $(\boldsymbol{P})\leqslant\frac{5}{2}$, unless the 3 points in $X-A$ are an antichain and all three are on the same side of A. Note that this statement is not true for ordinary dimension.

To complete the proof, it is sufficient to show that the following poset has fractional dimension at most $n-1+1/(n-1)$. The antichain A is the set of minimal elements of $\boldsymbol{P}$. The maximal elements of $\boldsymbol{P}$ are the n points in $Y=X-A=\{y_1,y_2,\dots,y_n\}$. Furthermore, for each nonempty subset $S\subset\{1,2,\dots,n\}$ with $S\neq\{1,2,\dots,n-1\}$, there is an element $a_S\in A$ with $a_S<y_i$ in P if and only if $i\in S$. Note that by requiring $S\neq\{1,2,\dots,n-1\}$, we are excluding the appearance of $\boldsymbol{S}_n$ as a subposet of $\boldsymbol{P}$.

Here is the basic idea. We describe two families of linear extensions $\mathcal{M}=\{M_1,M_2,\dots,M_n\}$ and $\mathcal{R}=\{R_1,R_2,\dots,R_{n-1}\}$ such that $\mathcal{F}=\mathcal{M}\cup(n-2)\mathcal{R}$ is a $(n-1)$-fold realizer of $\boldsymbol{P}$.

To define these extensions, we first describe how they order the points of Y. Once this is done, we have the 'gaps' between these points into which the points of the antichain A will be inserted. These insertions will be done according to a specified set of rules. Finally, we will provide a rule for ordering elements of A which are inserted into the same gap.

Let $a_n=a_{\{n\}}$; note that $a_n<y_i$ in P if and only if $i=n$. Here are the rules for constructing $\mathcal{M}=\{M_1,M_2,\dots,M_n\}$:

(1) For each $i=1,2,\dots,n$, the point y_i is below all other elements of Y in M_i.

(2) The largest element of Y in M_1 is y_n.

(3) For each $i=2,3,\dots,n$, we insert $a\in A-\{a_n\}$ into the highest gap in M_i consistent with the partial order P.

(4) For each $i=1,2,\ldots,n$, if $a,a'\in A-\{a_n\}$, both belong to the same gap in M_i, and $|U(a)|>|U(a')|$, then $a>a'$ in M_i.

(5) The point a_n is the second largest element in M_1 and the first element of M_i for each $i=2,3,\ldots,n$.

Here are the rules for constructing $\mathscr{R}=\{R_1,R_2,\ldots,R_{n-1}\}$:

(1) For each $i=1,2,\ldots,n-1$, the point y_i is below all other elements of Y in R_i.

(2) For each $i=1,2,\ldots,n-1$, the point y_n is the second lowest element of Y in R_i.

(3) For each $i=1,2,\ldots,n-1$ and each $a\in A$, we insert a into the highest gap in M_i consistent with the partial order P.

(4) For each $i=1,2,\ldots,n-1$, if $a,a'\in A$, both belong to the same gap in R_i, and $|U(a)|>|U(a')|$, then $a>a'$ in R_i.

We leave it to the reader to verify that a critical pair of the form (a_S,a_n) with $n\in S$ is reverted in M_i for $i=2,\ldots,n$, while every other critical pair (x,y) is reversed in a linear extension $M\in\mathscr{M}$ and in a linear extension $R\in\mathscr{R}$. This shows that $\mathscr{F}$ is a $(n-1)$-fold realizer. Since $\mathscr{F}$ consists of $(n-1)^2+1$ linear extensions we conclude that $\mathrm{fdim}(\boldsymbol{P})\leqslant n-1+1/(n-1)$, as claimed. $\square$

It is interesting to note that we have been able to provide a discrete 'jump' in the fractional dimension in the characterization of equality in Theorem 7.2. However, there is no such jump in the characterizations of equality for Theorem 3.3.

Remark 7.3. For all $k\geqslant 2$ and every $\varepsilon>0$, there exists a poset $\boldsymbol{P}$ with $\Delta_D(\boldsymbol{P})=k$ and $k+1-\varepsilon<\mathrm{fdim}(\boldsymbol{P})<k+1$.

Let $\boldsymbol{P}$ be the incidence poset of a k-regular, k-partite hypergraph, where the sides if the k-partition all are of size n for some large $n=n_\varepsilon$.

