4.1. Ideal-variety correspondence

The correspondence between algebra and geometry about to be discussed is the core of the area called *algebraic geometry*, which uses geometric intuition on one hand and algebraic formalism on the other. Computations in polynomial rings is what drives the effective methods in algebraic geometry.

In this section we will consider the polynomial ring $R = k[x] = k[x_1, \ldots, x_n]$ over an <u>*algebraically closed*</u> field k, i.e., a field such that every univariate polynomial with coefficients in k has its roots in k. From the list of popular fields that we considered $(\mathbb{Q}, \mathbb{R}, \mathbb{C})$ only the field of complex numbers is algebraically closed; this follows from the *fundamental theorem of algebra*. Note that, e.g., polynomial x^2+1 has coefficients in $\mathbb{Q} \subset \mathbb{R}$, but its roots are not real. Therefore, neither $\mathbb{Q}$ nor $\mathbb{R}$ are not algebraically closed.

In the rest of the text we assume $k = \mathbb{C}$ and will describe the classical (complex) algebraic geometry, an area where the main object of study is a <u>*variety*</u> (as we defined above) sometimes referred to by the full name of <u>*complex affine variety*</u>.

We have already introduces the concept of a <u>*variety*</u> $\mathbb{V}(F)$ given by a system of polynomials $F \subset R$ (see §1.2). Here we would like to regard the operation $\mathbb{V}(\ldots)$ as a map

$$\begin{array}{rccl} \mathbb{V}: & \{\text{ideals}\} & \to & \{\text{varieties}\} \\ & I & \mapsto & \mathbb{V}(I) = \{\, x \in k^n \mid f(x) = 0, \text{ for all } f \in I \,\} \end{array}$$

Exercise 1.2.2 implies that $\mathbb{V}(I) = \mathbb{V}(F)$ for any generating set F of the ideal I.

In the opposite direction we have a map

$$\begin{array}{rccl} \mathbb{I}: & \{\text{varieties}\} & \to & \{\text{ideals}\} \\ & V & \mapsto & \mathbb{I}(V) = \{\, f \in R \mid f(x) = 0, \text{ for all } x \in V \,\} \end{array}$$

We refer to $\mathbb{I}(V)$ as <u>*the ideal*</u> of V. (Fuller names are <u>*vanishing ideal*</u> and <u>*defining ideal*</u>.)

EXERCISE 4.1.1. *Show that for an arbitrary set $V \subset k^n$ (not necessarily a variety) $I(V)$ is an ideal.*

EXERCISE 4.1.2. *Show that both $\mathbb{V}$ and $\mathbb{I}$ are inclusion-reversing, i.e,*
- $I \subseteq J \implies \mathbb{V}(J) \subseteq \mathbb{V}(I)$,
- $V \subseteq W \implies \mathbb{I}(W) \subseteq \mathbb{I}(V)$.

A variety V is called a <u>*hypersurface*</u> if $\mathbb{I}(V)$ is a proper principal ideal, i.e., it is defined by one nonconstant polynomial.

EXAMPLE 4.1.3. $\mathbb{V}\big(x^2+y^2+z^2-1\big)$ *is a hypersurface in k^3 with coordinates x, y, z and $\mathbb{V}\big(x^2+y^2-1\big)$ is a hypersurface in k^2 with coordinates x, y.*

If I is an ideal and $f \in R$, the variety

$$\mathbb{V}(I + \langle f \rangle) = \{\, x \in \mathbb{V}(I) \mid f(x) = 0 \,\}$$

is simply the intersection of the variety $\mathbb{V}(I)$ with the hypersurface $\mathbb{V}(f)$.

