Precalculus Exam #4

July 17, 2008

Name:_____________________ Section:___________________

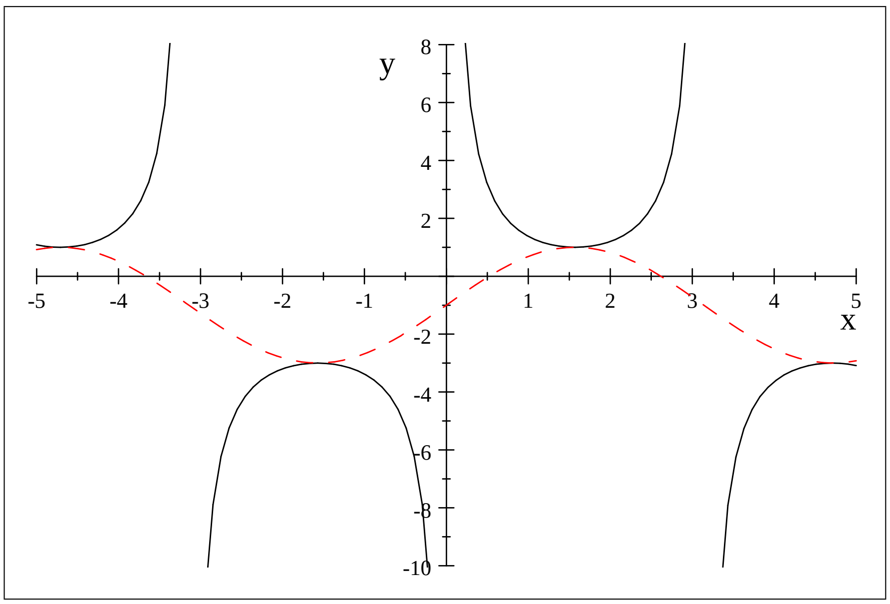
The instructions for the exam are listed below.

- **General Policies**
  - Calculators, cell phones, iPods, other electronic devices, and other mechanical calculating devices are NOT allowed during the exam.
  - You are NOT allowed any additional paper other than the test paper (e.g. no note sheets).
  - Make certain that all cell phones are off before the exam begins.
  - DO NOT remove the staple from the exam (unless you are able to staple it back with your own stapler). We will NOT accept tests that are not stapled.
  - The last page of the exam is blank in case you need some scratch paper.
- **During the Exam**
  - Make sure that you read through the whole exam before starting to answer. You should pick out the problems you find easiest first so that you get the maximum score possible.
  - Write your answers NEATLY on the test. If your answers are not legible and your answer is not easily seen to be correct, then you will NOT be given the benefit of the doubt on the problem's correctness. This could result in a lower score. Please note that if a grading error is made because of poor handwriting, then you will NOT be granted a regrade.
  - If you have a question, then please raise your hand during the exam.
- **About the Grading**
  - **Unless the statement of a problem SPECIFICALLY says that work is not required, you MUST show all of your work! Answers that are not supported by justification will receive NO credit (even if the answer is correct).**
  - Please understand that anything you write on the exam will be graded. Points will be taken off for incorrect statements/work that is not crossed out even if you have the rest of the problem correct.
  - If you have two conflicting statements as an answer to a problem, then we will take the incorrect one as your answer. If we detect repeated attempts to answer a single problem different ways, then you will be reported to the Dean of Students for cheating.
  - This exam is worth 57 points, however there are a total of 60 points possible on the exam. If you score more than 57 points on the exam, then the grade will be recorded as a 57.
  - **Do NOT forget to sign the signature line below.** Failure to do so may result in your test not being graded.

By signing you acknowledge that you have read and understand the instructions above, and you will not violate Georgia Tech's honor code.

Signature: _______________________________ Date:______________

1. *(15 points)* Sketch a graph of the equation $y = 2\csc(x) - 1$ over one period.



