Math 2605, **Second Exam**, 18 Mar 2010, 3pm, **NAME:** Solutions.

Neither cheatsheets nor calculators are allowed.

1. (40 points)

(a) Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$$

$$\det(A - \mu I) = \begin{vmatrix} 1-\mu & -2 \\ -2 & 1-\mu \end{vmatrix} = \mu^2 - 2\mu - 3 = 0$$

$$\mu_1 = 3, \quad \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} \vec{v}_1 = 0, \quad \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\mu_2 = -1, \quad \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \vec{v}_2 = 0, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(b) Write down an orthogonal matrix U such that $U^T A U$ is diagonal.

$$\vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad U = [\vec{u}_1, \vec{u}_2] = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

(c) Perform the first step of Jacobi's digonalization algorithm for the matrix

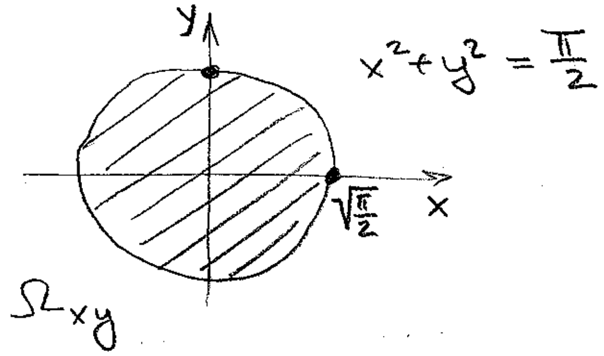
$$A = \begin{pmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

$$G_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}; \qquad AG_1 = \begin{bmatrix} -\frac{3}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 1 \\ \frac{3}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 1 \\ 0 & \frac{2}{\sqrt{2}} & 3 \end{bmatrix}$$

$$G_1^T A G_1 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & \sqrt{2} \\ 0 & \sqrt{2} & 3 \end{bmatrix}.$$

2. (20 points) Consider the solid bounded below by the xy-plane and above by the surface $z = \cos(x^2 + y^2)$ and containing a part of z-axis.

(a) Sketch the projection Ω of the solid onto the xy-plane.



(b) Find the volume of the solid using polar coordinates.

$$\iint_{\Omega_{r\theta}} \cos(r^2)\cdot r \, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^{\sqrt{\frac{\pi}{2}}} r\cos(r^2)\, dr$$

$$= 2\pi \left[\frac{1}{2}\sin(r^2)\right]_0^{\sqrt{\frac{\pi}{2}}} = \underline{\underline{\pi}}$$

$$\Omega_{r\theta}: \begin{cases} 0 \le r \le \sqrt{\frac{\pi}{2}} \\ 0 \le \theta \le 2\pi \end{cases}$$

3. (20 points) The region Ω in the plane is bounded by the lines $x + y = 0$, $x + y = 1$, $3x - 2y = 0$, and $3x - 2y = 2$. Calculate the following integral

$$I = \iint_\Omega (4x - y)\, dx\, dy.$$

Subsitition: $\begin{cases} u = x+y \\ v = 3x-2y \end{cases}$ Old coord's in terms of new: $\begin{cases} x = \frac{2u+v}{5} \\ y = \frac{3u-v}{5} \end{cases}$

$$J(u,v) = \left|\det(\text{Jacobian of transform})\right|$$

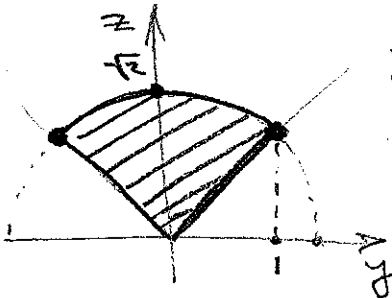
$$= \left|\det\begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{3}{5} & -\frac{1}{5} \end{bmatrix}\right| = \left|-\frac{1}{5}\right| = \frac{1}{5}.$$

$$I = \frac{1}{5}\int_0^2\int_0^1 (u+v)\,du\,dv = \frac{1}{5}\int_0^2 \left[\frac{u^2}{2}+vu\right]_0^1 dv = \frac{1}{5}\int_0^2\left(\frac{1}{2}+v\right)dv$$

$$= \frac{1}{5}\left[\frac{v^2}{2}+\frac{v}{2}\right]_0^2 = \underline{\underline{\frac{3}{5}}}.$$

4. (20 points) Consider the solid T in the upper half-space bounded by the sphere $x^2 + y^2 + z^2 = 2$ and the cone $x^2 + y^2 - z^2 = 0$.

(a) Sketch the intersection of the solid with yz-plane.



$X = 0 \Rightarrow y^2 + z^2 = 2;$
$y^2 - z^2 = 0.$

Set up the triple integral $\iiint_T y\,dx\,dy\,dz$ (do not evaluate)

(b) in spherical coordinates:

$$\Omega_{\rho\theta\varphi}: \begin{cases} 0 \le \rho \le \sqrt{2} \\ 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \frac{\pi}{4} \end{cases} \quad \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \int_0^{\sqrt{2}} \underbrace{\rho \sin\varphi \sin\theta}_{y} \underbrace{\rho^2 \sin\varphi}_{\text{multiplier}} d\rho\, d\varphi\, d\theta$$

(c) in cylindrical coordinates:

$$\Omega_{r\theta z}: \begin{cases} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \\ r \le z \le \sqrt{2 - r^2} \end{cases} \quad \int_0^{2\pi} \int_0^{1} \int_r^{\sqrt{2-r^2}} \underbrace{r \sin\theta}_{y} \cdot \underbrace{r}_{\text{multiplier}}\, dz\, dr\, d\theta$$

$r^2 - z^2 = 0$ $\qquad$ $r^2 + z^2 = 2$

5. (extra credit) Calculate the integral

$$\iiint_T \frac{z}{c}\, dx\, dy\, dz,$$

where T is the solid bounded by ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$ in the upper halfspace.[1]

Substitute: $\begin{cases} z = cZ \\ y = bY \\ x = aX \end{cases}$ $J(X,Y,Z) = abc,$ $\iiint_{X^2+Y^2+Z^2 \le 1} Z\, abc\, dX\, dY\, dZ =$

$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{1} \rho\cos\varphi \cdot abc \cdot \rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi = abc \cdot \frac{\pi}{4}$

[1] Use the other side of the page for Problem 5 if necessary.