Math 2605, **Second Exam**, 18 Mar 2010, 10am, **NAME:**

Neither cheatsheets nor calculators are allowed.

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1. (40 points) (a) Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 10 & -6 \\ -6 & 10 \end{pmatrix}$$

$$\det(A - \mu I) = \begin{vmatrix} 10-\mu & -6 \\ -6 & 10-\mu \end{vmatrix} = \mu^2 - 20\mu + 64 = 0$$

$$\mu_1 = 16: \quad \begin{bmatrix} 6 & -6 \\ -6 & 6 \end{bmatrix} \vec{v}_1 = 0, \quad \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix};$$

$$\mu_2 = 4: \quad \begin{bmatrix} -6 & -6 \\ -6 & -6 \end{bmatrix} \vec{v}_2 = 0; \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(b) Write down an orthogonal matrix U such that $U^T A U$ is diagonal.

$$\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}; \quad U = [u_1, u_2] = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

(c) Find the singular value decomposition, $A = VDU^T$ for the matrix

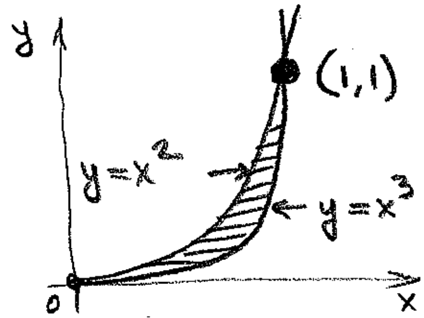
$$A = \begin{pmatrix} 3 & -1 \\ 0 & 0 \\ 1 & -3 \end{pmatrix}$$

$$\underline{\underline{D}} = \begin{bmatrix} \sqrt{\mu_1} & 0 \\ 0 & \sqrt{\mu_2} \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}; \quad \underline{\underline{U}} =$$

$$A = VDU^T \Rightarrow \underline{\underline{V}} = AUD^{-1} = \begin{bmatrix} 3 & -1 \\ 0 & 0 \\ 1 & -3 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} = \frac{1}{4\sqrt{2}} \begin{bmatrix} 3 & -1 \\ 0 & 0 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{4\sqrt{2}} \begin{bmatrix} -4 & 4 \\ 0 & 0 \\ -4 & -4 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ -1 & -1 \end{bmatrix}$$

2. (20 points) (a) Scketch the region Ω bounded by $y = x^3$ and $y = x^2$ in the first quadrant of the plane.



(b) Find the area of Ω via a repeated integral of Type I:

$$\int_0^1 \int_{x^3}^{x^2} dy\,dx = \int_0^1 (x^2 - x^3)dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{3} - \frac{1}{4} = \underline{\underline{\frac{1}{12}}}$$

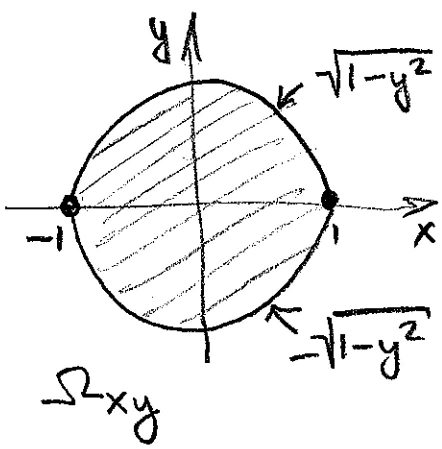
(b) Find the area of Ω via a repeated integral of Type II:

$$\int_0^1 \int_{\sqrt{y}}^{\sqrt[3]{y}} dx\,dy = \int_0^1 (\sqrt[3]{y} - \sqrt{y})\,dy = \left[\frac{3}{4}y^{\frac{4}{3}} - \frac{2}{3}y^{\frac{3}{2}}\right]_0^1$$

$$= \frac{3}{4} - \frac{2}{3} = \underline{\underline{\frac{1}{12}}}$$

3. (20 points) Calculate using polar coordinates

$$\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} e^{-(x^2+y^2)} dx\,dy.$$



$$\iint_{\Omega_{r\theta}} e^{-r^2} dr\,d\theta = \int_0^{2\pi} d\theta \int_0^1 e^{-r^2} \cdot r\,dr$$

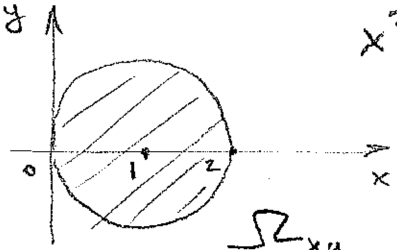
$$= 2\pi \left[\frac{-e^{-r^2}}{2}\right]_0^1 = \underline{\underline{2\pi\left(\frac{1}{2} - \frac{1}{2e}\right)}}.$$

$$\Omega_{r\theta}: \begin{cases} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{cases}$$

4. (20 points) The solid T is bounded above by the paraboloid of revolution $x^2 + y^2 = z$, below by xy-plane, and on the sides by the cylinder $x^2 + y^2 = 2x$.

(a) Draw the projection of T onto xy-plane.

$$x^2+y^2=2x \iff (x-1)^2+y^2=1$$



Ω_{xy}

(b) Find the volume of the solid.[1]

In polar coordinates: $\Omega_{r\theta}\ \begin{cases} -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ 0 \le r \le 2\cos\theta \end{cases}$

$$V = \int_{\Omega_{xy}} (x^2+y^2)\,dx\,dy = \int_{\Omega_{r\theta}} r^2 \cdot r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^3\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta = \int_{-\pi/2}^{\pi/2} 4\cos^4\theta\,d\theta = 4\cdot\frac{3}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

$$= 3\cdot\frac{1}{2}\int_{-\pi/2}^{\pi/2} d\theta = \underline{\underline{\frac{3}{2}\pi}}$$

5. (extra credit) The curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is called an *astroid*. Find the area bounded by astroid in the first quadrant.[2]

Substitute: $\begin{cases} x = aX^3 \\ y = aY^3 \end{cases}$ $\qquad J(X,Y) = 9a^2X^2Y^2$

$$\iint_{\Omega_{XY}} 9a^2X^2Y^2\,dX\,dY = \int_0^{2\pi}\int_0^1 9a^2r^5\sin^2\theta\cos^2\theta\,dr\,d\theta$$

$$= \frac{9a^2}{6}\int_0^{2\pi}\frac{\sin^2 2\theta}{4}\,d\theta = \frac{3a^2}{8}\int_0^{2\pi}\frac{1-\cos 4\theta}{2}\,d\theta = \underline{\underline{\frac{3a^2}{8}}}.$$

[1] You may use the formula $\int \cos^n(x)\,dx = \frac{\cos^{n-1}(x)\sin(x)}{n} + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$.

[2] Use the other side of the page for Problem 5 if necessary.