8. The complements of antichains II

There is another inequality bounding the dimension of the complement of an antichain. This is another result of Trotter [14]. Let $A\subset X$ be an antichain in $\boldsymbol{P}$ with $X-A\neq\emptyset$, and let $\boldsymbol{Q}$ be the subposet of $\boldsymbol{P}$ induced by $X-A$.

Theorem 8.1. *Let $\boldsymbol{P}=(X,P)$ be a poset, with A and $\boldsymbol{Q}$ as above. Then* $\dim(\boldsymbol{P})\leqslant 1+2\,\mathrm{width}(\boldsymbol{Q})$.

In [16] it is shown that this inequality is best possible. The proof makes use of the *Product Ramsey Theorem* (see [9, Ch. 5]). We now show, that in the case of fractional dimension the situation changes slightly. For a poset $\boldsymbol{P}=(X,P)$ and a subset $Y\subseteq X$, we let $P(Y)$ denote the restriction of P to Y, so that $\boldsymbol{Q}=(Y,P(Y))$ is the subposet of $\boldsymbol{P}$ induced by Y.

Theorem 8.2. *Let $\boldsymbol{P}=(X,P)$ be a poset, let A be an antichain with $X-A$ nonempty, and let $\boldsymbol{Q}=(Y,P(Y))$. Then let $w=\mathrm{width}(\boldsymbol{Q})$.*

(1) *If* $w=1$ *or* $w=2$, *then* $\mathrm{fdim}(\boldsymbol{P})<1/2+2w$.
(2) *If* $w=3$, *then* $\mathrm{fdim}(\boldsymbol{P})<1/4+2w$.
(3) *If* $w\geqslant 4$, *then* $\mathrm{fdim}(\boldsymbol{P})\leqslant 2w$.
Furthermore, for $w=1$ *and* $w\geqslant 4$, *the inequalities are asymptotically best possible.*

Proof. Let $C_1, C_2, \ldots, C_w$ be a chain partition of $\boldsymbol{Q}$. With C_i^+ we denote the subchain of C_i above A and with C_i^- the subchain below A. We may assume that $C_i = C_i^+ \cup C_i^-$ for all i. Also let $Y=X-A$ and define $Y^+ = Y\cap U(A)$ and $Y^- = Y\cap D(A)$.

To prove the theorem we will construct a multirealizer $\mathscr{F}$ of $\boldsymbol{P}$. This multirealizer is the union of three blocks of linear extensions, i.e., $\mathscr{F}=\mathscr{B}_1\cup\mathscr{B}_2\cup\mathscr{B}_3$.

Block 1. For each $i=1,\ldots w$, let L_i and M_i be linear extensions with $C_i/(X-C_i)$ and $(X-C_i)/C_i$. Let $\mathscr{L}=\{L_1,\ldots,L_w\}$ and $\mathscr{M}=\{M_1,\ldots,M_w\}$. Define $\mathscr{B}_1$ as $p\mathscr{L}\cup p\mathscr{M}$, the value of $p=p(w)$ to be specified later.

Block 2. In $\mathscr{B}_2$ we have for each $i=1,\ldots,w$ a subblock $\mathscr{C}_i$ of linear extensions, i.e., $\mathscr{B}_2=\mathscr{C}_1\cup\ldots\cup\mathscr{C}_w$. Each linear extension in $\mathscr{C}_i$ has $(Y^+-C_i^+)/C_i^+$ and $C_i^-/(Y^- - C_i^-)$. Moreover, for every subset B of A with $|B|=|A|/2$ (we may assume $|A|$ to be even) there is a linear extension L_B in $\mathscr{C}_i$ in which elements of B are as low as possible, while elements of $A-B$ are as high as possible. Let $q=\binom{|A|}{|A|/2}$ denote the number of linear extensions in $\mathscr{C}_i$.

Block 3. In $\mathscr{B}_3$ we assemble r copies of a linear extension which simultaneously reverses all critical pairs (x,y) with $x,y\in A$. Assuming that there are no interchangeable pairs this can easily be done.

To count how often a critical pair (x,y) of $\boldsymbol{P}$ is reversed in $\mathscr{F}$, we have to distinguish three types.

Type 1. $x\in C_i$ and $y\in C_j$. The pair is reversed whenever $C_i/(X-C_i)$ and when $(X-C_j)/C_j$. Since $i\neq j$, this gives $2p$ reversals in $\mathscr{B}_1$.

