   2. Show your work and explain your answers and reasoning.
   3. Calculators may be used, but pay particular attention to instruction 2. 
      **To receive credit, you must show your work.** Unexplained answers, 
      and answers not supported by the work you show, will not receive 
      credit.
   4. Express your answers in simplified form.

1. (25) Find a basis for the image and a basis for the kernel of the matrix

   \[ \begin{pmatrix} 1 & 2 & 1 & 0 & 1 \\ 2 & 4 & 3 & 0 & 4 \\ 0 & 0 & -1 & 2 & 2 \\ 1 & 2 & 2 & 0 & 3 \end{pmatrix} \]

   Compute the dimension of the image and the dimension of the kernel.

2. (25) Find the QR factorization of the matrix

   \[ \begin{pmatrix} 1 & 0 & 2 \\ 2 & 2 & 0 \\ 1 & 2 & 4 \end{pmatrix} \]

   by applying the Gram-Schmidt process to the columns of A.

3. (25) A matrix A has QR factorization

   \[ A = QR = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \\ -1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \]

   Use this QR factorization to find

   a. The least-squares solution to \( A x = \begin{pmatrix} 3 \\ -3 \\ -1 \\ 7 \end{pmatrix} \)

   b. The matrix for the orthogonal projection onto the image of A.

4. (25) Find a diagonal matrix D and an invertible matrix V such that

   \[ A = \begin{pmatrix} 5 & -4 \\ 3 & -2 \end{pmatrix} = VDV^{-1} \]

   and use them to calculate \( A^8 \).
Answers.

1. A basis for the image can be found by row reducing $A$ to identify the pivot columns

and then using the pivot columns of $A$. Thus a basis is

$$\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix},$$

and the image of $A$ has dimension 3. A basis for the kernel of $A$ is found from the general

solution to $Ax = 0$, and is

$$\begin{bmatrix} 1 \\ 0 \\ -2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. $$

The kernel has dimension 2.

2. $R = \begin{bmatrix} \sqrt{6} & \sqrt{6} & \sqrt{6} \\ 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & 2\sqrt{3} \end{bmatrix}$, $Q = \begin{bmatrix} 1 \\ \sqrt{2} \\ \sqrt{3} \\ \sqrt{6} \\ 1 \\ \sqrt{2} \\ \sqrt{3} \\ \sqrt{6} \\ 1 \end{bmatrix}$

3. a. Solve $Rx = Q^T b$ for least squares solution $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

b. $P = QQ^T = \frac{1}{4} \begin{bmatrix} 2 & 0 & -2 & 0 \\ 0 & 2 & 0 & -2 \\ -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \end{bmatrix}$

4. $S = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $A^8 = \begin{bmatrix} 1021 & -1020 \\ 765 & -764 \end{bmatrix}$