EXERCISE 4.1.4. *The variety $V = \mathbb{V}\big(y - x^2, z - x^3\big) \subset k^3$ is called the* <u>twisted cubic</u>. *Find the ideal of the projection of V onto*

(1) *the xy-plane;*
(2) *the xz-plane;*

(3) *the yz-plane. (Hint: Look at implicitization procedure in §6.2.1.)*

4.1.1. Hilbert's Nullstellensatz. "Nullstellensatz" translates as "theorem about zeros of functions" from German.

THEOREM 4.1.5 (Weak Nullstellensatz). *If $I \subseteq R$ is an ideal with $\mathbb{V}(I) = \emptyset$, then $I = R$.*

PROOF. add a reference □

THEOREM 4.1.6 (Hilbert's Nullstellensatz). *Let $I \subseteq R$ be an ideal and $f \in R$ be a polynomial vanishing at every point of the variety $\mathbb{V}(I)$.*
Then there exists $m > 0$ such that $f^m \in I$.

PROOF. add a reference □

Hilbert's Nullstellensatz is stronger than the weak Nullstellensatz: indeed, any function vanishes at every point of an empty set, hence, some power of $f = 1$ belongs to I if $\mathbb{V}(I) = \emptyset$.

EXERCISE 4.1.7. *Let $V \subseteq k^n$ be a variety and $I \subseteq R$ be an ideal. Show that*

(1) $\mathbb{V}(\mathbb{I}(V)) = V$;
(2) $I \subseteq \mathbb{I}(\mathbb{V}(I))$ *(In general, $I \neq \mathbb{I}(\mathbb{V}(I))$; e.g., see Exercise 4.1.8.)*

EXERCISE 4.1.8. *Draw the real points of the varieties $\mathbb{V}\big(x^2 + y^2 - 1\big)$, $\mathbb{V}(x - 1)$, and $V = \mathbb{V}(I)$, where $I = \big\langle x^2 + y^2 - 1, x - 1\big\rangle$.*
Show that $\mathbb{I}(V) \neq I$.

4.1.2. Radical ideals. The *radical* of an ideal $I \subseteq R$ is

$$\sqrt{I} = \{\, f \in R \,|\, f^m \in I \text{ for some } m \,\}\,.$$

An ideal $I \subseteq R$ is a *radical ideal* if $\sqrt{I} = I$.

PROPOSITION 4.1.9. *If I is an ideal, then the set $\sqrt{I}$ is an ideal.*

PROOF. Take $f, g \in \sqrt{I}$. Then there exist a, b such that $f^a, g^b \in I$.
Each term in the binomial expansion

$$(f + g)^{a+b} = \sum_{i=0}^{a+b} \binom{a+b}{i} f^i g^{a+b-i}$$

has a factor of either f^a or g^b. Therefore, $(f + g)^{a+b} \in I$ and $f + g \in \sqrt{I}$.
Also, for all $h \in R$, the multiple $hf \in \sqrt{I}$, since $(hf)^a \in \sqrt{I}$. □

Two immediate corollaries follow.

COROLLARY 4.1.10. *If I is a proper ideal, then $\sqrt{I}$ is.*

PROOF. The element $1 \notin \sqrt{I}$, since all powers of 1 are not in I. □

COROLLARY 4.1.11. *Nontrivial ideals that are* maximal *(with respect to inclusion) are radical.*

PROOF. Let $\mathfrak{m} \subsetneq R$ be a maximal ideal. Then $\mathfrak{m} \subseteq \sqrt{\mathfrak{m}} \subsetneq R$, which forces $\mathfrak{m} = \sqrt{\mathfrak{m}}$. □

EXERCISE 4.1.12. *Show that for every ideal I,*

(1) $\mathbb{V}\left(\sqrt{I}\right) = \mathbb{V}(I)$,
(2) $\sqrt{\sqrt{I}} = \sqrt{I}$.

EXERCISE 4.1.13. *For a point $a \in k^n$, show that the ideal*

$$\mathfrak{m}_a = \mathbb{I}(\{a\}) = \langle x_1 - a_1, \dots, x_n - a_n \rangle$$

is maximal.

Prove that every maximal ideal (in the polynomial ring $R = \mathbb{C}[x]$) has this form.