Type 2. $x\in C_i$ and $y\in A$. The pair is reversed whenever $C_i/(X-C_i)$, giving p reversals in $\mathscr{B}_1$. However $x\in C_i^-$ and the pair is also reversed in a linear extension $L_B\in\mathscr{C}_i$ when $y\in B$. Together this gives $p+q/2$ reversals. By the dual argument, the same number counts the reversals when $x\in A$ and $y\in C_i$.

Type 3. $x\in A$ and $y\in A$. The pair is reversed r times in $\mathscr{B}_3$. Also each $\mathscr{C}_i$ gives more than $q/4$ reversals, together this makes more than $r+(wq)/4$.

Note that $|\mathscr{F}|=(2p+q)w+r$. We now give the proportions of p and r relative to q.

$$w=1 \rightarrow p=0, \quad r=q/4,$$

$$w=2 \rightarrow p=q/2, \quad r=q/2,$$

$$w=3 \rightarrow p=q/2, \quad r=q/4,$$

$$w\geqslant 4 \rightarrow p=q/2, \quad r=0.$$

A simple calculation shows that the multirealizers built with these proportions have the right values. Note, that in the first three cases, the values p and r where chosen as to balance the proportion of reversals of Type 3 with the proportion of reversals of Types 1 and 2. In the calculation we only counted $(wq)/4$ reversals of Type 3 in $\mathscr{B}_3$. As already noted, the actual number is slightly larger. This proves that the given bounds cannot be attained.

We now sketch the argument showing that for every $\varepsilon>0$, there is a poset $\boldsymbol{P}$ containing an antichain A, such that width $(P-A)=w$ and the fractional dimension of $\boldsymbol{P}$ is at least $2w-\varepsilon$.

For given w and n, let $\boldsymbol{P}=(X,P)$ be the poset defined by:

(1) A is an antichain in $\boldsymbol{P}$.

(2) $C_1,\dots,C_w$ is a chain partition of $X-A$.

(3) $|C_i^+|=|C_i^-|=n$ for all i.

(4) C_i^+ is incomparable to C_j^+, C_i^- is incomparable to C_j^- and $C_i^-<C_j^+$ for all $i\neq j$.

(5) For every down-set $S\subseteq D(A)$ and every up-set $T\subseteq U(A)$ there is an element $a_{S,T}\in A$ with $D(a_{S,T})=S$ and $U(a_{S,T})=T$.

Now let $\mathscr{F}$ be an optimal multirealizer of $\boldsymbol{P}$, in what follows we consider $\mathscr{F}$ as a probability space. We choose a $\delta=\delta_\varepsilon$ and round all probabilities to multiples of δ. Consider an antichain B consisting of one point from C_i^+ for each i. For each π in the symmetric group on w objects, let p_π be the (rounded) probability that in a linear extension in $\mathscr{F}$ the elements of B appear in the order specified by π. There are fewer than $(1+1/\delta)^{w!}$ possible functions $\pi\to p_\pi$ and we color the antichains by these functions. Provided n is large, the product Ramsey theorem guarantees the existence of a two-element subchain E_i in each C_i^+, such that all antichains consisting of one point from each E_i receive the same color. Note that this implies that in all but a negligible portion of linear extensions in $\mathscr{F}$ the subchains E_i behave as points relative to one another, i.e, if $x\in E_i, y\in E_j$ and $x<y$, then $E_i<E_j$.

The same argument allows us to choose two element subchains F_i in C_i^- for each i, such that the F_i behave as points relative to each other almost all linear extensions in $\mathscr{F}$.

Now let $a\in A$ be the element with $a<\max E_i$, $a\parallel\min E_i$, $a\parallel\max F_i$ and $a>\min F_i$ for all $i=1,\dots,w$. Observe that when the E_i and F_i behave as points in a linear extension L, then only one of the $2w$ events $a>\min E_i$, $a<\max F_i$ for $i=1,\dots,w$ can hold in L. Hence the probability of the least probable of these events is at most $(1-\gamma)/2w+\gamma$, where γ, the probability that the E_i or the F_i do not behave as points, only depends on the accuracy of the rounding, i.e., on δ. Now δ can be chosen so that $(1-\gamma)/2w+\gamma<1/(2w-\varepsilon)$. This completes the proof. $\square$

Note, that the examples given in the proof of Theorem 8.2 have already been used in [16] to show that the inequality of Theorem 8.1 is tight.

