Instructions:  
2. Show your work and explain your answers and reasoning.
3. Calculators may be used, but pay particular attention to instruction 2.
   **To receive credit, you must show your work.** Unexplained answers, and answers not supported by the work you show, will not receive credit.
4. Express your answers in simplified form.

1. (35) Let \( A = \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix} \)

   a. Find the characteristic polynomial \( p(\lambda) = |\lambda I - A| \) of \( A \) and use it to calculate the eigenvalues of \( A \).

   b. Find a norm one eigenvector for each of the eigenvalues you found in part a and sketch these eigenvectors on the axes provided.

   c. Find a rotation matrix \( R \) and a diagonal matrix \( D \), for which \( R^T A R = D \).

2. (30) An unknown matrix \( A \) has QR factorization with \( R = \begin{bmatrix} \frac{1}{\sqrt{2}} & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \) and

\[
Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}
\]

Use this QR factorization to find the least squares solution, \( \hat{x} \) of \( A \hat{x} = \hat{b} \). Please note that actually finding the matrix \( A \) is a counterproductive waste of time. Don't do it!!!
3. (35) Let $A = \begin{pmatrix} 5 & 1 & 0 & 2 \\ 1 & 3 & 4 & 0 \\ 5 & 1 & 0 & 2 \\ 1 & 3 & 2 & 2 \end{pmatrix}$ and let $v = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$.

a. Compute $Av$ and verify that $v$ is a norm one eigenvector for $A$. What's its eigenvalue?

b. Find a Householder reflection $H$ that takes $v$ to $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

c. Compute $H^t A H$.

d. You've probably noticed by now that you've just completed the first step for finding the Schur decomposition of $A$. Describe the next step words, not computations.

Answers

1. a. $p(l) = l^2 + 4l + 21$, $l = 7, 1$

b. $\begin{pmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{pmatrix}$, $\begin{pmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{pmatrix}$

c. $\begin{pmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{pmatrix}$ is one of four correct answers.

2. 

3. a. The eigenvalue is 2.
b. \[ H = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \]

c. \[ A_1 = \frac{1}{4} \begin{bmatrix} 8 & 8 & 16 & 8 \\ 8 & 16 & 0 & 0 \\ 16 & 16 & 8 & 0 \\ 8 & 16 & 0 & 0 \end{bmatrix} \]

d. Next, multiply \( A_1 \) on both sides (as in part c) by a matrix
\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & * & * & * \\
0 & * & * & * \\
0 & * & * & *
\end{bmatrix},
\]
where the starred submatrix is a reflection that takes a norm one eigenvector of
\[
\begin{bmatrix}
8 & 16 & 0 & 0 \\
16 & 16 & 8 & 0 \\
8 & 16 & 0 & 0 \end{bmatrix}
\]