MATH 1501 J3/J4/J5 Test III

Fall 2008

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There are 5 questions on this exam on 6 pages (not counting this coverpage). Be sure to explain your answers, *as answers that are not accompanied by explanations/work may receive no credit*. You are to complete this exam completely alone, *without the aid of notes, texts, calculators, cellular telephones, personal digital assistants, or any other mechanical or digital calculating device.*

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Student signature: ______________________________

Question	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
Total:	25	
Average:	5	

1. (5 points) Evaluate the indefinite integral

$$\int \frac{e^{4t}}{e^{2t} + 3e^{t} + 2}\, dt.$$

Note: This is a multistep problem. You may use subsitution and then find out you need apply another method to the new integral.

Solution: First we apply the substitution $u = e^t$ which has $du = e^t dt$. So, we get (after applying long division to the rational function)

$$\begin{aligned} \int \frac{e^{4t}}{e^{2t} + 3e^{t} + 2}\, dt &= \int \frac{u^3}{u^2 + 3u + 2}\, du \\ &= \int \left[(u-3) + \frac{-1}{u+1} + \frac{8}{u+2} \right] du \\ &= \frac{1}{2}u^2 - 3u - \ln(u+1) + 8\ln(u+2) + C \\ &= \frac{1}{2}e^{2t} - 3e^t - \ln(e^t+1) + 8\ln(e^t+2) + C. \end{aligned}$$

2. (5 points) Consider the region bounded by the curves $y = \cos x$, $y = 1$, $x = 0$, and $x = \frac{\pi}{2}$ and the solid generated by rotation the region about the x axis. Write as a definite integral the expression for the volume V of the solid by (i) using the washer method and (ii) using the shell method. Evaluate exactly *one* of these integrals. *Note: Your answer should have two (2) integrals, one evaluated and one unevaluated. You might need the formula* $\sin^2 u = \frac{1-\cos(2u)}{2}$.

Solution: First, we set up the integral for volumes using the shell method. For this, the outer radius is 1 and the inner radius is $\cos x$. Thus, we have the volume is

$$V = \int_0^{\frac{\pi}{2}} \pi \left(1^2 - \cos^2 x\right) dx = \int_0^{\frac{\pi}{2}} \pi \left(1 - \cos^2 x\right) dx.$$

Next, we use the shell method, but we need to get the functions in terms of x. The curves that bound the region are $x = \frac{\pi}{2}$ and $x = \arcsin y$. So, we get the volume is

$$V = 2\pi \int_0^1 y \arcsin y \, dy.$$

The first integral is solved by integrating $\sin^2 x$ and the second integral is solved using parts. The final answer is $V = \frac{\pi^2}{4}$.

3. (5 points) Let m and n be positive integers and prove that

$$\int_0^1 x^n(1-x)^m\,dx = \int_0^1 x^m(1-x)^n\,dx.$$

Solution: Here, we just use the substitution $u = 1 - x$ which gives $du = -dx$. Observe if $x = 0$, then $u = 1$; also, if $x = 1$, then $u = 0$. Finally, we note that $u = 1 - x$ implies that $x = 1 - u$, thus

$$\begin{aligned}\int_0^1 x^n(1-x)^m\,dx &= -\int_1^0 (1-u)^n u^m\,(-du) \\ &= \int_0^1 (1-u)^n u^m\,du.\end{aligned}$$

Replacing u by x gives the result. (Note: We can do this because u is just a dummy variable in the previous expression. If that doesn't convince you, then do a new substitution $u = x$ which gives $du = dx$ and convice yourself that way.)

4. (5 points) Evaluate the following indefinite integral

$$\int \left(x \ln \sqrt{x} + \sin^5 x\right) \, dx.$$

Solution: On the first integral we use parts. We let $u = \ln \sqrt{x}$ and $dv = xdx$ which implies $du = \frac{1}{2x}dx$ and $v = \frac{1}{2}x^2$. So, we get

$$\begin{aligned} \int x \ln \sqrt{x} \, dx &= \frac{1}{2}x^2 \ln \sqrt{x} - \int \frac{1}{4}x \, dx \\ &= \frac{1}{2}x^2 \ln \sqrt{x} - \frac{1}{8}x^2 + C. \end{aligned}$$

For the second part of the integral, we take out a $\sin x$ from the expression and replace the rest with cosines. So, we have

$$\begin{aligned} \int \sin^5 x dx &= \int \sin^4 x \sin x dx \\ &= \int (1 - \cos^2 x)^2 \sin x dx \\ &= \int (1 - 2\cos^2 x + \cos^4 x) \sin x dx \\ &= \int (1 - 2u^2 + u^4)(-du) \\ &= -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + C \\ &= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C. \end{aligned}$$

5. (5 points) Find the volume of the ellipsoid obtained by rotating the region bounded by the curve $y = \frac{1}{2}\sqrt{4 - x^2}$ and the x axis, about the x axis.

Solution: Probably the easiest way to solve this problem is to use the shell method. Here, we only have one function bound and so the volume is

$$\begin{aligned} V &= \int_{-2}^{2} \pi(\frac{1}{2}\sqrt{4 - x^2})^2 \, dx \\ &= \frac{\pi}{4} \int_{-2}^{2} (4 - x^2) \, dx \\ &= \frac{\pi}{2} \int_{0}^{2} (4 - x^2) \, dx \\ &= \frac{\pi}{2} \left(\frac{16}{3} \right) \\ &= \frac{8}{3}\pi. \end{aligned}$$

Scratch Work