Math 2605, **First Exam**, 18 Feb 2010, 3pm, **NAME:** Solution Key

Neither cheatsheets nor calculators are allowed.

1. (20 points) Given a parametric representation of the line $\boldsymbol{x}(t) = (1, -1, 1)^T + t(2, 0, 1)^T$, write down a representation in terms of the system of equations of two planes.

A possible solution:

$\vec{x}_0 = (1, -1, 1)^T$ // base point

$\left.\begin{array}{l} \vec{a}_1 = (1, 0, -2)^T \\ \vec{a}_2 = (0, 1, 0)^T \end{array}\right\}$ // two linearly independent vectors orthogonal to $(2, 0, 1)^T$

$$\begin{cases} \vec{a}_1 \cdot (\vec{x} - \vec{x}_0) = 0; \\ \vec{a}_2 \cdot (\vec{x} - \vec{x}_0) = 0. \end{cases}$$

2. (20 points) Find the distance from the point $\boldsymbol{x}_0 = (2, 1, 3)$ to the line

$$\boldsymbol{x}(t) = (1, 2, 3)^T + t(1, -1, 1)^T$$

$\vec{w} = (2, 1, 3)^T - (1, 2, 3)^T = (1, -1, 0)^T;$

$\vec{v} = (1, -1, 1)^T; \quad \vec{u} = \frac{1}{\sqrt{3}} \vec{v}.$

$\vec{w}_{\parallel} = (\vec{w} \cdot \vec{u}) \vec{u} = \frac{2}{3} \vec{v}$

$\vec{w}_{\perp} = \vec{w} - \vec{w}_{\parallel} = \left(\frac{1}{3}, -\frac{1}{3}, -\frac{2}{3}\right)$

distance $= \|\vec{w}_{\perp}\| = \frac{\sqrt{6}}{3}$

3. (20 points) Consider the graph of $f(x,y) = \sqrt{(x-3)^2 + (y-2)^2}$. Find a point on this surface where the tangent plane is horizontal (i.e. parallel to the xy plane).

$$\nabla f = \begin{bmatrix} (x-3)/f(x,y) \\ (y-2)/f(x,y) \end{bmatrix}$$

$$\nabla f = 0 \overset{?}{\Rightarrow} \begin{cases} x = 3 \\ y = 2 \end{cases}$$

At $\vec{x} = (3,2)$ the gradient is not defined, since the denominator is zero

Write an equation for the tangent plane at that point.

The tangent plane at $(3,2)$ does not exist: the graph is a part of $z^2 = (x-3)^2 + (y-2)^2$, which is a cone (singular at $(3,2)$).

The problem was not intended to be tricky.

4. (20 points) Let $f(x,y) = 2x^2 + 2y^2 + 3xy - x + y - 1$.
Find the gradient of f,

$$\nabla f = \begin{bmatrix} 4x + 3y - 1 \\ 4y + 3x + 1 \end{bmatrix}$$

the critical points,

$$\left[\begin{array}{cc|c} 4 & 3 & 1 \\ 3 & 4 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & 3 & 1 \\ 0 & \frac{7}{4} & -\frac{7}{4} \end{array}\right] \sim \left[\begin{array}{cc|c} 4 & 0 & 4 \\ 0 & 1 & -1 \end{array}\right] \qquad \begin{array}{l} x = 1 \\ y = -1 \end{array}$$

and Hessian at these points.

$$H_f = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$$

For each point determine whether it is a local max, local min, or saddle point.

$$\left.\begin{array}{l} D = \det H_f = 7 > 0 \\ \dfrac{\partial^2 f}{\partial x^2} = 4 > 0 \end{array}\right\} \Rightarrow \text{min at } (1,-1)$$

5. (20 points) Set up Newton's method for solving the system of equations: $x^2+y^2-4=0$ and $xy-1=0$. Use as an initial guess the point $\boldsymbol{x}_0=(2,1)^T$ and calculate the next approximation $\boldsymbol{x}_1$.

$$J_F = \begin{bmatrix} 2x & 2y \\ y & x \end{bmatrix}, \quad J = J_F(\vec{x}_0) = \begin{bmatrix} 4 & 2 \\ 1 & 2 \end{bmatrix},$$

$$J\vec{x}_1 = J\vec{x}_0 - F(\vec{x}_0) = \begin{bmatrix} 10-1 \\ 4-1 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 4 & 2 & 9 \\ 1 & 2 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & -6 & -3 \\ 1 & 2 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 2 & 1 \\ 1 & 0 & 2 \end{array}\right] \quad \begin{array}{l} y = \frac{1}{2} \\ x = 2 \end{array}$$

Answer: $\vec{x}_1 = \begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix}$

6. (extra credit) Prove that for nonnegative $x_1, \ldots, x_n$

$$(x_1 \cdots x_n)^{\frac{1}{n}} \le \frac{x_1 + \cdots + x_n}{n}$$

using Lagrange's method.
(Hint: restrict $(x_1 \cdots x_n)^{\frac{1}{n}}$ to level curves of $x_1 + \cdots + x_n$.)

$$f(\vec{x}) = (x_1 \ldots x_n)^{\frac{1}{n}}, \quad g(\vec{x}) = x_1 + \ldots + x_n = k$$

$$\nabla f = \left(\tfrac{1}{n}(x_1 \ldots x_n)^{\frac{1}{n}-1}(x_2 \ldots x_n), \ \ldots, \ \tfrac{1}{n}(x_1 \ldots x_n)^{\frac{1}{n}-1}(x_1 \ldots x_n)\right)^T$$

$\nabla g = (1, \ldots, 1)$; $\nabla f = \lambda \nabla g$ implies, in particular,

$$\tfrac{1}{n}(x_1 \ldots x_n)^{\frac{1}{n}-1}(x_1 \ldots x_n) = \lambda x_i \quad \text{for all } i.$$

$\Rightarrow \lambda x_i - \lambda x_j = 0 \Rightarrow x_1 = \ldots = x_n$ at $\vec{x}_0$ (critical point)

$f(\vec{x}_0) = (x_1^n)^{\frac{1}{n}} = x_1$ and $g(\vec{x}_0) = n x_1 = k$

Conclusion: For any k, at a critical point $x_1 = \frac{k}{n}$;
for any other $\vec{x}$: $f(\vec{x}) = (x_1 \ldots x_n)^{\frac{1}{n}} \le f(\vec{x}_0) = \frac{k}{n} = \frac{x_1 + \ldots + x_n}{n}$