2. *(7 points)* Find all x-intercepts of the graph of $y = \sin(\pi x) + \cos(\pi x)$. *Hint: There are infinitely many x-intercepts.*

The x intercepts of the graph are the points where $y = 0$ so we solve the equation

$$\begin{aligned} 0 &= \sin(\pi x) + \cos(\pi x) \\ -\cos(\pi x) &= \sin(\pi x) \\ -1 &= \frac{\sin(\pi x)}{\cos(\pi x)} \\ -1 &= \tan(\pi x) \end{aligned}$$

So, now we need to solve the equation $\tan(\theta) = -1$ in the interval $[0, 2\pi)$ which implies that $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$. Thus, all solutions of $\tan\theta = -1$ are found by $\theta = \frac{3\pi}{4} + 2\pi n$ and $\theta = \frac{7\pi}{4} + 2\pi n$ where $n \in \mathbb{N}$. We have $\theta = \pi x$ and so all of the solutions to the original equation are $x = \frac{3}{4} + 2n$ and $x = \frac{7}{4} + 2n$.

3. *(15 points)* Verify the indentity

$$\cos^4 x - \sin^4 x = \cos(2x)$$

where x is a real number. *Hint: You should try factoring the expression on the left.*

This identity relies on the fact that $a^2 - b^2 = (a - b)(a + b)$. So, we let $a = \cos^2 x$ and $b = \sin^2 x$ and get

$$\begin{aligned} \cos^4 x - \sin^4 x &= \left(\cos^2 x - \sin^2 x\right)\left(\cos^2 x + \sin^2 x\right) \\ &= \left(\cos^2 x - \sin^2 x\right)(1) \\ &= \cos(2x) \quad \textit{(double angle formula).} \end{aligned}$$

4. *(4 points for each part, 8 points total)* During takeoff, an airplane's angle of ascent is $16°$ and its speed is 200 feet per second.

   (a) Find the plane's altitude after 2 minutes.

   Observe that the plane's ascent can be thought of as the hypotenuse of a right triangle (with length $200\,(60)\,(2) = 24000$ *feet). Thus, we want the opposite side of the angle and so we use sine to get*

$$\begin{aligned}\sin 16° &= \frac{altitude}{24000 \text{ ft}} \\ altitude &= 24000 \sin 16° \text{ ft.}\end{aligned}$$

   (b) How long will it take the plane to climb to an altitude of 1,000 feet?

   We know that the distance traveled along the hypotenuse is $200t$ *where* t *is measured in seconds. We want to find a* t *such that the altitude is* 1000 *ft. So, we want to solve the equation*

$$\begin{aligned}\sin 16° &= \frac{1000 \text{ ft}}{200t} \\ 200t &= \frac{1000 \text{ ft}}{\sin 16°} \\ t &= \frac{5 \text{ s}}{\sin 16°}.\end{aligned}$$

5. *(5 points for each part, 15 points total)* Find all solutions to the following equations in the indicated intervals:

   (a) $\cos(2x)(2\sin x - 1) = 0$
   interval: $\mathbb{R}$

   It is important to note here that the equation is already factored and so we only need to solve $\cos(2x) = 0$ *and* $2\sin x - 1 = 0$ *in the interval* $\mathbb{R}$. *Observe, we first solve* $\cos y = 0$ *in the interval* $[0, 2\pi)$ *which gives* $y = \frac{\pi}{2} + 2\pi n, \frac{3\pi}{2} + 2\pi n$ *where* $n \in \mathbb{N}$. *Thus we have* $x = \frac{\pi}{4} + \pi n, \frac{3\pi}{4} + \pi n$ *where* $n \in \mathbb{N}$. *Next, we need to solve the equation* $2\sin x - 1 = 0$ *which is the same as* $\sin x = \frac{1}{2}$. *The solutions here are* $x = \frac{\pi}{6} + 2\pi n, \frac{5\pi}{6} + 2\pi n$ *where* $n \in \mathbb{N}$.

(b) $\sec^2 x + \tan x - 3 = 0$
interval: $[0, 2\pi)$
Hint: You may not be able to evaluate an inverse trigonometric function.

