Math 2605
Exam 2

Name: Key

By signing my name below I pledge that I have neither given nor received help on this exam.

Pledged:________________________

Problem	Possible	Earned
1	15	
2	10	
3	15	
4	10	
5	10	
Total	60	

1. (15 pts) Use the method of Lagrange multipliers to find the maximum and minimum values of

$$f(x, y) = x + y$$

subject to the constraint that

$$x^2 + y^2 = 1.$$

Set $g(x,y) = x^2 + y^2 - 1$

$\nabla f(x,y) = (1, 1)$

$\nabla g(x,y) = (2x, 2y)$

Lagrange multipliers gives

$$\nabla f = \lambda \nabla g \iff \begin{array}{l} 1 = 2\lambda x \\ 1 = 2\lambda y \end{array}$$

$$\Rightarrow \begin{array}{l} y = 2\lambda xy \\ \quad\quad \shortparallel \\ x = 2\lambda xy \end{array} \quad \Rightarrow x = y.$$

Substitute into $g(x,y) \equiv 0$ $\quad 2x^2 = 1 \quad \Rightarrow x^2 = \frac{1}{2}$

$\Rightarrow x = \pm \frac{1}{\sqrt{2}}$

Candidate Points: $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

Max: $\frac{2}{\sqrt{2}} = f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ $\quad$ Min: $-\frac{2}{\sqrt{2}} = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

2. (10 pts)

(a) Find an equation of the tangent plane to the surface

$$z = e^x \ln y$$

at $(3, 1, 0)$.

(b) Find the normal line to the surface at the point $(3, 1, 0)$.

Define $g(x, y, z) = z - e^x \ln y$

$\nabla g(x, y, z) = \left(-e^x \ln y, -\frac{e^x}{y}, 1\right)$

$\nabla g(3,1,0) = \left(-e^3 \ln 1, \frac{-e^3}{1}, 1\right) = (0, -e^3, 1)$.

Normal Line:

$\ell(t) = (3, 1, 0) + t(0, -e^3, 1)$

Tangent Plane: $\nabla g(3,1,0) \cdot (\hat{x} - (3,1,0)) = 0$

$= (0, -e^3, 1) \cdot (x-3, y-1, z) = 0$

$-e^3(y-1) + z = 0.$

3. (15 pts) Find the local maximum, minimum and saddle points of the following function:

$$f(x, y) = x^4 + y^4 - 4xy + 1.$$

$$\nabla f(x,y) = (4x^3 - 4y,\ 4y^3 - 4x)$$

$$\nabla f(x,y) = \hat{0} \iff 4x^3 = 4y \quad \text{and} \quad 4y^3 = 4x$$

$$x^3 = y \quad \text{and} \quad y^3 = x$$

$$\Rightarrow (y^3)^3 = y \quad \Rightarrow y^9 = y$$

$$\Rightarrow y(y^8 - 1) = 0 \qquad y = 0, 1, -1$$

Critical Points: $(0,0), (1,1), (-1,-1)$.

$$\text{Hessian}(f) = \begin{pmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{pmatrix}$$

$$\text{Hessian}(f)(0,0) = \begin{pmatrix} 0 & -4 \\ -4 & 0 \end{pmatrix} \Rightarrow \det < 0 \Rightarrow \text{Saddle}$$

$$\text{Hessian}(f)(1,1) = \begin{pmatrix} 12 & -4 \\ -4 & 12 \end{pmatrix} \Rightarrow \det = 144 - 16 > 0 \Rightarrow \text{Min}$$

Since $\partial_x^2 f > 0$

$$\text{Hessian}(f)(-1,-1) = \begin{pmatrix} 12 & -4 \\ -4 & 12 \end{pmatrix} \Rightarrow \det = 144 - 16 > 0 \Rightarrow$$

So at $(0,0)$ we have a saddle

$(1,1)$ we have local min.

$(-1,-1)$

4. (10 pts) For the function

$$\mathbf{F}(x,y) = \left(\sin(xy) - x, x^2y - 1\right),$$

(a) Compute the Hessian of the function $\mathbf{F}(x,y)$ at the point (x,y).

(b) Compute the Hessian of the function $\mathbf{F}(x,y)$ at the point $(1,0)$.

(c) Give the best linear approximation to the function $\mathbf{F}(x,y)$ at the point $(1,0)$.

$$H_f(x,y) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\[2ex] \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{pmatrix}$$

$$= \begin{pmatrix} y\cos(xy) - 1 & x\cos(xy) \\ 2xy & x^2 \end{pmatrix}$$

$$H_f(1,0) = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}$$

$$\hat{F}(1,0) = (1, -1) = (1,-1)$$

$$\hat{L}(x,y) = \hat{F}(1,0) + J_{\hat{F}}(1,0)(\hat{x} - (1,0))$$

$$= (1,-1) + \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x-1 \\ y \end{pmatrix}.$$

5. (10 pts) Evaluate the following quantities:

(a) Find the directional derivative of the function $f(x, y, z) = xy^2 + z^3$ at the point $(1, -2, 1)$ in the direction $\mathbf{u} = (\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

(b) Use the chain rule to find $\frac{\partial z}{\partial t}$ if

$$z = xe^y + ye^{-x}, \quad x = e^t, \quad y = 2t^2.$$

(a) $\nabla f(x,y,z) = (y^2, 2xy, 3z^2)$

$\nabla f(1,-2,1) = (4, 2(1)(-2), 3) = (4, -4, 3)$

Directional Derivative:

$\nabla f(1,-2,1) \cdot \hat{u} = (4,-4,3) \cdot (\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$

$= \frac{4}{\sqrt{3}} + \frac{4}{\sqrt{3}} + \frac{3}{\sqrt{3}} = \boxed{\frac{11}{\sqrt{3}}}$

(b) $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \boxed{\left(e^{2t^2} - 2t^2 e^{-e^t}\right)e^t + 4t\left(e^{-e^t} + e^{t+2t^2}\right)}$

$\frac{\partial z}{\partial x} = e^y - ye^{-x}$ $\qquad \left.\frac{\partial z}{\partial x}\right|_{x,y} = e^{2t^2} - 2t^2 e^{-e^t}$

$\frac{\partial z}{\partial y} = xe^y + e^{-x}$ $\qquad \left.\frac{\partial z}{\partial y}\right|_{x,y} = e^t e^{2t^2} + e^{-e^t}$

$\frac{dx}{dt} = e^t$ $\qquad \frac{dy}{dt} = 4t$