EXERCISE 4.1.14. *Show that for every $V \subseteq k^n$ the ideal $\mathbb{I}(V)$ is radical.*

In fact, restricted to radical ideals, the maps $\mathbb{I}$ and $\mathbb{V}$ establish a one-to-one correspondence

$$\{\text{varieties}\} \leftrightarrow \{\text{radical ideals}\}.$$

EXERCISE 4.1.15. *Show that if the ideal I is radical and $f \notin I$ then $\mathbb{V}(I + \langle f \rangle)$, the intersection of $V = \mathbb{V}(I)$ with the hypersurface $\mathbb{V}(f)$, is strictly smaller than V.*

4.1.3. Irreducible varieties and prime ideals. A variety V is *irreducible* if it can not be decomposed as $V = V_1 \cup V_2$ where $V_1, V_2 \subsetneq V$ are strictly smaller varieties.

An ideal I is *prime* if for every pair $f, g \in R$,

$$fg \in I \implies f \in I \text{ or } g \in I.$$

PROPOSITION 4.1.16. *Prime ideals and irreducible varieties are in one-to-one correspondence.*

PROOF. Let V be an irreducible variety and consider $I = \mathbb{I}(V)$, which is radical by Exercise 4.1.14. Suppose $f, g \in R$ are such that $fg \in I$, but $f, g \notin I$. Then both $V_1 = \mathbb{V}(I + \langle f \rangle) \subsetneq \mathbb{V}(I)$ and $V_2 = \mathbb{V}(I + \langle g \rangle) \subsetneq \mathbb{V}(I)$ by Exercise 4.1.17. On the other hand, a point $x \in V$ belongs either to V_1 or to V_2, since $fg \in I$. Therefore, $V = V_1 \cup V_2$ contradicts the irreducibility of V. We conclude that I is prime.

Let I be prime. If $V = \mathbb{V}(I)$ is reducible, then $V = V_1 \cup V_2$ for $V_1, V_2 \subsetneq V$. We can find $f, g \in R$ such that $f \in \mathbb{I}(V_1) \setminus \mathbb{I}(V_2)$ and $g \in \mathbb{I}(V_2) \setminus \mathbb{I}(V_1)$. Now $fg \in I$ (as fg vanishes on V), but $f, g \notin I$, which can not happen, since I is prime. Therefore, V is irreducible. □

EXERCISE 4.1.17. *Show that*

(1) *every maximal ideal is prime;*
(2) *ideal $\langle x_1, \dots, x_m \rangle$ is prime for any m;*
(3) *the hypersurface $\mathbb{V}(f)$ is irreducible iff f does not factor.*

Define an *r-plane* in k^n to be a variety $\mathbb{V}(\ell_1, \cdots, \ell_{n-r})$ where $f_i \in R_{\leq 1}$ are linearly independent linear functions. One can use Exercise 4.1.17 to show that an r-plane is irreducible. An $(n-1)$-plane is called a *hyperplane*.

4.2. Zariski topology, irreducible decomposition, and dimension

We shall introduce a *topology* on the space k^n that is weaker than the usual topology induced by the distance metric.

4.2.1. Varieties as Zariski closed sets. The *closed sets* of *Zariski topology* are varieties $V \subseteq k^n$. The *open sets* are their complements $U = k^n \setminus V$, where V is closed. The geometric intuition dictated by the definition of a variety makes the axioms of a topolgy hold. One can use basic operations discussed in §4.3 to rigorously check that this indeed is a topology:

(1) The trivial subsets and k^n are closed sets.
(2) The intersection of a collection of closed sets is a closed set.
(3) The union of a finite number of closed sets is a closed set.

Given any subset $S \subseteq k^n$, we define the Zariski closure of S

$$\overline{S} = \mathbb{V}(\mathbb{I}(S)).$$

Exercise 4.2.1. *Show that the unit ball $B = \{\, x \in \mathbb{C} \mid |x| \leq 1 \,\} \subseteq \mathbb{C}$ is neither a closed nor an open set in Zariski topology.*

What is the Zariski closure of B?

4.2.2. Irreducible decomposition of a variety. Every Zariski closed set can be decomposed into a finite union of irreducible components.

