Bonus Exam

July 24, 2008

Name:____________________ Section:__________________

The instructions for the exam are listed below.

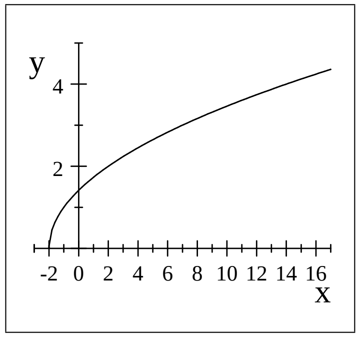
- **General Policies**
  - Calculators, cell phones, iPods, other electronic devices, and other mechanical calculating devices are NOT allowed during the exam.
  - You are NOT allowed any additional paper other than the test paper (e.g. no note sheets).
  - Make certain that all cell phones are off before the exam begins.
  - DO NOT remove the staple from the exam (unless you are able to staple it back with your own stapler). We will NOT accept tests that are not stapled.
  - The last page of the exam is blank in case you need some scratch paper.
- **During the Exam**
  - Make sure that you read through the whole exam before starting to answer. You should pick out the problems you find easiest first so that you get the maximum score possible.
  - Write your answers NEATLY on the test. If your answers are not legible and your answer is not easily seen to be correct, then you will NOT be given the benefit of the doubt on the problem's correctness. This could result in a lower score. Please note that if a grading error is made because of poor handwriting, then you will NOT be granted a regrade.
  - If you have a question, then please raise your hand during the exam.
- **About the Grading**
  - **Unless the statement of a problem SPECIFICALLY says that work is not required, you MUST show all of your work! Answers that are not supported by justification will receive NO credit (even if the answer is correct).**
  - Please understand that anything you write on the exam will be graded. Points will be taken off for incorrect statements/work that is not crossed out even if you have the rest of the problem correct.
  - If you have two conflicting statements as an answer to a problem, then we will take the incorrect one as your answer. If we detect repeated attempts to answer a single problem different ways, then you will be reported to the Dean of Students for cheating.
  - This exam is completely optional and worth 60 points. This exam grade will replace your lowest test score unless you score lower on this exam than your lowest test score. Also, you cannot score more than 100% on the exam you replace (so if you replace an exam that was worth 57 points and you get a 60 on this test, then it will be recorded as a 57).
  - **Do NOT forget to sign the signature line below.** Failure to do so may result in your test not being graded.

By signing you acknowledge that you have read and understand the instructions above, and you will not violate Georgia Tech's honor code.

Signature: ______________________________ Date:______________

1. *(10 points)* Does the function $h(t) = \sqrt{t+2}$ have an inverse? If h has an inverse, then compute h^{-1}. Finally, state the range of h and say how it relates to the domain of h^{-1}. *Note: If you use the horizontal line test to prove or disprove the existence of an inverse, then you need to draw the graph of the function. Also, if the function has no inverse, then write no inverse in the answer space.*

The function does have an inverse because it passes the horizontal line test.



To find the inverse we let $y = h(t)$ and then switch y and t. Observe

$$\begin{aligned} t &= \sqrt{y+2} \\ t^2 &= y+2 \\ t^2 - 2 &= y. \end{aligned}$$

Thus, we have $h^{-1}(t) = t^2 - 2$. The range of h is $[0, \infty)$ *as can be seen by the graph. We know that the range of h* is the same as the domain of h^{-1}. *Thus, the domain of h^{-1} is* $[0, \infty)$.

2. *(10 points)* Solve the equation $\sin\theta = -1$ where $0 \le \theta \le 4\pi$ (so θ is in radians).

This just requires knowledge of the unit circle. Here we know $\sin\left(\frac{3\pi}{2}\right) = -1$*, but we need all solutions between 0 and 4π. Thus, we keep adding 2π until we get something past 4π. So, we have the solutions are $\theta = \frac{3\pi}{2}, \frac{7\pi}{2}$.*

3. *(10 points)* Solve the following system of equations

$$\begin{cases} x - 3y + 3z = -4 \\ 2x + 3y - z = 15 \\ 4x - 3y - z = 19 \end{cases}$$

Here we need to use Gaussian elimination and label the equations $(1), (2),$ *and* (3) *in the order they appear above. First, we compute* $-2 * (1) + (2) \Rightarrow (2)$ *which gives the new equation* (2)*:* $9y - 7z = 23$*. Similarly, we take* $-4*(1)+(3) \Rightarrow (3)$*:* $9y - 13z = 35$*. Thus, we have the equivalent system of equations*

