Instructions: 1. Closed book, calculators may be used, but I would prefer all answers to be expressed exactly (no decimals).
2. Please show your work and justify your answers.
3. Please do any four of the five problems. Attach this cover sheet to your exam, and indicate clearly on the table below which problem you do not want graded.
4. Please begin each problem on a new sheet of paper.

1. For each matrix $A$, determine whether or not there exists an invertible matrix $S$ such that $S^{-1}AS = D$ is real and diagonal and whether or not there exists an orthogonal matrix $Q$ such that $Q^{T}AQ = D$ is real and diagonal. Note that while you might not need to calculate $S$, $D$, or $Q$, you must explain your reasoning.

   a. $A = \begin{pmatrix} 3 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 3 \end{pmatrix}$

   b. $A = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{pmatrix}$

   c. $A = \begin{pmatrix} -1 & 8 & -41 \\ 0 & 1 & -10 \\ 0 & 0 & -1 \end{pmatrix}$

   d. $A = \begin{pmatrix} -1 & 8 & -40 \\ 0 & 1 & -10 \\ 0 & 0 & -1 \end{pmatrix}$

2. With $A = \begin{pmatrix} -1 & -32 & 0 \\ 0 & 1 & 0 \\ 0 & -10 & -1 \end{pmatrix}$ find an invertible matrix $S$ and a real diagonal matrix $D$ such that $S^{-1}AS = D$.

3. Let $x$ be an eigenvector for the square matrix $A$ with eigenvalue $\lambda$.

   a. Prove that $x$ is an eigenvalue for $A^2$ with eigenvalue $\lambda^2$.

   b. Prove that $x$ is an eigenvector for $A - cI$ with eigenvalue $\lambda - c$. 
c. Prove that if $A - cI$ is nonsingular, then $x$ is an eigenvector for $(A - cI)^{-1}$ with eigenvalue $\frac{1}{\lambda - c}$.

4. Prove that if $A$ is a real, symmetric, positive definite matrix, then there exists a real, symmetric, positive definite matrix $B$ such that $B^2 = A$. (Suggestion: First show the result is correct if $A$ is diagonal and positive definite.)

5. The function

$$f(x, y, z) = 3x^2 + 2xy + 3y^2 + 3z^2 - 2xz - 2yz$$

has a critical point at $(0,0,0)$. Calculate the Hessian matrix and use it to determine if $f$ has a local minimum, local maximum, or saddle point at $(0,0,0)$. 