Proposition 4.2.2. *Let V be a variety. Then there exist varieties $V_i \subseteq V$, $i = 1, \dots, r$, (called* irreducible components *of V) such that*

- *each V_i is irreducible,*
- *V_i is not contained in V_j for $i \neq j$, and*
- *$V = V_1 \cup \cdots \cup V_r$.*

Moreover, such decomposition (called irreducible decomposition*) is unique.*

Proof. Consider a sequence of decompositions

$$V = V_1^{(i)} \cup \cdots \cup V_{r_i}^{(i)}$$

starting with $V = V_1^{(1)}$ where $V_1^{(1)} = V$. Given the i-th decomposition, if some component is reducible, then it can be replaced with a union of two strictly smaller varieties producing a finer decomposition at step $i + 1$.

Suppose this process does not terminate, i.e., each decomposition contains at least one reducible component. Then we can construct an infinite descending chain of varieties

$$V_{j_1}^{(1)} \supsetneq V_{j_2}^{(2)} \supsetneq V_{j_3}^{(3)} \supsetneq \cdots$$

which translates into the (strictly) ascending chain of their ideals

$$\mathbb{I}\left(V_{j_1}^{(1)}\right) \subsetneq \mathbb{I}\left(V_{j_2}^{(2)}\right) \subsetneq \mathbb{I}\left(V_{j_3}^{(3)}\right) \subsetneq \cdots$$

However, since the polynomial ring R is Noetherian, this can not happen. Therefore, an irreducible decomposition exists.

Suppose there are two irreducible decompositions: $V = V_1 \cup \cdots \cup V_r$ and another decomposition containing an irreducible component W that is distinct from all V_i. Then $V = W \cup W'$, where W' is the union of the rest of irreducible components in the second decomposition. Now note that $V_i = (V_i \cap W) \cup (V_i \cap W')$, but V_i is irreducible and $W \neq V_i$, therefore, $V_i \subseteq W'$. It follows that $V \subseteq W'$ contradicting the assumption that W is an irreducible component in a decomposition. We conclude that the irreducible decomposition is unique. □

4.2.3. Dimension. First, we define the *dimension* of an irreducible variety $V \neq \emptyset$ to be the maximal length d of a chain

$$\emptyset \neq V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_d = V$$

where V_i are irreducible varieties. Note that the dimension is finite due to Noetherianity.

The *dimension* of an arbitrary variety V is defined as the maximal dimension of its irreducible components.

EXERCISE 4.2.3. *Show that a finite set of points is 0-dimensional.*

Note: The geometric notion of dimension defined above corresponds to the algebraic notion of *Krull dimension*: the dimension of an ideal $I \subseteq R$ is the maximal length d of the chain

$$R \neq P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq P_d \supset I$$

where P_i are prime ideals.

For a variety $\dim V = \dim \mathbb{I}(V)$; for an ideal $\dim I = \dim \mathbb{V}(I)$.

The *dimension reduction principle* says that intersecting of a variety V, $\dim V = m$, with a generic r-plane L of $\operatorname{codim} L = n - r \leq m$, reduces the dimension by $n - r$:

$$\dim(V \cap L) = \dim V - \operatorname{codim} L = m - n + r.$$

Note that if $\operatorname{codim} L > m$ then a generic r-plane L misses V, i.e., $V \cap L = \emptyset$.

REMARK 4.2.4. *A variety $V \subseteq k^n$ and a generic r-plane, where $r = \operatorname{codim} V = n - \dim V$, intersect at finitely many points.*

In fact, this generic intersection is 0-dimensional iff $r = \operatorname{codim} V$. This observation gives a way to determine $\dim V$.