$$\begin{cases} x - 3y + 3z = -4 \\ 9y - 7z = 23 \\ 9y - 13z = 35 \end{cases}.$$

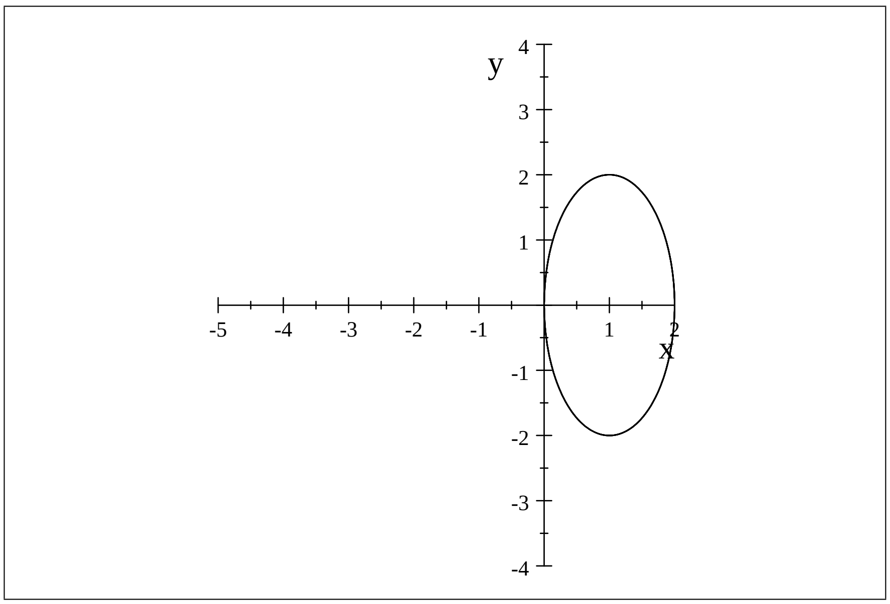
Next, we take $-1 * (2) + (3) \Rightarrow (3)$*:* $-6z = 12$ *which gives the solution* $z = -2$*. Substituting* $z = -2$ *into* (2) *we get* $9y - 7(-2) = 23$ *which implies* $y = 1$*. Finally, using* $y = 1$ *and* $z = -2$ *in* (1) *gives* $x - 3(1) + 3(-2) = -4$ *which implies* $x = 5$*. Thus, the solution to this system of equations is* $(x, y, z) = (5, 1, -2)$*.*

4. *(11 points)* Consider the equation $4x^2 - 8x + y^2 = 0$. For this equation, state what kind of conic section that the equation represents and also state the center of the conic.

We can already see that the equation will give an ellipse since the coefficients on the x^2 *and* y^2 *are the same sign, but different numbers. So, we need to complete the square to get the equation in standard form. Observe*

$$\begin{aligned} 4x^2 - 8x + y^2 &= 0 \\ 4\left(x^2 - 2x + 1 - 1\right) + y^2 &= 0 \\ 4\left(x - 1\right)^2 - 4 + y^2 &= 0 \\ \frac{\left(x-1\right)^2}{1} + \frac{y^2}{4} &= 1. \end{aligned}$$

So, we arrive at the equation of an ellipse with center $(1, 0)$.



Type of Conic: *Ellipse*

Center of Conic: $(1, 0)$

5. Given that $\sin\theta = \frac{1}{2}$ and $\tan\theta > 0$ find the following:

(a) *(3 points)* $\csc\theta = 2$

We know $\csc\theta = \dfrac{1}{\sin\theta} = \dfrac{1}{\frac{1}{2}} = 2.$

(b) *(3 points)* $\sec\theta = \dfrac{2\sqrt{3}}{3}$

Observe that $\sin\theta = \frac{1}{2}$ *gives a reference angle of* $\frac{\pi}{6}$. *So that means that* $\cos\theta = \pm\frac{\sqrt{3}}{2}$ *where we determine whether it is positive or negative depending on any additional information. The fact that* $\tan\theta = \dfrac{\sin\theta}{\cos\theta} > 0$ *and since we already know* $\sin\theta > 0$ *we must also have* $\cos\theta > 0$. *Thus, we have* $\cos\theta = \frac{\sqrt{3}}{2}$ *and finally we arrive at* $\sec\theta = \frac{1}{\cos\theta} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$.

(c) *(3 points)* $\tan\theta = \dfrac{\sqrt{3}}{2}$

We are already given that $\tan\theta > 0$ *so this should help us double check our answer. Observe we have*

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{3}.$$

6. *(10 points)* Verify the indentity

$$\cos^4 x - \sin^4 x = \cos(2x)$$

where x is a real number. *Hint: You should try factoring the expression on the left.*

This identity relies on the fact that $a^2 - b^2 = (a-b)(a+b)$*. So, we let* $a = \cos^2 x$ *and* $b = \sin^2 x$ *and get*

$$\begin{aligned} \cos^4 x - \sin^4 x &= \left(\cos^2 x - \sin^2 x\right)\left(\cos^2 x + \sin^2 x\right) \\ &= \left(\cos^2 x - \sin^2 x\right)(1) \\ &= \cos(2x) \quad \textit{(double angle formula).} \end{aligned}$$

Formulas You Might Need

Sum and Difference Formulas

$$\sin(u+v) = \sin u \cos v + \cos u \sin v \qquad \sin(u-v) = \sin u \cos v - \cos u \sin v$$
$$\cos(u+v) = \cos u \cos v - \sin u \sin v \qquad \cos(u-v) = \cos u \cos v + \sin u \sin v$$
$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v} \qquad \tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$$

Double-Angle Formulas

$$\begin{aligned} \sin(2u) &= 2\sin u \cos u \\ \cos(2u) &= \cos^2 u - \sin^2 u \\ \tan(2u) &= \frac{2\tan u}{1-\tan^2 u} \end{aligned}$$

Power-Reducing Formulas

$$\begin{aligned} \sin^2 u &= \frac{1-\cos(2u)}{2} \\ \cos^2 u &= \frac{1+\cos(2u)}{2} \\ \tan^2 u &= \frac{1-\cos(2u)}{1+\cos(2u)} \end{aligned}$$

Half-Angle Formulas

$$\begin{aligned} \sin\left(\frac{u}{2}\right) &= \pm\sqrt{\frac{1-\cos u}{2}} \\ \cos\left(\frac{u}{2}\right) &= \pm\sqrt{\frac{1+\cos u}{2}} \\ \tan\left(\frac{u}{2}\right) &= \frac{1-\cos u}{\sin u} = \frac{\sin u}{1+\cos u} \end{aligned}$$