Georgia Institute of Technology

Math 2605 - Calculus III for Computer Science

Solving systems of polynomial equations by homotopy continuation

Newton's method. In order to find an approximation to a solution of the system of polynomial equations

$$F(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = \mathbf{0} \tag{1}$$

we have discussed Newton's method. This method amounts to picking a start point $\boldsymbol{x}_0 = (x_0, y_0)$ and applying Newton's iterator –

$$N_F(\boldsymbol{x}) = \boldsymbol{x} - J(\boldsymbol{x})^{-1}F(\boldsymbol{x}),\ \ J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}$$

– to obtain a sequence of refinements $\boldsymbol{x}_{i+1} = N_F(\boldsymbol{x}_i)$ approximating a solution of the system. The algorithm is terminated when $||\boldsymbol{x}_{i+1} - \boldsymbol{x}_i|| \leq \varepsilon$ for a given bound $\varepsilon > 0$ on the desired error.

A major deficiency of this method is that it produces only one solution of the system and we have no control over which solution it is in general, since the only input we can vary is the starting point $\boldsymbol{x}_0$.

Assignment 1. *Code a function* `Newton`$(F, \boldsymbol{x}_0, \varepsilon)$ *that takes a system of two polynomials of two variables, a starting point, and the error tolerance as input and executes Newton's method returning the final approximation. Make sure the function gives up after 100 refining steps returning an error message. Try to find all solutions of the system*

$$F(x,y) = \begin{pmatrix} x^2 - 2xy + y^2 + 5x - 5y + 6 \\ x^3 + x^2y - xy^2 - y^3 + 2xy + 2y^2 - x - y \end{pmatrix} = \mathbf{0} \tag{2}$$

using `Newton`.

Homotopy continuation. One could attempt to find **all** solutions of the *target* system $F = (f_1, f_2)$ as in (1) by taking a *start* system $G = (g_1, g_2)$ of a special form and connecting the start to the target by means of a *homotopy* for $t \in [0,1]$:

$$H(x,y,t) = tF(x,y) + (1-t)G(x,y) = \begin{pmatrix} tf_1(x,y) + (1-t)g_1(x,y) \\ tf_2(x,y) + (1-t)g_2(x,y) \end{pmatrix} = \mathbf{0}. \tag{3}$$

Note that $H(x,y,0) = G(x,y)$ and $H(x,y,1) = F(x,y)$.

Assuming that the solutions to the start system G are easy to obtain we propose the following method to get the target solutions: follow the path $t \mapsto \boldsymbol{x}_t$ connecting a start solution $\boldsymbol{x}_0$ of G as t changes from 0 to 1 to obtain a target solution $\boldsymbol{x}_1$.

To follow the path choose the step $\Delta t = 1/n$ for some positive integer n and (starting with $\tilde{\boldsymbol{x}}_0 = \boldsymbol{x}_0$) increment t updating approximate value $\tilde{\boldsymbol{x}}_t$ of $\boldsymbol{x}_t$ as follows:

$$\tilde{\boldsymbol{x}}_{t+\Delta t} = \texttt{Newton}(H(x,y,t+\Delta t), \tilde{\boldsymbol{x}}_t, \varepsilon).$$

Assignment 2. *Code a function* `followpath`$(F, G, \boldsymbol{x}_0, n, \varepsilon)$ *that takes two systems of two polynomials of two variables, a starting point, the number of steps, and the error tolerance as input and implements the procedure described above producing* $\tilde{\boldsymbol{x}}_1$ *as output.*

The following statement enables us to find all solutions systematically:

Proposition. *Let* g_1 *and* g_2 *be univariate polynomials with no multiple roots such that* $\deg g_1 = \deg f_1$ *and* $\deg g_2 = \deg f_2$*. Then the homotopy (3) with the start system* $G = (\gamma g_1(x), \gamma g_2(y))$*, where* γ *is a random complex number, produces all solutions of* F *from the solutions of* G *with high probability.*

Assignment 3. *Let* F *be the system (2) and* $G = (i(x^2 - x), i(y^3 - y))^T$*. Write out all solutions* S *of* G *and execute* `followpath`$(F, G, \boldsymbol{x}_0, 100, 10^{-6})$ *for every* $\boldsymbol{x}_0 \in S$*. What target solutions do you get? What paths seem to diverge?*

Note: you have to make `Newton` *and* `followpath` *work with complex numbers.*

Another note: here we set $\gamma = i$*, which is not truly random; in general, for the numerical stability purposes, a random complex number is picked on a unit circle:* $|\gamma| = 1$*.*

Bonus Assignment:

1. *Graph the projections (onto the complex* x*- and* y*-planes) of points on the paths that* `followpath` *produces.*

2. *Explore what happens for various values of* γ *above. What problems are encountered if* γ *is real? Can you explain why?*