EXERCISE 4.2.5. *Using the dimension reduction principle show that*

(1) $\mathbb{V}(\langle x_1, \dots, x_m \rangle) \subseteq k^n$ *has dimension $n - m$ (codimension m);*
(2) *the dimension of an r-plane is r;*
(3) *a hypersurface $\mathbb{V}(f)$ in k^n has dimension $n - 1$ (codimension 1).*

A polynomial system $F = (f_1, \dots, f_c) \subseteq R$ is a *regular sequence* if

$$\dim \mathbb{V}(f_1, \dots, f_m) = n - m, \quad m = 1, \dots, c.$$

Note: In algebraic terms, F is a regular sequence if f_m is not a zero-divisor in $R/\langle f_1, \dots, f_{m-1} \rangle$, $m = 1, \dots, c$.

A system $F = (f_1, \dots, f_c) \subseteq R$ is a *local regular sequence* with respect to an irreducible variety V if

$$\dim(\text{irreducible component of } \mathbb{V}(f_1, \dots, f_m) \text{ that contains } V) = n - m, \quad m = 1, \dots, c.$$

Note: Our definition of a local regular sequence is equivalent to the definition of *set-theoretic local regular sequence* (at a generic point of an irreducible variety V) given in *commutative algebra*.

An irreducible variety V is a *local complete intersection* if there is a local regular sequence F (with respect to V), such that V is an irreducible component of $\mathbb{V}(F)$.

The following algorithm gives a way to construct such F given polynomials that define V.

Algorithm 4.2.1 $F = \text{LocalCompleteIntersection}(G, V)$

Require: $G = (g_1, \ldots, g_r) \subset R = \mathbb{C}[x_1, \ldots, x_n]$, a system of polynomials;
V, an irreducible component of $\mathbb{V}(G)$.
Ensure: F is a local regular sequence with respect to V is an irreducible component of $\mathbb{V}(F)$.

$F \leftarrow \emptyset$
$W \leftarrow \mathbb{C}^n$
for $i = 1$ to r **do**
  if there is an irreducible component W' of $\mathbb{V}(F \cup \{g_i\})$ such that

$$V \subseteq W' \subsetneq W$$

  then
    $F \leftarrow F \cup \{g_i\}$
    $W \leftarrow W'$
  end if
end for

Proof of correctness of Algorithm 4.2.1. At every step of the algorithm when a polynomial gets appended to the system F the new irreducible component W' containing V is strictly smaller than the old (irreducible variety) W. Hence, the dimension of W' is smaller than that of W; in fact,

$$\dim W' = \dim W - 1,$$

since we add only one more polynomial.

Let $c = |F|$ when algorithm terminates and

$$W_{i_1} \supsetneq W_{i_2} \supsetneq \cdots \supsetneq W_{i_c}$$

be the sequence of irreducible varieties produced (the values that W takes at the end of the loop at steps $1 \leq i_1 < i_2 < \cdots < i_c \leq r$). Showing that $W_{i_c} = V$ would conclude this proof.

Suppose $W_c \neq V$. Then there exists $g_i \in G$ such that $W_i' = W_{i-1}$ (the case when g_i does not get appended to F) and $W_{i_c} \cap \mathbb{V}(g_i) \neq V$. But then $W_i' \subseteq W_{i-1} \cap \mathbb{V}(g_i) \neq W_{i-1}$, which produces a contradiction. □

Note that it is implied that the number of polynomials in the system F that the algorithm outputs equals $c = \operatorname{codim} V$.

4.3. Basic computational operations

Here we shall discuss what operations on varieties correspond to basic operations on ideals

4.3.1. Intersection of varieties: sum of ideals. Let $V, W \subseteq k^n$ be varieties; $V = \mathbb{V}(I)$ and $W = \mathbb{V}(J)$ for some ideals $I, J \in R$. Then $V \cap W = \mathbb{V}(I + J)$, since $V \cap W$ is precisely the set of points on which both polynomials of I and J vanish.

Note that even if I and J are radical ideals, $I + J$ is not radical in general.

EXAMPLE 4.3.1. *The ideals $I = \langle y \rangle$ and $J = \langle y - x^2 \rangle$ in $k[x, y]$ are radical, since they have irreducible principle generators. However,*

$$I + J = \langle y, y - x^2 \rangle = \langle x^2, y \rangle$$

is not radical as $\sqrt{I + J} = \mathfrak{m}_0 = \langle x, y \rangle$.