9. Hiraguchi's inequality and removal theorems

We now give two further characterizations for inequalities involving fractional dimension. Both results will be easy corollaries of previous results, and both show that some problems are much easier when stated in terms of fractional dimension. The first example is known as Hiraguchi's inequality [10]. In this case, when $|X|\geqslant 8$, the characterizing posets are the same for both ordinary dimension and fractional dimension. For ordinary dimension, the proof is only moderately complicated when $|X|$ is even (see [1]). However, for ordinary dimension, there is no transparent proof when $|X|$ is odd. The most frequently cited source for the proof is Kimble [11], but to the best of our knowledge, the arguments therein are incomplete.

By way of contrast, the proof of the characterization for fractional dimension is very easy in both cases (for an analogous situation, see the characterization given for *interval dimension* in [2]).

Theorem 9.1. *If* $\boldsymbol{P}=(X,P)$ *and* $|X|\geqslant 4$, *then* $\mathrm{fdim}(\boldsymbol{P})\leqslant\lfloor |X|/2\rfloor$. *Furthermore, if* $|X|\geqslant 6$, *then* $\mathrm{fdim}(\boldsymbol{P})=\lfloor |X|/2\rfloor$ *only if* $|X|$ *is even and* $\boldsymbol{P}$ *is a standard example, or* $|X|$ *is odd and* $\boldsymbol{P}$ *contains a standard example on* $|X|-1$ *points.*

Proof. If P contains an antichain A with $|A|>|X|/2$, then Theorem 7.2 implies $\mathrm{fdim}(P)\leqslant|X-A|<|X|/2$. Now assume that there is no antichain of size more than $|X|/2$, then width $(P)\leqslant\lfloor |X|/2\rfloor$. In this case Theorem 6.3 implies $\mathrm{fdim}(P)\leqslant\lfloor |X|/2\rfloor$. The characterization is an immediate consequence of the characterization of Theorem 6.3. □

Theorem 9.2. *If* $\boldsymbol{P}=(X,P)$ *is a poset and* $x\in X$, *then* $\mathrm{fdim}(X-x,P(X-x))\geqslant \mathrm{fdim}(\boldsymbol{P})-1$. *Furthermore* $\mathrm{fdim}(X-x,P(X-x))=\mathrm{fdim}(\boldsymbol{P})-1$ *for each* $x\in X$ *only if* $\boldsymbol{P}$ *is a standard example.*

Note, that the condition $\dim(X-x,P(X-x))=\dim(\boldsymbol{P})-1$ for each $x\in X$ gives a definition for irreducible posets. A characterization of irreducible posets has only been obtained for dimensions two and three. An indication for the abundancy of irreducible posets is the following estimate: *For each* $t\geqslant 4$, *if* $n>10t$, *then the number of* t*-irreducible posets on* n *points is at least as large as the number of posets on* $n/3$ *points having dimension at most* $t-1$.

Proof of Theorem 9.2. In [3] it is shown that a poset $\boldsymbol{P}=(X,P)$ on n elements contains an element x, such that $\mathrm{fdim}(X-x,P(X-x))\leqslant((n-2)/n)\mathrm{fdim}(\boldsymbol{P})$. Hence, when $\mathrm{fdim}(\boldsymbol{P})<|X|/2$, there is an element $x\in X$, such that $\mathrm{fdim}(X-x,P(X-x))> \mathrm{fdim}(\boldsymbol{P})-1$. The result now follows from Theorem 9.1. □

10. Open problems for fractional dimension

In most instances, we have been able to show that our inequalities for fractional dimension are best possible. In two cases, there is still some uncertainty.

Problem 10.1. Are the following two inequalities of Theorem 8.2 best possible?

Let $\boldsymbol{P}=(X,P)$ be a poset, let A be an antichain with $X-A$ nonempty, and let $\boldsymbol{Q}=(Y,P(Y)$. Then let $w=\text{width}(\boldsymbol{Q})$.

(1) If $w=2$, then $\text{fdim}(\boldsymbol{P})<1/2+2w$.

(2) If $w=3$, then $\text{fdim}(\boldsymbol{P})<1/4+2w$.

For the inequality of Theorem 6.3 (see the remarks at the end of Section 7), we make the following conjecture.

Conjecture 10.2. (1) For each $\varepsilon>0$, there is an integer w_ε so that for every $w>w_\varepsilon$, there exists a poset $\boldsymbol{P}$ so that $w-\varepsilon<\text{fdim}(P)<w=\text{width}(\boldsymbol{P})$.

