2.1 #42. a. We're given that v_1 people live in the city and v_2 people live in the suburbs. We're told that 60% of the city dwellers drive cars and 30% of the suburb dwellers drive cars. Therefore, the total number of people who drive cars is

$$u_1 = .6v_1 + .3v_2.$$

Hence the first row of the matrix B is going to be $[.6 \quad .3]$, because we then have that

$$\begin{bmatrix} .6 & .3 \\ * & * \\ * & * \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ * \\ * \end{bmatrix}.$$

We use similar reasoning to figure out how to compute u_2 and u_3 from v_1 and v_2, and end up with

$$B = \begin{bmatrix} .6 & .3 \\ .3 & .5 \\ .1 & .2 \end{bmatrix}.$$

2.1 #54. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence rank$(AB) = 1$ while rank$(BA) = 0$.

2.3 #6. Since $AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 0 \\ -1 & 0 & -1 \end{bmatrix}$, we see that B is NOT the inverse of A.

2.3 #32. We are given the 2nd, 4th, and 6th columns of A. That is, we know a_2, a_4, and a_6, and our task is to find a_1, a_3, a_5.

The second column of R is twice the first column. Hence the same is true of A, i.e., $a_2 = 2a_1$. Now we know a_1.

The third column plus the fourth column of R equals first column: $r_1 = r_3 + r_4$. Since the columns of A have exactly the same relationships as the columns of R, this tells us that $a_1 = a_3 + a_4$. We already know a_1 and a_4, so this tells us a_3.

Finally, $r_4 + r_5 = r_6$, so $a_4 + a_5 = a_6$. Since a_4 and a_6 are known, this tells us a_5.

Putting it all together, we find that $A = \begin{bmatrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 2 & 4 & -1 & 3 & -4 & -1 \\ 3 & 6 & 4 & -1 & 0 & -1 \\ -1 & -2 & -2 & 1 & 1 & 2 \end{bmatrix}$.

2.4 #50. By definition, A similar to B means that $B = P^{-1}AP$ for some invertible matrix P. Since we're told that A is invertible, B is therefore a product of invertible matrices and hence is itself invertible. Moreover, taking the inverse of both sides of $B = P^{-1}AP$ tells us:

$$B^{-1} = (P^{-1}AP)^{-1} = P^{-1}A^{-1}(P^{-1})^{-1} = P^{-1}A^{-1}P.$$

Now, we're hoping to show that A^{-1} is similar to B^{-1}. That means we have to show that

$$B^{-1} = Q^{-1}A^{-1}Q$$

for some invertible matrix Q. We've already shown that $B^{-1} = P^{-1}A^{-1}P$. So the Q we need is $Q = P$ (but there's no reason that it HAS to be the same matrix, just that there is SOME invertible Q that works).

2.6 #14. We're given that T is linear with $T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ and $T\left(\begin{bmatrix} -4 \\ 0 \end{bmatrix}\right) = \left(\begin{bmatrix} -5 \\ 1 \end{bmatrix}\right)$, and we have to determine $T\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right)$. Note that $\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -4 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Since T is linear, we therefore have

$$T\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -4 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) + T\left(\begin{bmatrix} -4 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -5 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}. \quad \square$$

2.6 #28. Since $T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we have that $T(0) \neq 0$, so T cannot possibly be linear.

2.6 #52. T_A is the linear transformation defined by the rule $T_A(x) = Ax$ for every $x \in \mathbf{R}^n$. T_B is the linear transformation defined by the rule $T_B(v) = Bv$ for every $v \in \mathbf{R}^p$. T_{AB} is the linear transformation defined by the rule $T_{AB}(v) = (AB)v$ for every $v \in \mathbf{R}^p$. Hence

$$T_{AB}(v) = (AB)v = A(Bv) = T_A(Bv) = T_A(T_B(v)).$$

2.7 #6. The standard matrix is $A = \begin{bmatrix} 5 & -4 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. The columns of A are a spanning set for range(T).

2.7 #23. Find a spanning set for the nullspace of $T\colon \mathbf{R}^4 \to \mathbf{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 + x_3 - x_4 \\ x_1 + x_2 + 2x_3 + 2x_4 \\ x_1 - x_3 - 3x_4 \end{bmatrix}.$$

Solution

The nullspace is the set of all vectors x that satisfy $T(x) = 0$. So you just have to find the general solution to the system

$$\begin{aligned} 2x_1 + x_2 + x_3 - x_4 &= 0, \\ x_1 + x_2 + 2x_3 + 2x_4 &= 0, \\ x_1 - x_3 - 3x_4 &= 0. \end{aligned}$$

Set up the matrix form of this system of linear equations and do the row reduction. The appropriate matrix A and its row reduced echelon form are

$$A = \begin{bmatrix} 2 & 1 & 1 & -1 \\ 1 & 1 & 2 & 3 \\ 1 & 0 & -1 & -3 \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore the general solution to $T(x) = 0$ in parametric form is

$$x = \begin{bmatrix} x_3 + 3x_4 \\ -3x_3 - 5x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -5 \\ 0 \\ 1 \end{bmatrix}, \qquad x_3, x_4 \in \mathbf{R}.$$

The nullspace is the SET of all these solutions, i.e.,

$$\mathrm{Null}(T) = \left\{ x_3 \begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -5 \\ 0 \\ 1 \end{bmatrix} : x_3, x_4 \in \mathbf{R} \right\} = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

So the two vectors on the last line above are the spanning vectors that we seek. Note that the spanning set is just those two vectors, i.e., it is the set

$$\left\{ \begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\},$$

while the nullspace itself is the set of infinitely many linear combinations of these two vectors. Note also that since those two vectors are independent (why?), their span is a plane. Thus the nullspace of this T is a plane within $\mathbf{R}^4$. $\square$