4.3.2. Union of varieties: intersection or multiplication of ideals. Both intersection and multiplication of ideals correspond to taking union of the corresponding varieties.

EXERCISE 4.3.2. *Let $V, W \subseteq k^n$ be varieties; $V = \mathbb{V}(I)$ and $W = \mathbb{V}(J)$ for some ideals $I, J \in R$.*

(1) *Show that $V \cup W = \mathbb{V}(IJ) = \mathbb{V}(I \cap J)$;*
(2) *Show that if $I$and J are radical, $I \cap J$ is.*
(3) *Find an example of radical I and J such that IJ is not.*

Let $I = \langle f_1, \ldots, f_r \rangle$ and $J = \langle g_1, \ldots, g_s \rangle$. While finding a set of generators for IJ is straightforward, namely

$$IJ = \langle\, f_i g_j \mid i = 1, \ldots, r; j = 1, \ldots, s \,\rangle\,,$$

how would one construct generators of $I \cap J$?

PROPOSITION 4.3.3. *Given $I = \langle f_1, \ldots, f_r \rangle$ and $J = \langle g_1, \ldots, g_s \rangle$, ideals in $R = k[x] = k[x_1, \ldots, x_n]$, define*

$$K = tI + (1 - t)J = \langle tf_1, \ldots, tf_r, (1 - t)g_1, \ldots, (1 - t)g_s \rangle \subset k[x, t].$$

Then $K \cap k[x] = I \cap J$.

PROOF. The inclusion $I \cap J \subseteq K \cap k[x]$ is straightforward: if $f \in I \cap J \subseteq k[x]$ then $f = tf + (1 - t)f$ is in K (and still in $k[x]$).

Suppose $f \in K \cap k[x]$, i.e., $f = g + h$ where $g \in tI$ and $h \in (1 - t)J$. Then $g|_{t=0} = 0$, therefore,

$$f = f|_{t=0} = g|_{t=0} + h|_{t=0} = h|_{t=0} \in J.$$

Similarly,

$$f = f|_{t=1} = g|_{t=1} + h|_{t=1} = g|_{t=1} \in I.$$

We conclude that $f \in I \cap J$. □

Note that Proposition 4.3.3 provides an algorithmic way to compute generators of $I \cap J$ via elimination.

4.3.3. Difference of varieties: colon ideal. The difference of two varieties V and W is not a variety in general: for instance take $V = \mathbb{V}(x)$ and $W = \mathbb{V}(y)$ as varieties in k^2. Then $V \setminus W = V \setminus \{(0,0)\}$ is not Zariski closed: $\overline{V \setminus W} = V$ as every polynomial that vanishes on $V \setminus W$ must vanish on $\{(0,0)\}$ as well.

A good questions is: how to find $\overline{V \setminus W}$ in general?

The answer is not hard to give if one can construct an irreducible decomposition

$$V = V_1 \cup \cdots \cup V_r.$$

In this case,

$$\overline{V \setminus W} = \bigcup_{\{\, i \,|\, V_i \not\subseteq W \,\}} V_i.$$

In algebraic language (without resorting to decomposition algorithms), the construction of <u>*colon ideal*</u> (also called <u>*quotient ideal*</u>) provides an answer. Let $I, J \subseteq R$ be ideals, define

$$I : J = \{\, f \in R \,|\, fJ \subseteq I \,\}\,.$$

EXERCISE 4.3.4. *Show that $I : J$ is an ideal.*

PROPOSITION 4.3.5. *If $I, J \subseteq R$ and I is a radical ideal, then*

$$\overline{\mathbb{V}(I) \setminus \mathbb{V}(J)} = \mathbb{V}(I : J).$$

Moreover, $\mathbb{I}\Big(\overline{\mathbb{V}(I) \setminus \mathbb{V}(J)}\Big) = I : J.$

PROOF. Let $V = \mathbb{V}(I) = V_1 \cup \cdots \cup V_r$ be the irreducible decomposition, $W = \mathbb{V}(J)$, and set

$$V' = \bigcup_{\{\, i \,|\, V_i \not\subseteq W \,\}} V_i.$$

Taking $f \in I : J$ we can show that f vanishes on all $V_i \not\subseteq W$: pick a polynomial $g \in J$ such that $g \notin \mathbb{I}(V_i)$ then $fg \in fJ \subseteq I$. Since g does not vanish on V_i, the polynomial f has to.