(2) For every positive integer $w\geqslant 2$, there is an $\varepsilon_w>0$ so that $\text{fdim}(\boldsymbol{P})\leqslant w-\varepsilon_w$, for every poset $\boldsymbol{P}$ with $\text{fdim}(\boldsymbol{P})<w=\text{width}(\boldsymbol{P})$.

Our results and techniques suggest that many dimension theoretic problems have analogous versions for fractional dimension which may be somewhat more tractable. Here are two such problems. The first problem is just a restatement of a problem first posed by Peter Fishburn (see [7]).

Problem 10.3. For each $t\geqslant 3$, let $f(t)$ be the minimum number of incomparable pairs in a poset $\boldsymbol{P}$ with $\text{fdim}(\boldsymbol{P})\geqslant t$. Is it true that $f(t)=t^2$?

We note that the 'chevron' [17] has dimension 3, and has only 7 incomparable pairs. However, an easy exercise shows that a poset with fractional dimension at least 3 must have at least 9 incomparable pairs. The case $t=4$ was resolved by Qin [12]. He showed that a poset must have at least 16 incomparable pairs in order to have dimension 4.

Problem 10.4. Given rational numbers p and q, what is the minimum value of the fractional dimension of $\boldsymbol{P}\times\boldsymbol{Q}$, where $\text{fdim}(\boldsymbol{P})=p$ and $\text{fdim}(\boldsymbol{Q})=q$?

We were unable to make any progress on determining whether a poset on three or more points always contains a pair whose removal decreases the fractional dimension by at most 1. As is the situation with ordinary dimension, this appears to be a very difficult problem. Perhaps the following restricted version is accessible.

Problem 10.5. Does there exist an absolute constant $\varepsilon>0$ so that any poset with 3 or more points always contains a pair whose removal decreases the fractional dimension by at most $2-\varepsilon$?

Acknowledgement

The authors would like to express their appreciation to Peter Winkler for numerous stimulating conversations on the topics discussed in this paper. He also gave a careful reading to a preliminary version and made many helpful suggestions for improving the exposition.

References

[1] K.P. Bogart and W.T. Trotter, Maximal dimensional partially ordered sets II, Discrete Math. 5 (1973) 33–44.
[2] K.P. Bogart and W.T. Trotter, Maximal dimensional partially ordered sets III: A characterization of Hiraguchi's inequality for interval dimension, Discrete Math. 15 (1976) 389–400.
[3] G. Brightwell and E. Scheinerman, Fractional dimension of partial orders, Order 9 (1992) 139–158.
[4] R.L. Brooks, On colouring the nodes of a network, Proc. Cambridge Philos. Soc. 37 (1941) 194–197.
[5] R.A. Brualdi, H.C. Jung and W.T. Trotter, On the poset of all posets on n elements, Discrete Appl. Math. 50 (1994) 111–123.
[6] R.P. Dilworth, A decomposition theorem for partially ordered sets, Ann. Math. 51 (1950) 161–165.
[7] P.C. Fishburn and W.T. Trotter, Posets with large dimension and relatively few critical pairs, Order 10 (1993) 317–328.
[8] Z. Füredi and J. Kahn, On the dimension of posets of bounded degree, Order 3 (1986) 15–20.
[9] R.L. Graham, J. Spencer and B.L. Rothschild, Ramsey Theory (Wiley, New York, 1991).
[10] T. Hiraguchi, On the dimension of partially ordered sets, Sci. Rep. Kanazawa Univ. 1 (1951) 77–94.
[11] R. Kimble, Extremal problems in dimension theory for partially ordered sets, Ph.D. Thesis, M.I.T., 1973.
[12] J. Qin, Some problems involving the dimension theory for ordered sets and the first-fit coloring algorithm, Ph.D. Thesis, Arizona State University, 1991.
[13] I. Rabinovitch and I. Rival, The rank of distributive lattices, Discrete Math. 25 (1979) 275–279.
[14] W.T. Trotter, Inequalities in dimension theory for posets, Proc. Amer. Math. Soc. 47 (1975) 311–316.
[15] W.T. Trotter, On the construction of irreducible posets, University of South Carolina Technical Memorandum, 1975.
[16] W.T. Trotter, A forbidden subposet characterization of an order dimension inequality, Math. Systems Theory 10 (1976) 91–96.
[17] W.T. Trotter, Combinatorics and Partially Ordered Sets: Dimension Theory, The Johns Hopkins University Press, Baltimore, Maryland, 1992.