Here we use the identity $\sec^2 x = \tan^2 x + 1$ *to get a polynomial in* $\tan x$. *Thus we have*

$$\begin{aligned} \tan^2 x + 1 + \tan x - 3 &= 0 \\ \tan^2 x + \tan x - 2 &= 0 \\ (\tan x + 2)(\tan x - 1) &= 0. \end{aligned}$$

Thus, we solve the two equations $\tan x = -2$ *and* $\tan x = 1$ *in the interval* $[0, 2\pi)$. *Observe that the second equation gives* $x = \frac{\pi}{4}, \frac{5\pi}{4}$. *The first equation cannot be solved with the unit circle and so we must use inverse trig functions. Observe that* $\tan x = -2 > 0$ *implies that there should be a solution in Quadrants II and IV. Oberve that* $\tan^{-1}(-2)$ *gives an angle in the IV quadrant, but it is in the interval* $\left(-\frac{\pi}{2}, 0\right)$. *Thus, we need to add* 2π *to* $\tan^{-1}(-2)$ *to get the correct solution between 0 and* 2π *that is in Quadrant IV. Now, to get the solution to the second quadrant, we take* $\tan^{-1}(-2)$ *and add* π *to it to get the correct angle. So, the solutions are* $x = \frac{\pi}{4}, \frac{5\pi}{4}, \tan^{-1}(-2) + \pi$, *and* $\tan^{-1}(-2) + 2\pi$.

(c) $\sin(2x)\sin x = \cos x$
interval: $[-\pi, \pi]$

The strategy here is to use the double angle formula for $\sin 2x$. *Thus we have*

$$\begin{aligned} \sin 2x \sin x &= \cos x \\ 2 \sin x \cos x \sin x &= \cos x \\ 2\sin^2 x \cos x - \cos x &= 0 \\ \cos x \left(2\sin^2 x - 1\right) &= 0. \end{aligned}$$

Thus we solve $\cos x = 0$ *and* $2\sin^2 x - 1 = 0$ *in the interval* $[-\pi, \pi]$. *Here observe that* $\cos x = 0$ *implies that* $x = -\frac{\pi}{2}, \frac{\pi}{2}$. *Next, we solve* $\sin^2 x = \frac{1}{2}$ *which gives* $\sin x = \pm\frac{\sqrt{2}}{2}$. *Thus, the angles that satisfy this equation are* $x = \frac{\pi}{4}, \frac{3\pi}{4}, -\frac{3\pi}{4}$, *and* $-\frac{\pi}{4}$.

Formulas You Might Need

Sum and Difference Formulas

$$\sin(u+v) = \sin u \cos v + \cos u \sin v \qquad \sin(u-v) = \sin u \cos v - \cos u \sin v$$
$$\cos(u+v) = \cos u \cos v - \sin u \sin v \qquad \cos(u-v) = \cos u \cos v + \sin u \sin v$$
$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v} \qquad \tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$$

Double-Angle Formulas

$$\begin{aligned} \sin(2u) &= 2\sin u \cos u \\ \cos(2u) &= \cos^2 u - \sin^2 u \\ \tan(2u) &= \frac{2\tan u}{1-\tan^2 u} \end{aligned}$$

Power-Reducing Formulas

$$\begin{aligned} \sin^2 u &= \frac{1-\cos(2u)}{2} \\ \cos^2 u &= \frac{1+\cos(2u)}{2} \\ \tan^2 u &= \frac{1-\cos(2u)}{1+\cos(2u)} \end{aligned}$$

Half-Angle Formulas

$$\begin{aligned} \sin\left(\frac{u}{2}\right) &= \pm\sqrt{\frac{1-\cos u}{2}} \\ \cos\left(\frac{u}{2}\right) &= \pm\sqrt{\frac{1+\cos u}{2}} \\ \tan\left(\frac{u}{2}\right) &= \frac{1-\cos u}{\sin u} = \frac{\sin u}{1+\cos u} \end{aligned}$$