On the other hand, take $f \in \mathbb{I}(V')$ then $fg \in I$ for every $g \in J$ and, hence, $f \in I : J$. □

EXERCISE 4.3.6. *If $I = \langle f_1, \ldots, f_r \rangle$ and $J = \langle g \rangle$ one can compute the generators h_i of the intersection,*

$$I \cap J = \langle h_1, \ldots, h_s \rangle\,.$$

Show that $I : J = \langle h_1/g, \ldots, h_s/g \rangle$.

EXERCISE 4.3.7. *Show that $I : (J + K) = (I : J) \cap (I : K)$ for any ideals I, J, K.*

The exercises above give a way of constructing generators of $I : J$ algorithmically for any $I = \langle f_1, \ldots, f_r \rangle$ and $J = \langle g_1, \ldots, g_s \rangle$:

(1) compute generators of $K_i = I : \langle g_i \rangle$ using the conslusion of Exercise 4.3.6,
(2) compute generators of $K_1 \cap \cdots \cap K_s$.

The output of the second step is generators of $I : J$ by Exercise 4.3.7.

Note: One can find algorithmically $\overline{\mathbb{V}(I) \setminus \mathbb{V}(J)}$ without the assumption of radicality of I that Proposition 4.3.5 makes. To that end, one needs to compute the *saturation* of I with respect to J,

$$I : J^{\infty} = \{\, f \in R \mid fJ^m \subseteq I \text{ for some } m \,\}\,.$$

The saturation ideal $I : J^{\infty}$ is the ideal at which the chain

$$I : J \subseteq I : J^2 \subseteq I : J^3 \subseteq \cdots$$

stabilizes.

4.3.4. Projection of variety: intersection with a subring. Define $\pi_m : k^n \to k^m$ to be the projection map that sends $(x_1, \ldots, x_n)$ to $(x_1, \ldots, x_m)$.

PROPOSITION 4.3.8. *Let $V = \mathbb{V}(I) \subseteq k^n$ be a variety given by some ideal $I \subseteq k[x_1, \ldots, x_n]$.*

Then $\overline{\pi_m(V)} = \mathbb{V}(I_m)$, where $I_m = I \cap k[x_1, \ldots, x_m]$.

PROOF. A point $(a_1, \ldots, a_m) \in \pi_m(V)$ clearly satisfies all polynomials in $I \cap k[x_1, \ldots, x_m]$. Hence, $\overline{\pi_m(V)} \subseteq \mathbb{V}(I)$

Let $f \in \mathbb{I}(\pi_m(V))$ then $f \in \mathbb{I}(V) = \sqrt{I}$. Since $\mathbb{I}(\pi_m(V)) \subseteq \sqrt{I}$, the inclusion reversing property of ideal-variety correspondence (Exercise 4.1.2) implies that $\mathbb{V}(I) \subseteq \overline{\pi_m(V)}$. □

Note that this proposition makes the closure of the projection of a variety computable via elimination.

Projections of varieties (more generally images of varieties under algebraic maps) are not Zariski closed.

EXAMPLE 4.3.9. *Consider $V = \mathbb{V}(xy - 1) \subset k^2$. The projection $\pi_x(V)$ to the coordinate x is the set $k \setminus \{0\}$, which is not Zariski closed.*

Note: In fact, projections of varieties (more generally, images of varieties under algebraic maps) are *constructible sets*, i.e., sets of the form

$$V = (V_1 \setminus W_1) \cup \cdots \cup (V_r \setminus W_r),$$

where V_i and W_i are Zariski closed.